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PRECISE ASYMPTOTICS FOR BETA ENSEMBLES[1]

Xingyuan Zeng and Zhenting Hou

School of Mathematics, Central South University,
Changsha 410075 *China*
e-mail: xyzeng@csu.edu.cn

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We consider the extremal (largest and smallest) eigenvalues of random matrices in the β-Hermite and β-Laguerre ensembles. Using the general β Tracy-Widom law together with Ledoux and Rider's small deviation inequalities for β-ensembles, we obtain some precise asymptotic results in both settings. This complements Su's results for the largest eigenvalue of Gaussian and Laguerre unitary ensembles.

Key words : Extremal eigenvalues; β-ensembles; general β Tracy-Widom law; small deviation inequalities.

1. INTRODUCTION

In the study of the spectral theory of random matrices, two kinds of problems are of the greatest concern. The first one is to consider the spectrum as a whole and study

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the global properties of the spectrum. The second one is to investigate local properties of spectrum, including the interactions between neighbouring eigenvalues and the behavior of extremal eigenvalues. In both settings, important progress and developments have taken place. For the global case, we refer the interested reader to the nice survey [3] by Bai for some important techniques and methodologies employed in this area as well as his recent book [4] coauthored with Silverstein for a comprehensive review. Besides the global spectrum properties, the very recent book [1] covers a detailed information on both the bulk and edge of the spectrum (local case) for Gaussian ensembles.

In this paper, we only investigate the edge of the spectrum of random matrices. In the past decade, much weight was put on the fluctuations of extremal eigenvalues of random matrices after the pioneer work [22, 23] by Tracy and Widom as well as the celebrated paper [5] by Baik, Deift and Johansson. A large class of models both within and outside of random matrices has been understood where the famous Tracy-Widom laws (see definition below) arise.

Recently, some non-asymptotic results for certain random matrix ensembles have also been developed with the motivation to consider the rate of concentration of those various objects about the limiting Tracy-Widom laws. We refer the interested reader to Ledoux's notes [14] as well as the recent paper [15] for further information on small deviation results for random matrices.

In this note, we add some precise asymptotic results for random matrices. We remark that Su [21] first studied that for the largest eigenvalues of Gussian unitary ensemble (GUE) and Leguerre unitary ensemble (LUE), β=2. And we extend Su's result for two cases. The first case is that we consider extreme eigenvalues including both the largest eigenvalue and the smallest eigenvalue. The other one is that we investigate both the β-Hermite and β-Laguerre ensembles. Here β can be any positive real value.

1.1 *Models and Results*

The random matrix ensembles that we shall consider in this paper are the so-called

β-Hermite ensemble and β-Laguerre ensemble.

For any $\beta > 0$, consider the probability density function of $\lambda_1, \lambda_2, \cdots, \lambda_N \in \mathbb{R}$ given by

$$P_N^\beta(\lambda_1, \lambda_2, \cdots, \lambda_N) = \frac{1}{Z_N^\beta} e^{-(\beta/4)\sum_{k=1}^N \lambda_k^2} \prod_{1 \le j < k \le N} |\lambda_j - \lambda_k|^\beta, \tag{1.1}$$

in which Z_N^β is a normalizing constant.

When $\beta = 1, 2, 4$, the joint density is shared by the eigenvalues of Gaussian orthogonal, unitary, or symplectic ensembles, G(O/U/S)E, of random matrix theory. The nice references are the classical book [16] by Mehta and the original paper [8] by Dyson. For these three special values of β, the above model is solvable: all finite dimensional correlation functions may be computed explicitly in terms of Hermite polynomials. Thus, the measure (1.1) is referred to the β-Hermite ensemble. We write it by H_β for short. The law (1.1) also describes a one-dimensional coulomb gas with logarithmic potential at temperature β, and thus of physical interest. Despite (1.1) being the focus of several branches of research, there appears to be no characterization of the correlation functions amenable to asymptotics in the case of $\beta \neq 1, 2, 4$. We refer the interested reader to the textbook [10] by Forrester for more information.

