JUSTIFICATION OF TWO DIMENSIONAL MODEL OF SHALLOW SHELLS USING GAMMA CONVERGENCE

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In this paper we derive the two dimensional model of elastic shallow shell using gamma convergence. We consider thin elastic shallow shells of "very small" thickness and we show that the sequence of functions minimizing the energy associated with the three-dimensional elastic shallow shells converges to the function which minimizes the energy associated with the two dimensional elastic shallow shell as the thickness of the shell goes to zero.

Key words: Shallow shell; gamma convergence; elasticity.

1. INTRODUCTION

Lower dimensional models are preferred to three dimensional models when the thickness of the shell is "very small". The main reasons for preferring lower dimensional models are their amenability to numerical computations and their simpler mathematical structure produces richer variety of results.
There are many approaches to justify the lower dimensional models. One way of doing is by formal asymptotic method. In this method, the three dimensional solution is first scaled in an appropriate manner so as to be defined in a fixed domain, then expanded as a formal series expansion in terms of small parameter, $\epsilon$, which is the thickness of the material. The formal series expansion of the scaled solution is then inserted into the three-dimensional problem, and sufficiently many factors of the successive powers of $\epsilon$ found in this fashion are equated to zero until the leading term of the expansion can be computed and hopefully, identified with the scaled solution of a known lower dimensional model (cf: [3, 4]).

Another approach is to justify using asymptotic analysis in which one shows that the three-dimensional scaled solution converge in some Hilbert space to the solution of the lower-dimensional model. Using this approach, the boundary value problems for linear elastic shells were justified by Ciarlet et al., ([2, 5, 6, 7, 16]) and the corresponding eigenvalue problems were justified by Kesavan and Sabu([10-14]).

Third approach to justify the lower dimensional model is through $\Gamma$-convergence. Here the main idea is to show that the sequence of energy functionals associated with the three dimensional models converges to the energy functional associated with the lower dimensional model. Using this, Bourquin et al., [1] have justified the two-dimensional model of elastic plates and Geneve [9] has justified the two-dimensional model of elastic membrane and flexural shells, LeDret and Raoult [12] have justified the nonlinear membrane case and Sabu [15] has justified an one dimensional model of elastic rods. In this paper we justify the two-dimensional model of elastic shallow shells using $\Gamma$ convergence. We first show that the scaled energy functional $J(\epsilon)$ associated with the three dimensional problem is weakly lower semi continuous. Then we construct a class of test functions for which the energy functional $J(\epsilon)$ converges to the energy functional $J(\nu)$ of the two dimensional problem and then we show the strong convergence of the displacements.
2. THE THREE-DIMENSIONAL PROBLEM

Throughout this paper, Latin indices vary over the set \{1, 2, 3\} and Greek indices over the set \{1, 2\} for the components of vectors and tensors. The summation over repeated indices will be used.

Let \( \omega \subset \mathbb{R}^2 \) be a bounded domain with a Lipschitz continuous boundary \( \gamma \) and let \( \omega \) lie locally on one side of \( \gamma \). Let \( \gamma_0 \subset \partial \omega \) with \( \text{meas}(\gamma_0) > 0 \). Let \( \gamma_1 = \partial \omega \setminus \gamma_0 \). For each \( \epsilon > 0 \), we define the sets

\[
\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \quad \Gamma_{-\epsilon} = \omega \times \{-\epsilon\}, \quad \Gamma_{0} = \gamma_0 \times (-\epsilon, \epsilon), \quad \Gamma_1 = \gamma_1 \times (-\epsilon, \epsilon).
\]

Let \( x^\epsilon = (x_1, x_2, x_3^\epsilon) \) be a generic point on \( \Omega^\epsilon \) and let \( \partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha} \) and \( \partial_3 = \partial_3^\epsilon. \)

We assume that for each \( \epsilon \), we are given a function \( \theta^\epsilon : \omega \to \mathbb{R} \) of class \( C^3 \). We then define the map \( \phi^\epsilon : \omega \to \mathbb{R}^3 \) by

\[
\phi^\epsilon(x_1, x_2) = (x_1, x_2, \theta^\epsilon(x_1, x_2)) \text{ for all } (x_1, x_2) \in \omega. \tag{2.1}
\]

At each point of the surface \( S^\epsilon = \phi^\epsilon(\omega) \), we define the normal vector

\[
a^\epsilon = (|\partial_1 \theta^\epsilon|^2 + |\partial_2 \theta^\epsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1 \theta^\epsilon, -\partial_2 \theta^\epsilon, 1).
\]

For each \( \epsilon > 0 \), we define the mapping \( \Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3 \) by

\[
\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1, x_2) + x_3^\epsilon a^\epsilon(x_1, x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon. \tag{2.2}
\]

It can be shown that there exists an \( \epsilon_0 > 0 \) such that the mapping \( \Phi^\epsilon : \Omega^\epsilon \to \Phi^\epsilon(\Omega^\epsilon) \) is a \( C^1 \) diffeomorphism for all \( 0 < \epsilon \leq \epsilon_0 \). The set \( \hat{\Omega}^\epsilon = \Phi^\epsilon(\Omega^\epsilon) \) is the reference configuration of the shell.

