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ON THE SOLVABILITY OF A NONLINEAR PSEUDOPARABOLIC PROBLEM

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This paper is concerned with the study of an initial boundary value problem for a nonlinear second order pseudoparabolic equation arising in the unidirectional flow of a thermodynamic compatible third grade fluid. We establish some a *priori* bounds for the solution and prove its existence.

Key words : Nonlinear flow; a priori estimate; existence of solution; nonlocal condition.

1. INTRODUCTION

The analysis of the flow of an incompressible non-Newtonian fluid has drawn much attention in the recent years. This is because of relevance of the applications of the non Newtonian fluids in industry and engineering. Examples of the non-Newtonian fluids include multi-grade oils, paints, food products, inks, glues, soaps, mud, certain polymers etc. The observed flow of the non-Newtonian fluids are markedly different from that of its Newtonian counterpart. The relationships between the shear stress and the flow field in the non-Newtonian fluids are more complicated in comparison to the Newtonian fluids. The governing equations of the non-Newtonian fluids are higher order and much nonlinear than equations of the Newtonain fluids. Besides all these challenges, several recent investigators [1-7], [14-16] have even carried out the analysis on various types of flows in the non-Newtonain fluid mechanics. Generally, the non-Newtonian fluids are classified under the three categories known as the differential type, rate type and integral type. A simplest subclass of the differential type fluid is called the second grade. This subclass can describe the normal stress effects and is not able to predict the shear thinning and shear thickening characteristics in the steady flows with rigid boundaries. The third grade fluids although can explain such features.

In this paper, we deal with an initial boundary value problem for a nonlinear second order equation. Such nonlinear equation appears when unidirectional flow of a third grade fluid is considered in a thermodynamic sense. The fluid is considered between the two non-porous plates. To obtain some a priori estimates for the solution of probelm (10)-(13) stated below, we apply the energy estimate method inspired from functional analysis, see for example [9-13]. The technique of deriving such a priori estimate is based on a conveniently chosen multiplier. From the resulted energy estimate, it is possible to establish the solvability of the posed problem.

2. Statement of the Problem

Let us examine the flow of an incompressible and homogeneous third grad fluid between two parallel stationary plates distant h apart. The x^* and y^* axes are chosen along and perpondicular to the channel walls. The flow is governed by the following equations

$$div\overline{V} = 0, \quad (1)$$

$$\rho \frac{d\overline{V}}{dt^*} = -p\overline{I} + div\overline{S}, \tag{2}$$

where $\overline{V}$ is the velocity, ρ the fluid density, d/dt the material derivative, p the pressure, $\overline{I}$ an identity tensor and an extra stress tensor $\overline{S}$ in a thermodynamic third grade fluid is described by the following expression [8]

$$\overline{S} = \left(\mu + \xi(tr\overline{A}_1^2)\right)\overline{A}_1 + \alpha_1\overline{A}_2 + \alpha_2\overline{A}_1^2, \tag{3}$$

with $\mu \geq 0, \xi \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\xi}$,

$$\overline{A}_1 = (\nabla\overline{V}) + (\nabla\overline{V})^{T^*},\ A_2 = \frac{d\overline{A}_1}{dt^*} + \overline{A}_1((\nabla\overline{V}) + (\nabla\overline{V})^{T^*}\overline{A}_1. \tag{4}$$

Here we refer μ as the dynamic viscosity of fluid, tr the trace, T^* the matrix transpose, $\alpha_i (i = 1, 2)$ and ξ the material parameters and $\overline{A}_i (i = 1, 2)$ the first two Rivlin-Ericksen tensors.

We define the velocity field as

$$\overline{V} = (u^*(y^*, t^*), 0, 0). \tag{5}$$

Now equation (1) is identically satisfied and equations $(2) - (5)$ in absence of modified pressure gradient yield

$$\rho \frac{\partial u^*}{\partial t^*} = \mu \frac{\partial^2 u^*}{\partial y^{*2}} + \alpha_1 \frac{\partial^3 u^*}{\partial y^{*2}\partial t^*} + \xi \left(\frac{\partial u^*}{\partial y^*}\right)^2 \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{6}$$