For all $\beta > 0$, Dumitriu and Edelman [7] discovered specific matrix models for the β-Hermite ensemble. Specifically, let $g_1, g_2, \cdots, g_N$ be independent Gaussian random variables with mean 0 and variance 2. Let also $\chi_\beta, \chi_{2\beta}, \cdots, \chi_{(N-1)\beta}$ be independent χ random variables indexed by the shape parameter. Then β-Hermite ensemble has the following random matrix construction

$$H_\beta = \frac{1}{\sqrt{\beta}} \begin{pmatrix} g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{N-1} & \chi_\beta \\ & & & \chi_\beta & g_N \end{pmatrix}. \tag{1.2}$$

Moreover, the N eigenvalues of the above tridiagonal matrix ensemble have joint density function given by (1.1).

Let $\lambda_{\max}(H_\beta) = \max_{1\le i\le N} \lambda_i$ and $\lambda_{\min}(H_\beta) = \min_{1\le i\le N} \lambda_i$ be the largest eigenvalue and the smallest eigenvalue for H_β respectively. Very recently, Ramírez, Rider and Virág [18] have shown that the fluctuations of the largest eigenvalue $\lambda_{\max}(H_\beta)$ around its expected value $2\sqrt{N}$ with the rate $(mean)^{1/3}$, i.e. $N^{1/6}$ and the asymptotic distribution exists. Precisely, for any t,

$$\lim_{N\to\infty} P\left(N^{1/6}(\lambda_{\max}(H_\beta) - 2\sqrt{N}) \le t\right) = F_\beta(t), \tag{1.3}$$

where F_β is the general β Tracy-Widom law (see below).

And the fluctuation result for the smallest eigenvalue of H_β is easily implied. For any t,

$$\lim_{N\to\infty} P\left(N^{1/6}(-2\sqrt{N} - \lambda_{\min}(H_\beta)) \le t\right) = F_\beta(t). \tag{1.4}$$

Through a random variational principle [18], the general Tracy-Widom law F_β ($\beta > 0$) is identified as

$$F_\beta = \sup_{f\in L}\left\{\frac{2}{\sqrt{\beta}}\int_0^\infty f^2(x)dB(x) - \int_0^\infty [(f'(x))^2 + xf^2(x)]dx\right\}, \tag{1.5}$$

in which $x \to B(x)$ is a standard Brownian motion and L is the space of functions f which satisfy

$$\begin{cases} f(0) = 0, \\ \int_0^\infty f^2(x)dx = 1, \\ \int_0^\infty [(f'(x))^2 + xf^2(x)]dx < \infty. \end{cases} \tag{1.6}$$

It is also implied in [18] that for all $\beta > 0$

$$\begin{cases} F_\beta(t) \sim \exp(\frac{1}{24}\beta t^3) & \text{as } t \to -\infty, \\ 1 - F_\beta(t) \sim \exp(-\frac{2}{3}\beta t^{3/2}) & \text{as } t \to \infty. \end{cases} \tag{1.7}$$

Now, we state our precise asymptotics results for the β-Hermite ensemble.

***Theorem* 1.1** — *Assume $\lambda_{\max}(H_\beta)$ is the largest eigenvalue for the above β-Hermite ensemble. For $\beta \geq 1$, we have*

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{\infty} P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) = \frac{3}{2}\int_0^{\infty} y^{1/2}\left(1-F_\beta(y)\right)dy \tag{1.8}$$

and

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{\infty} P\left(\lambda_{\max}(H_\beta) \leq 2\sqrt{N}(1-\epsilon)\right) = \frac{3}{2}\int_{-\infty}^{0} (-y)^{1/2}F_\beta(y)dy, \tag{1.9}$$

where $F_\beta(\cdot)$ is just the general β Tracy-Widom law.

The proof is deferred to Section 3. Based on the symmetry between the largest and the smallest eigenvalue of Hermitian ensemble, we can easily obtain the following result for the smallest eigenvalue of β-Hermitian ensemble.