We define vectors \( g_i^\epsilon \) and \( g_i^\epsilon \) by the relations

\[
g_i^\epsilon = \partial_i^\epsilon \Phi^\epsilon \text{ and } g_i^{\epsilon \xi} = \delta_i^\xi.
\]
which form the covariant and contravariant basis respectively of the tangent plane \( \Phi'(\Omega') \) at \( \Phi'(x') \). The covariant and contravariant metric tensors are given respectively by

\[
g'_{ij} = g'_{i} \cdot g'_{j} \quad \text{and} \quad g'^{ij,\epsilon} = g^{i,\epsilon} \cdot g^{j,\epsilon}.
\]

The Christoffel symbols are defined by

\[
\Gamma^{p,\epsilon}_{ij} = g^{p,\epsilon} \cdot \frac{\partial}{\partial x'} g^{p}_{i}.
\]

Note however that when the set \( \Omega' \) is of the special form \( \Omega' = \omega \times (-\epsilon, \epsilon) \) and the mapping \( \Phi' \) is of the form (2.2), the following relations hold,

\[
\Gamma^{3,\epsilon}_{\alpha 3} = \Gamma^{p,\epsilon}_{33} = 0.
\]

The volume element is given by \( \sqrt{g'} dx' \) where

\[
g' = \det(g'_{ij}).
\]

It can be shown that for \( \epsilon \) sufficiently small, there exist constants \( g_1 \) and \( g_2 \) such that

\[
0 < g_1 \leq g' \leq g_2. \tag{2.3}
\]

Let \( A^{ijkl,\epsilon} \) be the elastic tensor. We assume that the material of the shell is homogeneous and isotropic. Then the elasticity tensor is given by

\[
A^{ijkl,\epsilon} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}) \tag{2.4}
\]

where \( \lambda \) and \( \mu \) are the Lamè constant of the material.

This tensor satisfies the following coercive relations. There exists a constant \( c > 0 \) such that for all symmetric tensors \( (t_{ij}) \),

\[
...
\[ A_{ijkl,e} \delta_{kl} t_{ij} \geq c \sum_{i,j=1}^{3} (t_{ij})^2. \] (2.5)

Moreover we have the symmetries

\[ A_{ijkl,e} = A_{klji,e} = A_{ijkl,\epsilon}. \]

We define the space

\[ V^\epsilon = \{ v \in (H^1(\Omega))^{3}, v\big|_{\Gamma_0} = 0 \}. \] (2.6)

Then the variational form of the problem is to find \( u^\epsilon \in V^\epsilon \) such that

\[ a^\epsilon(u^\epsilon, v^\epsilon) = l^\epsilon(v^\epsilon) \text{ for all } v^\epsilon \in V^\epsilon \] (2.7)

where

\[ a^\epsilon(u^\epsilon, v^\epsilon) = \int_{\Omega^\epsilon} A_{ijkl,e} e_{k||l}(u^\epsilon) e_{i||j}(v^\epsilon) \sqrt{g^\epsilon} dx^\epsilon, \] (2.8)

\[ l^\epsilon(v^\epsilon) = \int_{\Omega^\epsilon} f^\epsilon : v^\epsilon \sqrt{g^\epsilon} dx^\epsilon, \] (2.9)

\[ e_{i||j}(v^\epsilon) = \frac{1}{2} \left( \frac{\partial v_i^\epsilon}{\partial x_j^\epsilon} + \frac{\partial v_j^\epsilon}{\partial x_i^\epsilon} \right) - \Gamma_{ij}^{\epsilon} v_p^\epsilon. \] (2.10)

Also \( u^\epsilon \) can be characterized as the minimizer of the following functional.

\[ J^\epsilon(u^\epsilon) = \min_{v^\epsilon \in V^\epsilon} J^\epsilon(v^\epsilon) \] (2.11)

where

\[ J^\epsilon(v^\epsilon) = \frac{1}{2} \left\{ \int_{\Omega^\epsilon} A_{ijkl,e} e_{k||l}(v^\epsilon) e_{i||j}(v^\epsilon) \sqrt{g^\epsilon} dx^\epsilon \right\} \]

\[ - \int_{\Omega^\epsilon} f^\epsilon v^\epsilon \sqrt{g^\epsilon} dx^\epsilon \forall v^\epsilon \in V^\epsilon. \] (2.12)
3. The Scaled Problem

We now perform a change of variable so that the domain no longer depends on $\epsilon$.

Let $\Omega = \omega \times (-1, 1)$. With $x = (x_1, x_2, x_3) \in \Omega$, we associate $x^\epsilon = (x_1, x_2, \epsilon x_3) \in \Omega^\epsilon$. Let

$$\Gamma_0 = \gamma_0 \times (-1, 1), \quad \Gamma_1 = \gamma_1 \times (-1, 1), \quad \Gamma^\pm = \omega \times \{ \pm 1 \}.$$ 

With the functions $\Gamma^{p, \epsilon}, g^\epsilon, A^{ijkl, \epsilon}$ we associate the functions $\Gamma^p(\epsilon), g(\epsilon), A^{ijkl}(\epsilon) : \Omega \to \mathbb{R}$ defined by

$$\Gamma^p(\epsilon)(x) := \Gamma^{p, \epsilon}(x^\epsilon), \quad g(\epsilon)(x) = g^\epsilon(x^\epsilon), \quad A^{ijkl}(\epsilon)(x) = A^{ijkl, \epsilon}(x^\epsilon).$$  (3.1)