The appropriate boundary and initial conditions are

$$u^*(0, t^*) = u^*(h, t^*) = 0, \tag{7}$$

$$u^*(y^*, 0) = g(y^*). \tag{8}$$

To explore the analysis in dimensionless form, we introduce the following variables

$$\begin{cases} u = \frac{u^*}{U_0},\ \ \eta = \frac{U_0 y^*}{\nu},\ \ t = \frac{U_0^2 t^*}{\nu} \\ \alpha = \frac{\alpha_1 U_0^2}{\rho\nu^2},\ \ \beta = \frac{6\xi U_0^4}{\rho\nu^3}, \end{cases} \tag{9}$$

where U_0 is the characteristic velocity and ν the kenematic viscosity. The non-dimensional problem can be written as

$$\begin{cases} u_t = u_{\eta\eta} + \alpha u_{\eta\eta t} + \beta u_\eta^2 u_{\eta\eta} \\ u(\eta,0) = \sigma(\eta), \\ u(l,t) = 0, \quad u(0,t) = 0, \end{cases} \tag{10}$$

where $\sigma(\eta) = g/U_0$ and $l = U_0 h/\nu$.

Let $T > 0$, $\Omega = (0,l)$, and

$$Q = \Omega \times (0,T) = \left\{(\eta,t) \in \mathbb{R}^2 : \eta \in \Omega,\ 0 < t < T\right\},$$

we consider the following nonlinear mixed problem

$$\mathcal{L}u = u_t - u_{\eta\eta} - \alpha u_{\eta\eta t} - \beta u_\eta^2 u_{\eta\eta} = f(\eta,t), \tag{11}$$

$$\ell u = u(\eta,0) = \sigma(\eta), \tag{12}$$

$$u(l,t) = 0, \quad u(0,t) = 0, \tag{13}$$

where $f(\eta,t)$, and $\sigma(\eta)$ are given functions and α and β are positive constants

For the investigation of this problem, we introduce the following function spaces.

Let $L^2(Q)$ be the Hilbert space of square integrable functions having the finite norm $\|u\|_{L^2(Q)}^2 = \int_Q u^2 d\eta$, and the associated inner product $(u,v)_{L^2(Q)} = \int_Q uvd\eta$. And $H^1(\Omega)$ is the Hilbert space with inner product $(u,v)_{H^1(\Omega)} = \int_\Omega uvd\eta + \int_\Omega u_\eta v_\eta d\eta$, and equipped with the norm $\|u\|_{L^2(\Omega)}^2 + \|u_\eta\|_{L^2(\Omega)}^2$.

We establish a *priori* bound and prove the existence of a solution of the problem (11)-(13). Let $Lu = \mathcal{F}$, where $L = (\mathcal{L}, \ell)$, and $\mathcal{F} = (\{, \sigma)$ be the operator equation corresponding to problem (11)-(13). The operator L with domain of definition $D(L) = \left\{u \in L^2(Q)/\ u_t,\ u_\eta,\ u_{\eta\eta}, u_{\eta\eta t} \in L^2(Q)\right\}$, satisfying conditions (13), acts from E to F defined as follows. The Banach space E consists of all functions $u(\eta,t)$ with the finite norm

$$\|u\|_E^2 = \sup_{0 \le \tau \le T} \|u(\eta,\tau)\|_{H^1(\Omega)}^2 + \|u_\eta\|_{L^2(Q)}^2. \tag{14}$$

The Hilbert space F consists of the vector valued functions $\mathcal{F} = (f, \sigma)$ with the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 + \|\sigma\|_{L^2(\Omega)}^2 . \tag{15}$$

We assume that the data function σ satisfies the conditions of the form (13),

$$\sigma(0) = \sigma(l) = 0. \tag{16}$$

We first establish a priori estimate for the solution of problem (11)-(13).

3. A PRIORI BOUND FOR THE SOLUTION

***Theorem* 3.1** — *For any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\sup_{0\le\tau\le T} \|u(\eta,\tau)\|_{H^1(\Omega)}^2 + \|u_\eta\|_{L^2(Q)}^2 \le c\left(\|f\|_{L^2(Q)}^2 + \|\sigma\|_{H^1(\Omega)}^2\right), \tag{17}$$

where

$$c = \gamma e^{\gamma T},\ \gamma = \frac{\max(\frac{\alpha+1}{2}, \frac{\beta}{12}, 1)}{\min(\frac{1}{3}, \alpha)}. \tag{18}$$