Corollary 1.2 — Assume $\lambda_{\min}(H_\beta)$ is the smallest eigenvalue for the above β-Hermite ensemble. For $\beta \geq 1$, we have

$$\begin{aligned}\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{\infty} P\left(\lambda_{\min}(H_\beta) \leq -2\sqrt{N}(1+\epsilon)\right)\\ = \frac{3}{2}\int_0^{\infty} y^{1/2}\left(1-F_\beta(y)\right)dy\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{\infty} P\left(\lambda_{\min}(H_\beta) \geq -2\sqrt{N}(1-\epsilon)\right)\\ = \frac{3}{2}\int_{-\infty}^{0} (-y)^{1/2}F_\beta(y)dy.\end{aligned} \tag{1.11}$$

As mentioned at the beginning of the introduction, we will also study the β-Laguerre ensemble which can also be considered as a general version of β-Hermite ensemble. Consider a density of the form (1.1) in which the weight $w(\lambda) = e^{-\beta\lambda^2/4}$ on $\mathbb{R}$ is replaced by $w(\rho) = \rho^{(\beta/2)(M-N+1)-1}e^{-\beta\rho/2}$, now restricted to $\mathbb{R}^+$, where M can be any real number strictly larger than $N-1$. More

precisely, we consider the joint density on points $\rho_1, \rho_2, \cdots, \rho_N \in \mathbb{R}^+$

$$P^{\beta}_{N,M}(\rho_1, \rho_2, \cdots, \rho_N) = \frac{1}{Z^{\beta}_{N,M}} \prod_{1 \le j < k \le N} |\rho_j - \rho_k|^{\beta} \times \prod_{i=1}^{N} \rho_i^{\frac{\beta}{2}(M-N+1)-1} e^{-\beta \rho_i/2}, \tag{1.12}$$

where $Z^{\beta}_{N,M}$ is a normalizing constant.

When M is an integer and $\beta = 1, 2$, or 4, (1.12) is the joint law of eigenvalues of Laguerre orthogonal, unitary or symptotic ensembles, L(O/U/S)E, of random matrix theory. Analogue to the three Gaussian ensembles G(O/U/S)E, the finite dimensional correlation functions for L(O/U/S)E can also be explicitly calculated in terms of Laguerre polynomials. For this reason, the measure (1.12) is referred to the β-Laguerre ensemble. We denote it by L_β.

Also, [7] provides a family of tridiagonal random matrices for the β-Laguerre ensemble. Take the $N \times N$ bidiagonal random matrix

$$W^{\beta}_{N,M} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \tilde{\chi}_{\beta M} & & & & & \\ \chi_{\beta(N-1)} & \tilde{\chi}_{\beta(M-1)} & & & & \\ & \ddots & \ddots & & & \\ & & \chi_{\beta 2} & \tilde{\chi}_{\beta(M-N+2)} & & \\ & & & \chi_{\beta} & \tilde{\chi}_{\beta(M-N+1)} & \end{pmatrix}, \tag{1.13}$$

where the entries are all independent χ variables of indicated parameter. Here $M > N-1$. Then, by [7], the eigenvalues of $(W^{\beta}_{N,M})^*(W^{\beta}_{N,M})$ have joint density (1.12).

Let $\rho_{\max}(L_\beta) = \max_{1 \le i \le N} \rho_i$ and $\rho_{\min}(L_\beta) = \min_{1 \le i \le N} \rho_i$ be the largest and the smallest eigenvalue for the above defined β-Laguerree ensemble L_β respectively. Using a tridiagonal model $(W^{\beta}_{N,M})^*(W^{\beta}_{N,M})$ for L_β, it is also proved in [18]: for $M + 1 > N \to \infty$, with $M/N \to \eta \ge 1$, and for any t, the following limit holds.

$$\lim_{N \to \infty} P\left(\frac{\rho_{\max}(L_\beta) - \mu_{M,N}}{\sigma_{M,N}} \le t \right) = F_\beta(t). \tag{1.14}$$

Here $\mu_{M,N} = (\sqrt{M} + \sqrt{N})^2$ and $\sigma_{M,N} = (\sqrt{M} + \sqrt{N})(\frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}})^{1/3}$.