**Assumption**: We assume that the shell is a shallow shell; i.e. there exists a function $\theta \in C^3(\omega)$ such that

$$\phi^\epsilon(x_1, x_2) = (x_1, x_2, \epsilon \theta(x_1, x_2)),$$  for all $(x_1, x_2) \in \omega$.  (3.2)

In this case, we make the following scalings on the unknowns.

$$u^\epsilon_\alpha(x^\epsilon) = \epsilon^2 u_\alpha(\epsilon)(x), \quad v_\alpha(x^\epsilon) = \epsilon^2 v_\alpha(x),$$  (3.3)

$$u^\epsilon_3(x^\epsilon) = \epsilon u_3(\epsilon)(x), \quad v_3(x^\epsilon) = \epsilon v_3(x).$$  (3.4)

With the applied body forces $f^\epsilon$, we associate the function $f(\epsilon)$ through the relation

$$f^\epsilon_\alpha(x^\epsilon) = \epsilon^2 f_\alpha(\epsilon)(x), \quad f^\epsilon_3(x^\epsilon) = \epsilon^3 f_3(\epsilon).$$  (3.5)

With the tensors $e^\epsilon_{ij \{ ij \}}$, we associate the tensors $e_{ij \{ ij \}}(\epsilon)$ through the relation
\[ e_{\epsilon i j}(v')(x') = \epsilon^2 e_{\epsilon i j}(\epsilon; v)(x). \] (3.6)

We define the space
\[ V = \{ v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0 \}. \] (3.7)

With the energy functional \( J^* \), we associate the energy functional \( J(\epsilon) \) as
\[ J'(v') = \epsilon^4 J(\epsilon)(v(\epsilon)). \] (3.8)

Then the scaled unknown \( u(\epsilon) \) satisfies
\[
\begin{align*}
\int_{\Omega} A^{ijkl}(\epsilon)e_{k l}(\epsilon)(\epsilon)(u(\epsilon))e_{i j}(\epsilon)(\epsilon)(v)\sqrt{\varphi(\epsilon)}dx \\
= \int_{\Omega} f_i(\epsilon)v_i\sqrt{\varphi(\epsilon)}dx \text{ for all } v \in V
\end{align*}
\] (3.9)

and \( u(\epsilon) \) is the minimizer of the functional
\[
J(\epsilon)(v) = \frac{1}{2} \left\{ \int_{\Omega} A^{ijkl}(\epsilon)e_{k l}(\epsilon)(\epsilon)(v)e_{i j}(\epsilon)(\epsilon)(v)\sqrt{\varphi(\epsilon)}dx \right\} \\
- \int_{\Omega} f_i(\epsilon)v_i(\epsilon)\sqrt{\varphi(\epsilon)}dx \text{ for all } v \in V. \] (3.10)

4. TECHNICAL PRELIMINARIES

In the sequel, we denote by \( C_1, C_2, \ldots, C_n \) various constants whose values do not depend on \( \epsilon \) but may depend on \( \theta \).

**Lemma 4.1** — The functions \( e_{\epsilon i j}(\epsilon, v) \) defined in (3.6) are of the form
\[
\begin{align*}
ee_{\alpha i j}(v) &= \bar{e}_{\alpha i j}(v) + \epsilon^2 \bar{e}_{\alpha i j}(\epsilon)(v), \quad (4.1) \\
e_{\alpha i \beta}(v) &= \frac{1}{\epsilon} \{ \bar{e}_{\alpha i \beta}(v) + \epsilon^2 \bar{e}_{\alpha i \beta}(\epsilon)(v) \}, \quad (4.2) \\
e_{3 i j}(v) &= \epsilon^2 \bar{e}_{3 i j}(v), \quad (4.3)
\end{align*}
\]
where

\[ \tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3 \partial_\alpha \theta = e_{\alpha\beta}(v) - v_3 \partial_\alpha \theta, \]  

(4.4)

\[ \tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha) = e_{\alpha 3}(v), \]  

(4.5)

\[ \tilde{e}_{3 3}(v) = \partial_3 v_3 = e_{3 3}(v), \]  

(4.6)

and there exists constant \( C_1 \) such that

\[ \sup_{0 < \epsilon \leq \epsilon_0} \max_{\alpha, \beta} \| e_{\alpha\beta}^4(\epsilon)(v) \|_{0, \Omega} \leq C_1 \| v \|_{1, \Omega} \text{ for all } v \in V. \]  

(4.7)

Also there exist constants \( C_2, C_3 \) and \( C_4 \) such that

\[ \sup_{0 < \epsilon \leq \epsilon_0} \max_{x \in \Omega} | g(x) - 1 | \leq C_2 \epsilon^2, \]  

(4.8)

\[ \sup_{0 < \epsilon \leq \epsilon_0} \max_{x \in \Omega} | A^{ijkl}(\epsilon) - A^{ijkl} | \leq C_3 \epsilon^2, \]  

(4.9)

where

\[ A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}), \]  

(4.10)

and

\[ A^{ijkl}(\epsilon) t_{kl} t_{ij} \geq C_4 t_{ij}, \]  

(4.11)

for \( 0 < \epsilon \leq \epsilon_0 \) and for all symmetric tensors \( (t_{ij}) \).