PROOF : For the equation (11) and $Q^\tau = \Omega \times (0, \tau)$, we have

$$\begin{aligned}(\mathcal{L}u, u)_{L^2(Q^\tau)} &= (u_t, u)_{L^2(Q^\tau)} - (u_{\eta\eta}, u)_{L^2(Q^\tau)} \\ &\quad -(\alpha u_{\eta\eta t}, u)_{L^2(Q^\tau)} - (\beta u_\eta^2 u_{\eta\eta}, u)_{L^2(Q^\tau)}.\end{aligned} \tag{19}$$

By using conditions (12) and (13), the right-hand side of (19) can be evaluated as follows

$$
\begin{aligned}
(u_t, u)_{L^2(Q^\tau)} &= \frac{1}{2}\left\|u(\eta,\tau)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|\sigma\right\|^2_{L^2(\Omega)}, && (20)\\
-(u_{\eta\eta}, u)_{L^2(Q^\tau)} &= \frac{1}{2}\left\|u_\eta\right\|^2_{L^2(Q^\tau)}, && (21)\\
-(\alpha u_{\eta\eta t}, u)_{L^2(Q^\tau)} &= \frac{\alpha}{2}\left\|u_\eta(\eta,\tau)\right\|^2_{L^2(\Omega)} - \frac{\alpha}{2}\left\|\sigma_\eta\right\|^2_{L^2(\Omega)}, && (22)\\
-(\beta u_\eta^2 u_{\eta\eta}, u)_{L^2(Q^\tau)} &= -\frac{\beta}{3}\int_0^\tau u_\eta^3 u \mid_0^l dt + \frac{\beta}{3}\int_{Q^\tau} u_\eta^4 d\eta dt. && (23)
\end{aligned}
$$

Equality (23) implies that

$$
-(\beta u_\eta^2 u_{\eta\eta}, u)_{L^2(Q^\tau)} = \frac{\beta}{3}\left\|u_\eta\right\|^4_{L^4(Q^\tau)}. \tag{24}
$$

Substituting (20)-(22) and (24) into (19), we obtain

$$
\begin{aligned}
& \frac{1}{2}\left\|u(\eta,\tau)\right\|^2_{L^2(\Omega)} + \left\|u_\eta\right\|^2_{L^2(Q^\tau)} + \frac{\alpha}{2}\left\|u_\eta(\eta,\tau)\right\|^2_{L^2(\Omega)} + \frac{\beta}{3}\left\|u_\eta\right\|^4_{L^4(Q^\tau)}\\
= & \frac{1}{2}\left\|\sigma\right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\|\sigma_\eta\right\|^2_{L^2(\Omega)} + (\mathcal{L}u, u)_{L^2(Q^\tau)}.
\end{aligned} \tag{25}
$$

If we discard the fourth term on the left-hand side of (25) and apply Cauchy ε inequality, we get

$$
\begin{aligned}
& \left\|u(\eta,\tau)\right\|^2_{H^1(\Omega)} + \left\|u_\eta\right\|^2_{L^2(Q^\tau)}\\
\leq & \gamma\left(\left\|\sigma\right\|^2_{H^1(\Omega)} + \left\|f\right\|^2_{L^2(Q^\tau)} + \left\|u\right\|^2_{L^2(Q^\tau)}\right)\\
\leq & \gamma\left(\left\|\sigma\right\|^2_{H^1(\Omega)} + \left\|f\right\|^2_{L^2(Q^\tau)} + \left\|u\right\|^2_{H^1(Q^\tau)}\right),
\end{aligned} \tag{26}
$$

where

$$
\gamma = \frac{\max(\frac{\alpha}{2}, \frac{1}{2})}{\min(\frac{1}{3}, \frac{\alpha}{2})}.
$$

Application of Gronwall's lemma [14] to the inequality (26), implies that

$$
\begin{aligned}
& \left\|u(\eta,\tau)\right\|^2_{H^1(\Omega)} + \left\|u_\eta\right\|^2_{L^2(Q)}\\
\leq & \gamma e^{\gamma T}\left(\left\|f\right\|^2_{L^2(Q)} + \left\|\sigma\right\|^2_{H^1(\Omega)}\right).
\end{aligned} \tag{27}
$$

As the right-hand side of the above inequality (27) is independent of τ, we take the least upper bound in its left-hand side with respect to τ from 0 to T, to obtain the desired inequality

$$\begin{aligned} & \sup_{0\leq\tau\leq T} \|u(\eta,\tau)\|^2_{H^1(\Omega)} + \|u_\eta\|^2_{L^2(Q)} \\ \leq \quad & \gamma e^{\gamma T}\left(\|f\|^2_{L^2(Q)} + \|\sigma\|^2_{H^1(\Omega)}\right). \end{aligned} \tag{28}$$

Let $R(L)$ be the range of the operator L. However, since we do not have any information about $R(L)$, except that $R(L) \subset F$, we must extend L, so that estimate (28) holds for the extension and its range is the whole space F. We first state the following proposition.