While not detailed there, the results of [18] imply that

$$\lim_{N\to\infty} P\left(\frac{\nu_{M,N} - \rho_{\min}(L_\beta)}{\delta_{M,N}} \le t\right) = F_\beta(t). \tag{1.15}$$

Here $\nu_{M,N} = (\sqrt{M} - \sqrt{N})^2$ and $\delta_{M,N} = (\sqrt{M} - \sqrt{N})(\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{M}})^{1/3}$.

We also obtain the precise asymptotics result for $\rho_{\max}(L_\beta)$ in the following theorem.

***Theorem* 1.3 —** *Assume $\rho_{\max}(L_\beta)$ is the largest eigenvalue for the above β-Laguerre ensemble. If $M = [\gamma N]$ with some $\gamma \ge 1$, for all $\beta \ge 1$, we have*

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon^{3/2} \sum_{N=1}^{\infty} P\left(\rho_{\max}(L_\beta) \ge (\sqrt{M} + \sqrt{N})^2(1+\epsilon)\right)\\ = \frac{3}{2(1+\sqrt{\gamma})\gamma^{1/4}} \int_0^\infty y^{1/2}\,(1 - F_\beta(y))\,dy\end{aligned} \tag{1.16}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon^{3/2} \sum_{N=1}^{\infty} P\left(\rho_{\max}(L_\beta) \le (\sqrt{M} + \sqrt{N})^2(1-\epsilon)\right)\\ = \frac{3}{2(1+\sqrt{\gamma})\gamma^{1/4}} \int_{-\infty}^0 (-y)^{1/2} F_\beta(y) dy.\end{aligned} \tag{1.17}$$

We also consider the precise asymptotics result for the smallest eigenvalue of β-Laguerre ensemble with the upper-tail case on the left of the mean. But the following result cannot be easily deduced by Theorem 1.3.

***Theorem* 1.4 —** *Assume $\rho_{\min}(L_\beta)$ is the smallest eigenvalue for the above β-Laguerre ensemble. If $M = [\gamma N]$ with some $\gamma \ge 1$, for all $\beta \ge 1$, we have*

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon^{3/2} \sum_{N=1}^{\infty} P\left(\rho_{\min}(L_\beta) \le (\sqrt{M} - \sqrt{N})^2(1-\epsilon)\right)\\ = \frac{3}{2\gamma^{1/4}(\sqrt{\gamma}-1)} \int_0^\infty y^{1/2}\,(1 - F_\beta(y))\,dy.\end{aligned} \tag{1.18}$$

1.2 *Some Remarks*

Remark 1.5 : From (1.7), it is clear that the general Tracy-Widom law F_β ($\beta > 0$) is asymmetric. Thus, it is not surprising that the upper tails on the right of the mean is different from the lower tails on the left of the mean in the above theorems and corollary. However, the rate is the same. They are both related to the order 1/3 of the mean.

Remark 1.6 : From the above results, we see that β is restricted to $\beta \geq 1$. Actually, the precise asymptotics results also remain valid to all $0 < \beta < 1$. In this paper, we would like to keep $\beta \geq 1$ since this case covers all the cases of classical interest and our proofs can be put cleaner.

Remark 1.7 : Our result is a generalization of that in [21] by Su who studied the precise asymptotics for two most famous ensembles of Hermitian random matrices. They are the GUE and LUE. Just as analyzed in [21] by Su, the study of the precise asymptotics for random matrices is in a sense similar to the precise asymptotics for sums of independent random variables in the context of the law of large numbers and complete convergence. The reader may be referred to the original paper [21] by Su for an extensive discussion and references.