**Proof:** A simple computation using the assumption (3.2) shows that

\[ g_\alpha(\epsilon) = \begin{pmatrix} \delta_{\alpha 1} - \epsilon^2 x_3 \partial_\alpha \theta + O(\epsilon^4) \\ \delta_{\alpha 2} - \epsilon^2 x_3 \partial_\alpha \theta + O(\epsilon^4) \\ \epsilon \partial_\alpha \theta + O(\epsilon^3) \end{pmatrix}, \]  

\[ g_3(\epsilon) = \begin{pmatrix} -\epsilon \partial_1 \theta + O(\epsilon^3) \\ -\epsilon \partial_2 \theta + O(\epsilon^3) \\ 1 + O(\epsilon^2) \end{pmatrix}, \]  

(4.12)
\[ g^9(\epsilon) = \begin{pmatrix} \delta_{\alpha1} + O(\epsilon^2) \\ \delta_{\alpha2} + O(\epsilon^2) \\ \epsilon \partial_\alpha \theta + O(\epsilon^2) \end{pmatrix}, \quad g^3(\epsilon) = \begin{pmatrix} -\epsilon \partial_1 \theta + O(\epsilon^3) \\ -\epsilon \partial_2 \theta + O(\epsilon^3) \\ 1 + O(\epsilon^2) \end{pmatrix}, \] (4.13)

\[
\begin{align*}
g_{\alpha\beta}(\epsilon) &= \delta_{\alpha\beta} + \epsilon^2 [\partial_\alpha \theta \partial_\beta \theta - 2x_3 \partial_\alpha \theta_3] + O(\epsilon^4), \\
g_{33}(\epsilon) &= O(\epsilon), \quad g_{33}(\epsilon) = 1 + O(\epsilon^2), \quad (4.14)
\end{align*}
\]

\[ \Gamma^\sigma_{\alpha\beta}(\epsilon) = O(\epsilon^2), \quad \Gamma^3_{\alpha\beta}(\epsilon) = \epsilon \partial_\alpha \theta(\theta) + O(\epsilon^3), \quad \Gamma^\sigma_{\alpha3}(\epsilon) = O(\epsilon). \] (4.15)

The announced results follows from the above relations.

**Lemma 4.2** — Let \( \theta \in C^3(\omega) \) be a given function and let the functions \( \widehat{\varepsilon}_{ij} \) be defined as in (4.4)-(4.6). Then there exists a constant \( C_5 \) such that the following generalised Korn’s inequality holds.

\[
||v||_{1,\Omega} \leq C_8 \left\{ \sum_{i,j} ||\varepsilon_{ij}(v)||_{0,\Omega}^2 \right\}^{\frac{1}{2}}
\] (4.16)

for all \( v \in V \) where \( V \) is the space defined in (3.7).

**Proof**: The proof is based on lemma 4.2 of [2].

**Definition** — Let \( V \) be a Banach space and \( (J(\epsilon))_{\epsilon>0} \) a sequence of functionals \( J(\epsilon) : V \to \mathbb{R} \cup \{ \infty \} \). We say that the functional \( J : V \to \mathbb{R} \) is the \( \Gamma \)-limit of the functionals \( J(\epsilon) \) if the following properties holds.

(i) If \( (v(\epsilon))_{\epsilon>0} \to v \) in \( V \) implies

\[ J(v) \leq \liminf_{\epsilon \to 0} J(\epsilon)(v(\epsilon)). \]

(ii) For every \( v \in V \), there exists a sequence \( (v(\epsilon))_{\epsilon>0} \in V \) such that

\[ (v(\epsilon))_{\epsilon>0} \to v \text{ and } J(\epsilon)(v(\epsilon)) \to J(v). \]

**Remark**: It can be shown that when the \( \Gamma \)-limit exists, it is unique.
The main result from $\Gamma$-convergence is the following, see Dal Maso [8].

**Theorem 4.3** — Assume that the sequence $(J(\epsilon))_{\epsilon > 0}$, $\Gamma$-converges to $J$, and assume that there exists a compact subset $U$ of $V$ independent of $\epsilon$ such that, for all $\epsilon > 0$, there exists $u(\epsilon)$ satisfying

$$u(\epsilon) \in U \quad \text{and} \quad J(\epsilon)(u(\epsilon)) = \inf_{v \in V} J(\epsilon)(v).$$

Then there exists $u \in U$ such that

$$u(\epsilon) \to u \quad \text{and} \quad J(u) = \inf_{v \in V} J(v).$$

Moreover, one has

$$J(\epsilon)(u(\epsilon)) \to J(u).$$

5. Convergence of the Scaled Solutions

Let $V_{KL}$ be the space defined by

$$V_{KL} = \{v = (v_i) \in (H^1(\Omega))^3; e_{i3}(v) = 0 \text{ in } \Omega \quad v_i = 0 \text{ on } \Gamma_0\}. \quad (5.1)$$

For any $v \in V$, define

$$J(v) = \begin{cases} \int_{\Omega} \left\{ \frac{2\lambda u}{\lambda + 2\mu} \tilde{e}_{\sigma\sigma}(v)\tilde{e}_{\tau\tau}(v) + 2\mu \tilde{e}_{\alpha\beta}(v)\tilde{e}_{\alpha\beta}(v) \right\} \, dx - \int_{\Omega} f_i v_i \, dx & \text{if } v \in V_{KL}, \\
\infty & \text{otherwise}. \end{cases}$$