Proposition 3.2 — The operator $L : E \to F$ admits a closure $\overline{L}$.

PROOF : The proof is similar to that in [13].

Let $\overline{L}$ be the closure of this operator, with domain of definition $D(\overline{L})$.

We define a strong solution of problem (10)-(13) as the solution of the operator equation: $\overline{L}u = (f,\sigma)$ for all $u \in D(\overline{L})$.

The a *priori* estimate (17) can be extended to strong solutions, i.e., we have the estimate

$$\begin{aligned} & \sup_{0\leq\tau\leq T} \|u(\eta,\tau)\|^2_{L^2(\Omega)} + \|u_\eta\|^2_{L^2(Q)} \\ \leq \quad & c^2\left(\|f\|^2_{L^2(Q)} + \|\sigma\|^2_{L^2(\Omega)}\right), \quad \forall u \in D(\overline{L}). \end{aligned} \tag{29}$$

It can be deduced from the a priori estimate estimate (29) that the range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and is equals to the closure $\overline{R(L)}$ of $R(L)$, that is $R(\overline{L}) = \overline{R(L)}$.

4. EXISTENCE OF SOLUTION

***Theorem* 4.1** — *For all $\mathcal{F} = (f, \sigma) \in F$, there exists a unique strong solution $u = \overline{L}^{-1}\mathcal{F} = \overline{\mathcal{L}^{-\infty}}\mathcal{F}$ of the problem (11)-(13).*

PROOF : From the fact that $R(\overline{L}) = \overline{R(L)}$, we deduce that to prove the existence of the strong solution, it is sufficient to show the range of the operator L is everywhere dense in the space F, that is $\overline{L}$ is injective. To this end, we first prove the following proposition.

Proposition 4.2 — Let $D_0(L)$ be the set of all $u \in D(L)$ vanishing in a neighbourhood of $t = 0$. If for $\phi \in L^2(Q)$ and for all $u \in D_0(L)$, we have

$$(\mathcal{L}u, \phi)_{L^2(Q)} = 0, \tag{30}$$

then the function ϕ vanishes almost everywhere in Q.

PROOF (of proposition 4.2) : Assume that (30) holds for any $u \in D_0(L)$. Using this fact, it can be expressed in a particular form. First define the function σ by the formula

$$\sigma(\eta, t) = \int_t^T \phi(\eta, s)ds. \tag{31}$$

Let $\partial u/\partial t$ be a solution of the equation

$$u_t(\eta, t) = \sigma(\eta, t). \tag{32}$$

And let

$$u(\eta, t) = \begin{cases} 0 & 0 \leq t \leq z \\ \int_z^t u_s ds & z \leq t \leq T. \end{cases} \tag{33}$$

It follows from above that

$$\phi(\eta, t) = -u_{tt}(\eta, t). \tag{34}$$

We have the following result:

Lemma 4.3 — The function u defined by (32) and (33) has derivatives with resect to t up to the second order belonging to $L^2(Q_z)$,where $Q_z = \Omega \times (z, T)$.

PROOF : For the proof, the reader should refer to [11].

To complete the proof of proposition 4.2, we replace $\phi((\eta, t)$ in (30) by its representation (34). We have

$$-(u_t, u_{tt})_{L^2(Q)} + (u_{\eta\eta}, u_{tt})_{L^2(Q)} + (\alpha u_{\eta\eta t}, u_{tt})_{L^2(Q)} \\ +(\beta u_\eta^2 u_{\eta\eta}, u_{tt})_{L^2(Q)} = 0. \tag{35}$$