Remark 1.8 : We remark that the precise asymptotics results for some random growth models are also discussed in [21]. The conclusions in our work can also be applied to the models considered by Su. Moreover, for certain models outside random matrices, our results are also applicable. To be more specific, the length of the largest increasing subsequence of a random permutation (Ulam's problem see e.g. [5]), directed last-passage percolation models in the plane [13], the totally asymptotic exclusion process for an initial condition with a step at the origin as well as certain polynuclear growth model in 1+1 dimension all have precise asymptotics results analogue to ours for β-ensembles. The reason is that there exists equivalence among the above mentioned models. We would like refer the interested reader to Johansson's nice paper [13] for detailed information (see also the recent survey [24] for the above mentioned models).

Remark 1.9 : It is expected to extend our results to all the models both within and outside random matrices where the famous Tracy-Widom law F_β occurs. Especially, due to the recent works [19, 20] and [17] by Soshnikov and Péché and [9] by Feldheim and Sodin, small deviation results for the extremal eigenvalues can be extended to a larger class of random matrices with not necessarily Gaussian entries.

Remark 1.10 : The theory of the general β Tracy-Widom law was recently developed by Ramírez, Rider and Virág [18] who obtained the limiting distribution for the largest eigenvalue of β-ensembles. To be noted, very recently, Jiang [12] also investigate the asymptotic distributions of the smallest and the largest eigenvalues for the β-Jacobi ensembles.

It is well known that the famous Tracy-Widom law F_β plays an essential role in the random matrix theory. And here we would like to make more remarks on it. For the special value $\beta = 2$, the Tracy-Widom law F_2 has two kinds of representations. One is in the form of Fredholm determinant, i.e.

$$F_2(t) = \det(I - K_A)|_{L^2(t,\infty)} \quad t \in (-\infty, +\infty) \tag{1.19}$$

of the integral operator associated with the Airy kernel K_A. The other representation for F_2 is

$$F_2(t) = \exp\left(-\int_t^\infty (x-t)q^2(x)dx\right), \tag{1.20}$$

where q satisfies the following Painlevé II equation.

$$q'' = tq + 2q^3, \quad q(t) \sim Ai(t) \quad \text{as } t \to +\infty \ . \tag{1.21}$$

And for the limit distributions F_β with $\beta = 1$ and 4, they satisfy the following relations respectively.

$$F_1(t) = \sqrt{F_2(t)}\exp\left(-\frac{1}{2}\int_t^\infty q(x)dx\right), \tag{1.22}$$

$$F_4(t/2^{2/3}) = \sqrt{F_2(t)}\cosh\left(-\frac{1}{2}\int_t^\infty q(x)dx\right). \tag{1.23}$$

It is also expected that the general Tracy-Widom law F_β has a representation in terms of Painlevé equation.

2. Preliminaries and the Main Technical Statements

The first tool for our later proofs is the famous Euler-Maclaurin formula which provides a powerful connection between integrals and sums. In the sequel, we use it to evaluate infinite series in terms of integrals. Now, let us recall it.

Lemma 2.1 (cf. (5.8.15) of [11]) — If n is a natural number and $f(x)$ is a smooth function defined for all real number between 0 and n, the Euler-Maclaurin formula is referred as

$$\begin{aligned}\sum_{i=0}^{n} f(i) &= \int_0^n f(x)dx - B_1[f(n)+f(0)] + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \\ &\quad \left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] + R_p, \end{aligned} \tag{2.1}$$

where B_k is the Bernoulli number and R_p is the reminder term in the form

$$R_p = n\frac{B_{2p+2}f^{(2p+2)}(\xi)}{(2p+2)!},$$

with $0 < \xi < n$.