**Theorem 5.1** — The functional $J$ is the $\Gamma$-limit of the functional $J(\epsilon)$ for the weak topology of the space $V$. 
PROOF: Note that the functional \( J(\varepsilon) \) can be written as

\[
J(\varepsilon)(v) = \frac{1}{2} \int_{\Omega} \frac{2\lambda\mu}{\lambda + 2\mu} g^{\alpha\beta}(\varepsilon) g^{\alpha\sigma}(\varepsilon) e_{\alpha|\beta}(\varepsilon)(v) e_{\sigma|\tau}(\varepsilon)(v) \sqrt{g(\varepsilon)} \, dx \\
+ \frac{1}{4} \int_{\Omega} \frac{2\mu g^{\alpha\sigma}(\varepsilon) g^{\alpha\tau}(\varepsilon) e_{\alpha|\beta}(\varepsilon)(v) e_{\sigma|\tau}(\varepsilon)(v) \sqrt{g(\varepsilon)} \, dx \\
+ \frac{1}{2} \int_{\Omega} \{(\lambda + 2\mu) \left[ \frac{\lambda}{\lambda + 2\mu} g^{\alpha\beta}(\varepsilon) e_{\alpha|\beta}(\varepsilon)(v) + g^{33}(\varepsilon) e_{3|3}(\varepsilon)(v) \right]^2 \} \\
+ \sqrt{g(\varepsilon)} \, dx \frac{1}{2} \int_{\Omega} 4\mu g^{\alpha\sigma}(\varepsilon) g^{33}(\varepsilon) e_{\alpha|\beta}(\varepsilon)(v) e_{\sigma|\tau}(\varepsilon)(v) \sqrt{g(\varepsilon)} \, dx \\
- \int_{\Omega} f_i(\varepsilon) v_i \sqrt{g(\varepsilon)} \, dx \\
(5.2)
\]

Step (i): We first show that

\[
v(\varepsilon) \rightharpoonup v \text{ in } V \Rightarrow J(v) \leq \liminf_{\varepsilon \rightarrow 0} J(\varepsilon)(v(\varepsilon)). \quad (5.3)
\]

If \( v \not\in V_{KL} \). Then from the definition of \( J \), it follows that \( J(v) = \infty \). Hence it is enough to show that

\[
\liminf_{\varepsilon \rightarrow 0} J(\varepsilon)(v(\varepsilon)) = \infty.
\]

Suppose that

\[
\liminf_{\varepsilon \rightarrow 0} J(\varepsilon)(v(\varepsilon)) < \infty.
\]

Then it follows that there exists a constant \( C > 0 \) and a subsequence \( (v(\varepsilon))_{\varepsilon > 0} \) (still denoted by \( \varepsilon \) ) such that

\[
\int_{\Omega} A^{ijkl}(\varepsilon) e_{k|l} v(\varepsilon)(v(\varepsilon)) e_{i|j}(\varepsilon)(v(\varepsilon)) \sqrt{g(\varepsilon)} \, dx \leq C_1. \quad (5.4)
\]

Hence from the relations (2.5), (4.8)-(4.11) and the lemma 4.2, it follows that

\[
\| e_{i|j}(\varepsilon)(v(\varepsilon)) \|_{0,\Omega} \leq C, \quad \| v_i(\varepsilon) \|_{1,\Omega} \leq C.
\]
Hence there exists subsequence \( (e_{i,j}(\epsilon)\nu(\epsilon))_{\epsilon>0} \) and functions \( e_{i,j} \in L^2(\Omega) \) and \( \nu \in H^1(\Omega) \) such that
\[
e_{i,j}(\epsilon)\nu(\epsilon) \rightharpoonup e_{i,j} \quad \text{weakly in } L^2(\Omega),
\]
\[
u_i(\epsilon) \rightharpoonup \nu_i \quad \text{weakly in } H^1(\Omega),
\]
(5.5) \hspace{1cm} (5.6)

Using the convergences (5.5)-(5.6), it is a standard argument (cf. [2]) to show that \( e_{i,j}(\nu) = 0, \) and hence \( \nu \in V_{KL} \) which is a contradiction. Hence
\[
\liminf_{\epsilon \to 0} J(\epsilon)(\nu(\epsilon)) = \infty.
\]

Assume next that \( \nu \in V_{KL} \). Suppose
\[
\liminf_{\epsilon \to 0} J(\epsilon)(\nu(\epsilon)) = \infty
\]
then (5.4) always holds. Suppose that
\[
\liminf_{\epsilon \to 0} J(\epsilon)(\nu(\epsilon)) < \infty.
\]

As in the first case, there exist subsequences \( (\nu(\epsilon))_{\epsilon>0} \) such that the convergences (5.5)-(5.6) holds and since \( \nu(\epsilon) \rightharpoonup \nu \) in \( H^1(\Omega) \), it follows from the definition that \( e_{i,j}(\epsilon)(\nu(\epsilon)) \rightharpoonup \bar{e}_{i,j}(\nu) \) in \( L^2(\Omega) \).