Invoking relations (32), (33) and the boundary conditions (13), and carrying out appropriate integrations by part of each term of (35), we obtain

$$-(u_t, u_{tt})_{L^2(Q)} = \frac{1}{2} \left\| u_t(\eta, z) \right\|^2_{L^2(\Omega)}, \tag{36}$$

$$(u_{\eta\eta}, u_{tt})_{L^2(Q)} = \left\| u_{t\eta} \right\|^2_{L^2(Q_z)}, \tag{37}$$

$$\begin{aligned} (\alpha u_{\eta\eta t}, u_{tt})_{L^2(Q)} &= \alpha \int_Q u_{\eta\eta t} u_{tt} d\eta dt = \alpha \int_z^T u_{\eta t} u_{tt} \mid_0^l dt - \alpha \int_{Q_z} u_{\eta t} u_{tt\eta} d\eta dt \\ &= -\alpha \int_0^l u_{\eta t}^2 \mid_z^T d\eta + \alpha \int_{Q_z} u_{\eta t} u_{tt\eta} d\eta dt. \end{aligned} \tag{38}$$

Equality (38) gives

$$(\alpha u_{\eta\eta t}, u_{tt})_{L^2(Q)} = \frac{\alpha}{2} \left\| u_{t\eta}(\eta, z) \right\|^2_{L^2(\Omega)}, \tag{39}$$

$$\begin{aligned} (\beta u_\eta^2 u_{\eta\eta}, u_{tt})_{L^2(Q)} &= \beta \int_Q u_\eta^2 u_{\eta\eta} u_{tt} d\eta dt \\ &= \beta \int_z^T u_\eta^3 u_{tt} \mid_0^l dt - \beta \int_{Q_z} u_\eta^3 u_{tt\eta} dx dt \\ &\quad -2\beta \int_{Q_z} u_\eta^2 u_{\eta\eta} u_{tt} d\eta dt. \end{aligned} \tag{40}$$

It follows from (40) that

$$\begin{aligned}(\beta u_\eta^2 u_{\eta\eta}, u_{tt})_{L^2(Q)} &= -\frac{\beta}{3}\int_{Q_z} u_\eta^3 u_{tt\eta} d\eta dt \\ &= -\frac{\beta}{3}\int_0^l u_\eta^3 u_{t\eta} \mid_z^T d\eta + \beta\int_{Q_z} u_\eta^2 u_{t\eta}^2 d\eta dt \\ &= \beta\int_{Q_z} u_\eta^2 u_{t\eta}^2 d\eta dt. \end{aligned} \quad (41)$$

Substitution of (36), (37), (39) and (41) into (35), yields

$$\begin{aligned}&\frac{1}{2}\left\|u_t(\eta,z)\right\|_{L^2(\Omega)}^2 + \left\|u_{t\eta}\right\|_{L^2(Q_z)}^2 + \frac{\alpha}{2}\left\|u_{t\eta}(\eta,z)\right\|_{L^2(\Omega)}^2 \\ &+\beta\int_{Q_z} u_\eta^2 u_{t\eta}^2 d\eta dt = 0. \end{aligned} \quad (42)$$

It follows from (42) that $\phi(\eta,t) = 0$ almost everywhere in Q_z. Proceeding in this way step by step, we prove that $\phi(\eta,t) = 0$ almost everywhere in Q. Therefore, the proof of Proposition 4.2 is complete.

Now consider the general case.

***Theorem* 4.4** — *The range $R(L)$ of the operator L coincides with the whole space F.*

PROOF : Assume that for some $G = (\varphi, g_0) \in \{R(L)\}^\perp$,

$$(Lu, G)_F = (\mathcal{L}u, \varphi)_{L^2(Q)} + (\ell u, g_0)_{L^2(\Omega)} = 0, \quad (43)$$

We must show that $G \equiv 0$.

Putting $u \in D_0(L)$ in (43), we obtain

$$(\mathcal{L}u, \varphi)_{L^2(Q)} = 0, \qquad u \in D_0(L).$$

Hence, Proposition 4.2 implies that $\varphi = 0$. Thus (43) takes the form

$$(\ell u, g_0)_{L^2(\Omega)} = 0, \qquad \forall u \in D(L). \quad (44)$$

As the range of the trace operator ℓ is everywhere dense in the Hilbert space $L^2(\Omega)$, then relation (44) implies that $g_0 = 0$. Hence, $G \equiv 0$, and thus $\overline{R(L)} = 0$.

Remark : The same analysis can be done to treat the problem

$$\begin{cases} \mathcal{L}u = u_t - u_{\eta\eta} - \alpha u_{\eta\eta t} - \beta u_\eta^2 u_{\eta\eta} = f(\eta, t) \\ u(\eta, 0) = 0, \\ u(0,t) = 1, \quad u(l,t) \to 0 \text{ when } l \to \infty \end{cases}$$

which reduces to Stokes' first problem.

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