The reminder term R_p is most easily expressed using the periodic Bernoulli polynomial $P_n(x)$, which is defined as

$$P_n(x) = B_n(x - [x]) \quad \text{for} \quad x > 0.$$

Here $[x]$ denotes the largest integer that is not greater than x and $B_n(x)$, $n = 0, 1, 2, \cdots$ are the so-called Bernoulli polynomials defined recursively as

$$B_0(x) = 1,$$

$$B_n'(x) = nB_{n-1}(x) \quad \text{and} \quad \int_0^1 B_n(x)dx = 0, \quad \text{for} \quad n \geq 1.$$

Then, in terms of $P_n(x)$, the reminder term R_p can be written as

$$R_p = (-1)\int_0^n f^{(2p)}(x)\frac{P_{2p}(x)}{(2p)!}dx.$$

In particular, if we take $2p = 1$, $n \to \infty$, then

$$R_p = (-1)\int_0^\infty f'(x)P_1(x)dx.$$

Thus, (2.1) can be reduced to

$$\begin{aligned}\sum_{i=0}^{\infty} f(i) &= \int_0^\infty f(x)dx - \int_0^\infty f'(x)P_1(x)dx \\ &= \int_0^\infty f(x)dx - \int_0^\infty f'(x)B_1(x-[x])dx \\ &= \int_0^\infty f(x)dx - \int_0^\infty (x-[x]-\frac{1}{2})f'(x)dx, \qquad (2.2)\end{aligned}$$

where we used $B_1(x) = x - \frac{1}{2}$.

In the sequel, we will see that the asymptotic fluctuation results for the extremal eigenvalues of L_β and H_β together with the following lemmas about non-asymptotics exponential deviation inequalities for β-ensembles are heavily relied on.

Lemma 2.2 (cf. Theorem 4 of [15]) — For all $N \geq 1$, $0 < \epsilon \leq 1$ and $\beta \geq 1$,

$$P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) \leq C_1 e^{-\beta N\epsilon^{3/2}/C_1} \qquad (2.3)$$

and

$$P\left(\lambda_{\max}(H_\beta) \leq 2\sqrt{N}(1-\epsilon)\right) \leq C_1^\beta e^{-\beta N^2\epsilon^3/C_1}, \qquad (2.4)$$

where C_1 is a numerical constant.

The above small deviation results was developed by Ledoux and Rider [15]. It is worthwhile to be noted, just as discussed in [14], the right-tail inequality for the GUE as well as LUE may be shown from the results of Johansson [13]. And

the left-tail inequality for the GUE and LUE can be implied from [6] where optimal estimates for Johansson's geometrical models [13] was established. And later, Auburn [2] obtained the above small deviation results for the largest eigenvalue of GUE by controlling the various Fredholm determinants by appropriate bounds on orthogonal polynomials.

Due to the symmetry between the smallest and the largest eigenvalue of β-Hermitian ensemble, we can easily get the following corollary for the smallest eigenvalue of H_β.

Corollary 2.3 — For all $N \geq 1$, $0 < \epsilon \leq 1$ and $\beta \geq 1$,

$$P\left(\lambda_{\min}(H_\beta) \leq -2\sqrt{N}(1+\epsilon)\right) \leq C_1 e^{-\beta N \epsilon^{3/2}/C_1} \tag{2.5}$$

and

$$P\left(\lambda_{\min}(H_\beta) \geq -2\sqrt{N}(1-\epsilon)\right) \leq C_1^\beta e^{-\beta N^2 \epsilon^3/C_1}. \tag{2.6}$$

For the extremal eigenvalues of β-Laguerre ensemble, analogue results are also obtained in [15].

Lemma 2.4 (cf. Theorem 2 of [15]) — For all $M+1 > N \geq 1$, $0 < \epsilon \leq 1$ and $\beta \geq 1$,

$$P\left(\rho_{\max}(L_\beta) \geq (\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right) \leq C_2 e^{-\beta\sqrt{MN}\epsilon^{3/2}(\frac{1}{\sqrt{\epsilon}}\ {}^{\mathrm{V}}\ (\frac{M}{N})^{1/4})/C_2} \tag{2.7}$$

and

$$P\left(\rho_{\max}(L_\beta) \leq (\sqrt{M}+\sqrt{N})^2(1-\epsilon)\right) \leq C_2^\beta e^{-\beta MN\epsilon^3(\frac{1}{\epsilon}\ {}^{\mathrm{V}}\ (\frac{M}{N})^{1/2})/C_2}. \tag{2.8}$$

Also C_2 is some numerical constant.