From the positive definiteness of \( A^{ijkl}(\epsilon) \) it follows that
\[
J(\epsilon)(\nu) \geq \frac{1}{2} \int_{\Omega} \left( \frac{2\lambda \mu}{\lambda + 2\mu} g^{\alpha\beta}(\epsilon) g^{\sigma\tau}(\epsilon) e_{i,j}(\epsilon)(\nu) e_{\alpha\beta}(\epsilon)(\nu) \sqrt{g(\epsilon)} \right) dx
\]
\[
+ \frac{1}{2} \int_{\Omega} \left( 2\mu g^{\alpha\sigma}(\epsilon) g^{\beta\tau}(\epsilon) e_{i,j}(\epsilon)(\nu) e_{\alpha\beta}(\epsilon)(\nu) \sqrt{g(\epsilon)} \right) dx
\]
\[
- \int_{\Omega} f_i(\epsilon) v_i(\epsilon) \sqrt{g(\epsilon)} dx
\]
(5.7)

With the convergence \( g^{ij}(\epsilon) \to \delta_{ij}, \quad g(\epsilon) \to 1 \) in \( C(\Omega) \) and the weak convergence \( e_{i,j}(\epsilon)(\nu(\epsilon)) \rightharpoonup \bar{e}_{i,j}(\nu) \), it follows that for any convergent subsequence
\( J(\varepsilon)(v(\varepsilon)) \) we have

\[
\lim_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)) \geq \frac{1}{2} \int_{\Omega} \left\{ \frac{2\lambda \mu}{\lambda + 2\mu} \tilde{e}_{\sigma \sigma}(v)\tilde{e}_{\tau \tau}(v) + 2\mu \tilde{e}_{\alpha \beta}(v)\tilde{e}_{\alpha \beta}(v) \right\} dx \\
- \int_{\Omega} f_{i} v_{i} dx = J(v)
\]

(5.8)

Hence (5.4) follows.

**Step (ii)**: We show that for any \( v \in V \), there exists a sequence \( (v(\varepsilon))_{\varepsilon > 0} \) such that

\[
v(\varepsilon) \to v \text{ in } V \text{ and } J(v) = \lim_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)).
\]

(5.9)

If \( v \notin V_{K,L} \) then by taking \( v(\varepsilon) = v \), it follows from step(i) and the definition of the functional \( J \) that

\[
J(v) = \liminf_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)) = \infty
\]

(5.10)

and hence the property (5.10) holds.

Define the space

\[
W = \{(\eta_{\alpha} - x_{3}\partial_{\alpha}\eta_{\alpha}), \eta_{3} \in H^{2}(\omega), \eta_{3} \in H^{2}(\omega), \eta_{\alpha} = \partial_{\alpha}\eta_{3} = 0, \text{ on } \gamma_{0}\}.
\]

(5.11)

Let \( v \in W \). Define \( v(\varepsilon) \in V \) by

\[
v_{\alpha}(\varepsilon) = v_{\alpha}, \quad v_{3}(\varepsilon) = \eta_{3} - \varepsilon^{2} \frac{\lambda}{\lambda + 2\mu} (x_{3}\partial_{\alpha}\eta_{\sigma} - \eta_{3}\partial_{\alpha}\theta) - \frac{x_{3}^{2}}{2} \Delta \eta_{3}
\]

(5.12)

Then as \( \varepsilon \to 0 \), we have

\[
v(\varepsilon) \to v
\]

(5.13)

\[
e_{\alpha||3}(v(\varepsilon)) = \varepsilon \frac{\lambda}{\lambda + 2\mu} \partial_{\alpha}(x_{3}\partial_{\sigma}\eta_{\sigma} - \eta_{3}\partial_{\alpha}\theta) - \frac{x_{3}^{2}}{2} \Delta \eta_{3} \to 0,
\]

(5.14)

\[
\frac{\lambda}{\lambda + 2\mu} e_{\sigma||3}(v(\varepsilon)) + e_{3||3}(v(\varepsilon)) = \varepsilon^{2} \left(\frac{\lambda}{\lambda + 2\mu}\right)^{2} \left(x_{3}(\partial_{\sigma}\eta_{\sigma} - \eta_{3}\partial_{\alpha}\theta) - \frac{x_{3}^{2}}{2} \Delta \eta_{3}\right) \to 0,
\]

(5.15)

Using the above convergences and relations (4.14) in (5.2) it follows that
\[ J(\varepsilon)(v(\varepsilon)) \rightarrow J(v) \]

Since the space \( W \) is dense in the space \( V_{KL} \), the above convergence hold for any \( v \in V_{KL} \).

**Theorem 5.2** — For each \( \varepsilon > 0 \), let \( (u(\varepsilon)) \) be the solution of the minimization problem (3.10). Then

\[ u_i(\varepsilon) \rightarrow u \text{ in } H^1(\Omega), \quad u \in V_{KL}, \quad (5.16) \]

and \( u \) is the solution of the minimization problem

\[ J(u) = \min_{v \in V} J(v) \quad (5.17) \]

**Proof:** It follows from the inequality (4.16) that \( ||u(\varepsilon)||_{1,\Omega} \) are bounded independent of \( \varepsilon \). Thus \( \{u(\varepsilon)\}_{\varepsilon > 0} \) belong to a weakly compact subset of \( V \). Moreover the weak limit \( u \) of \( u(\varepsilon) \) belongs to \( V_{KL} \) (cf. [2]). Then it follows from the above theorem that there exists a subsequence \( u(\varepsilon_k) \) such that \( u(\varepsilon_k) \rightharpoonup u \) in \( V \) and \( u \) satisfies

\[ J(u) = \inf_{v \in V} J(v). \]

Thus the function is unique and the whole sequence \( u(\varepsilon) \) converges weakly to \( u \) in \( V \).

To show that the family \( u(\varepsilon) \) converges strongly to \( u \) in \( H^1(\Omega) \), it is enough to show by virtue of (4.16) that

\[ \tilde{e}_{ij}(u(\varepsilon)) \rightarrow \tilde{e}_{ij}(u) \text{ in } L^2(\Omega). \quad (5.18) \]

Define

\[ K_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u(\varepsilon)), \quad K_{\alpha3}(\varepsilon) = \frac{1}{\varepsilon} \tilde{e}_{\alpha3}(u(\varepsilon)), \quad K_{33}(\varepsilon) = \frac{1}{\varepsilon^2} \tilde{e}_{33}(u(\varepsilon)) \quad (5.19) \]

and

\[ K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u), \quad K_{\alpha3} = 0, \quad K_{33} = -\frac{\lambda}{\lambda + 2\mu} \tilde{e}_{33}(u) \quad (5.20) \]
Claim: \( K(\epsilon) = (K_{ij}(\epsilon)) \to K = (K_{ij}) \) weakly in \( L^2(\Omega) \).

It follows from (2.5) and (3.9) that

\[
C \sum_{i,j} ||e_{i||j}(\epsilon; u(\epsilon))||^2_{0,\Omega} \leq \int_{\Omega} A_{ijkl}(\epsilon)e_{kl||i}(\epsilon; u(\epsilon))e_{i||j}(\epsilon; u(\epsilon)) \sqrt{q(\epsilon)} dx \\
\leq ||f||_{0,\Omega} ||u(\epsilon)||_{0,\Omega}
\]

(5.21)

Hence \( (e_{i||j}(\epsilon; u(\epsilon))) \) is bounded.

From the definition (5.20) and relations (4.1)-(4.6), we have

\[
||K(\epsilon)||^2_{\beta,\Omega} \leq 2 \sum_{i,j} ||e_{i||j}(\epsilon; u(\epsilon))||^2_{0,\Omega} + 2\epsilon^2 \sum_{\alpha} ||\varepsilon^2(\epsilon; u(\epsilon))||^2_{0,\Omega} + 4\epsilon^2 \sum_{\alpha} ||\varepsilon^2(\epsilon; u(\epsilon))||^2_{\alpha,\Omega}
\]

(5.22)

From the boundedness of \( (e_{i||j}(\epsilon; u(\epsilon))) \) and the relation (4.7) it follows that \( (K(\epsilon)) \) is bounded and hence \( K(\epsilon) \to K \) in \( (L^2(\Omega))^9 \) weakly.

We next note the following result:

\[
\int_{\Omega} u \partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0 \Rightarrow u = 0
\]

Clearly \( K_{\alpha\beta} = \bar{e}_{\alpha\beta}(u) \). Multiplying (3.9) by \( \epsilon \) and taking \( v_3 = 0 \) we get

\[
2 \int_{\Omega} A^{\alpha \sigma \tau}(0) K_{\alpha \beta}(\epsilon) \partial_3 v_\alpha dx = \epsilon R(\epsilon, K(\epsilon), u(\epsilon), v)
\]

(5.23)

where \( R(\epsilon, K(\epsilon), u(\epsilon), v) \) is bounded independent of \( \epsilon \). Passing to the limit as \( \epsilon \to 0 \) in (5.24) we get

\[
\int_{\Omega} K_{\alpha \beta} \partial_3 v_\alpha dx = 0 \text{ for all } v_\alpha.
\]

(5.24)

Hence \( K_{\alpha \beta} = 0 \). Multiplying (3.9) by \( \epsilon^2 \) and letting \( v_\alpha = 0 \) we get

\[
\int_{\Omega} \{ A^{3 \sigma \tau}(0) K_{\sigma \tau}(\epsilon) + A^{3333}(0) K_{33}(\epsilon) \} \partial_3 v_3 dx \\
= \int_{\Omega} \{ \lambda K_{\sigma \tau}(\epsilon) + (\lambda - 2\mu) K_{33}(\epsilon) \} \partial_3 v_3 dx \\
= \epsilon S(\epsilon, K(\epsilon), u(\epsilon), v)
\]

(5.25)
where \( S(\epsilon, K(\epsilon), u(\epsilon), v) \) is independent of \( \epsilon \). Letting \( \epsilon \to 0 \), we get
\[
\int_{\Omega} \{\lambda K_{\sigma\sigma} + (\lambda + 2\mu)K_{33}\}dx = 0. \quad (5.26)
\]
Hence \( K_{33} = -\frac{\lambda}{\lambda + 2\mu} \bar{e}_{\sigma\sigma}(u) \). Since \( \bar{e}_{i3}(u) = 0 \) and
\[
\sum_{i,j} ||\bar{e}_{ij}(u(\epsilon)) - \bar{e}_{ij}(u)||^2_{0,\Omega} = \sum_{\alpha\beta} ||K_{\alpha\beta}(\epsilon) - K_{\alpha\beta}||^2_{0,\Omega} + 2\epsilon^2 \sum_{\alpha} ||K_{\alpha\beta}(\epsilon)||^2_{0,\Omega} + \epsilon^2 ||K_{33}(\epsilon)||^2_{0,\Omega}, \quad (5.27)
\]
the convergence (5.18) is equivalent to showing that \( K(\epsilon) \to K \) strongly in \( L^2(\Omega) \).

For any two symmetric matrices \( S = (s_{ij}) \) and \( T = (t_{ij}) \), define
\[
AS : T = A^{ijkl}(0)t_{kl}t_{ij} = \lambda s_{pp}t_{qq} + 2\mu s_{ij}t_{ij}.
\]
Then
\[
\int_{\Omega} AK : Kdx = \int_{\Omega} \{\lambda K_{pp}K_{qq} + 2\mu K_{ij}K_{ij}\}dx
\]
\[
= \int_{\Omega} \frac{2\lambda}{\lambda + 2\mu} \bar{e}_{\sigma\sigma}(u)\bar{e}_{\tau\tau}(u) + 2\mu \bar{e}_{\alpha\beta}(u)\bar{e}_{\alpha\beta}(u)dx
\]
\[
= \int_{\Omega} f_iu_i\,dx \quad (5.28)
\]
Taking \( v = u(\epsilon) \) in (3.9), and using the relations (4.1)-(4.10), we get
\[
\int_{\Omega} AK(\epsilon) : K(\epsilon)dx = \int_{\Omega} f_iu_i(\epsilon)dx + \epsilon r(\epsilon, u(\epsilon)) \quad (5.29)
\]
where there exists a constant \( C \) such that
\[
sup_{0 < \epsilon \leq \epsilon_1} |r(\epsilon, u(\epsilon))| \leq C \quad (5.30)
\]
From the relations (5.28)-(5.29) and the weak convergence of \( u(\epsilon) \) we deduce that
\[
\int_{\Omega} AK(\epsilon) : K(\epsilon)dx \to \int_{\Omega} f_iu_i\,dx \text{ as } \epsilon \to 0. \quad (5.31)
\]
Also, using (5.27) and (5.30) it follows that

\[
2\mu|K(\varepsilon) - K|_{\Omega}^2 \leq \int_{\Omega} A(K(\varepsilon) - K) : (K(\varepsilon) - K)d\varepsilon
\]

\[
= \int_{\Omega} AK(\varepsilon) : K(\varepsilon)d\varepsilon + \int_{\Omega} AK : (K - 2K(\varepsilon))d\varepsilon
\]

\[
\rightarrow \int_{\Omega} f^* u_1 d\varepsilon - \int_{\Omega} AK : Kd\varepsilon
\]

\[
= 0
\]  \hspace{1cm} (5.32)

Hence the convergence (5.18) follows. \( \blacksquare \)

**Theorem 5.3** — Let \( u \) be the minimizer of the functional \( J \). Let

\[
V_H(\omega) = \{(\eta_1) \in (H^1(\omega))^2 : \eta_1 = 0 \text{ on } \gamma_0\}
\]  \hspace{1cm} (5.33)

\[
V_3(\omega) = \{\eta_3 \in H^2(\omega) : \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}
\]  \hspace{1cm} (5.34)

Then there exists \( (\zeta_i) \in V_H(\omega) \times V_3(\omega) \) such that

(a) \( u_\alpha(x) = \zeta_\alpha - x_3 \partial_\alpha \eta_3, \ u_3(x) = \zeta_3(x_3) \)

(b) \( (\zeta_i) \) solves the following variational equations:

\[
-\int_{\omega} m_{\alpha\beta} \partial_\alpha \eta_3 d\omega - \int_{\omega} n_{\alpha\beta} \partial_\alpha \beta \eta_3 d\omega = \int_{\omega} p_\alpha \beta \eta_3 d\omega - \int_{\omega} q_\alpha \eta_3 d\omega \ \forall \ \eta_3 \in V_3(\omega),
\]  \hspace{1cm} (5.35)

\[
\int_{\omega} n_{\alpha\beta} \partial_\beta \eta_\alpha d\omega = \int_{\omega} p_\alpha \eta_\alpha d\omega \ \forall \ \eta_\alpha \in V_H(\omega).
\]  \hspace{1cm} (5.36)

where

\[
m_{\alpha\beta} = -\left\{ \frac{4\lambda\mu}{3(\lambda + 2\mu)} \triangle \zeta_\alpha \delta_{\alpha\beta} + \frac{4}{3}\mu \partial_\alpha \beta \zeta_3 \right\}
\]  \hspace{1cm} (5.37)

\[
n_{\alpha\beta} = -\left\{ \frac{4\lambda\mu}{(\lambda + 2\mu)} \tilde{e}_{\alpha\beta}(\zeta)\delta_{\alpha\beta} + 4\mu \tilde{e}_{\alpha\beta}(\zeta) \right\}
\]  \hspace{1cm} (5.38)

\[
\tilde{e}_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha) - \zeta_\beta \partial_\alpha \beta + \zeta_3 \partial_\alpha \beta \partial_\beta \zeta_3
\]  \hspace{1cm} (5.39)

\[
p_\alpha = \int_{-1}^{1} f_i d\varepsilon \quad q_\alpha = \int_{-1}^{1} x_\alpha f_3 d\varepsilon.
\]  \hspace{1cm} (5.40)
PROOF: Since $u$ is the minimizer of the functional $J$, $u \in V_{KL}$ and satisfies
\[
\frac{1}{2} \int_{\Omega} \left( \frac{2\lambda \mu}{\lambda + \mu} \tilde{e}_{\sigma}(u) \tilde{e}_{\epsilon}(w) + 2\mu \tilde{e}_{\alpha\beta}(u) \tilde{e}_{\alpha\beta}(w) \right) dx = \int_{\Omega} f_i w_i dx \quad \forall w \in V_{KL}.
\]
Equations (5.34) and (5.35) follow by taking $\eta_\alpha = 0$ and $\eta_3 = 0$ respectively in the above equation.

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