Since the smallest eigenvalue of the β-Laguerre ensemble is much smaller in absolute value than the largest one, the state of affairs is different from the Hermitian case. But Ledoux and Rider still considered the right-tail upper bound for $\rho_{\min}(L_\beta)$. We now present it in the following.

Lemma 2.5 (cf. Corollary 13 of [15]) — For all $\beta \geq 1$ and all $0 < \epsilon \leq 1$, assume that $M \geq cN$ for $c > 1$, the following holds.

$$P\left(\rho_{\min}(L_\beta) \leq (\sqrt{M} - \sqrt{N})^2(1-\epsilon)\right) \leq C(c)e^{-\beta N\epsilon^{3/2}/C(c)}. \tag{2.9}$$

where $C(c)$ is a numerical number depended on c.

On the other hand, [15] also described an estimate with a less restriction on M and N but with additional constrain on ϵ.

Lemma 2.6 (cf. Theorem 12 of [15]) — For all $M - 1 \geq N \geq 1$, and $\beta \geq 1$,

$$\begin{aligned} &P\left(\rho_{\min}(L_\beta) \leq (\sqrt{M} - \sqrt{N})^2(1-\epsilon)\right) \\ &\quad \leq C_3 e^{-\beta(MN)^{1/4}(\sqrt{M}-\sqrt{N})\epsilon^{3/2}/C_3} \end{aligned} \tag{2.10}$$

for a numerical constant C_3 and all $0 < \epsilon \leq \sqrt{\frac{N}{M}}(\alpha^{14} \bigwedge \alpha^2 n^{-2/5})$ in which $\alpha = 1 - \sqrt{N/M}$.

3. PROOFS

PROOF [Proof of Theorem 1.1] : We shall first use the following decomposition

$$\begin{aligned} &\sum_{N=1}^{\infty} P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) \\ = &\sum_{N=1}^{\infty}\left[P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] \\ &+ \sum_{N=1}^{\infty}\left(1 - F_\beta(2\epsilon N^{2/3})\right). \end{aligned} \tag{3.1}$$

By the Euler-Maclaurin formula (see (2.2)), we have

$$\begin{aligned} \sum_{N=1}^{\infty}\left(1 - F_\beta(2\epsilon N^{2/3})\right) &= \int_1^{\infty}\left(1 - F_\beta(2\epsilon x^{2/3})\right)dx \\ &- \int_1^{\infty}\left(x - [x] + \frac{1}{2}\right)d\left(1 - F_\beta(2\epsilon x^{2/3})\right). \end{aligned} \tag{3.2}$$

We take a change of variables for the two terms in the right hand side (r.h.s.) of (3.2) respectively.

$$\int_1^\infty \left(1 - F_\beta(2\epsilon x^{2/3})\right) dx = \frac{3}{2(2\epsilon)^{3/2}} \int_{2\epsilon}^\infty y^{1/2} \left(1 - F_\beta(y)\right) dy \tag{3.3}$$

and

$$\begin{aligned} &\int_1^\infty \left(x - [x] + \frac{1}{2}\right) d\left(1 - F_\beta(2\epsilon x^{2/3})\right) \\ &\qquad = -\int_{2\epsilon}^\infty \left((\frac{y}{2\epsilon})^{3/2} - [(\frac{y}{2\epsilon})^{3/2}] + \frac{1}{2}\right) dF_\beta(y). \end{aligned} \tag{3.4}$$

Inserting (3.3) and (3.4) into (3.2), and then letting $\epsilon \to 0$, we get

$$\lim_{\epsilon \to 0} (2\epsilon)^{3/2} \sum_{N=1}^\infty \left(1 - F_\beta(2\epsilon N^{2/3})\right) = \frac{3}{2} \int_0^\infty y^{1/2} \left(1 - F_\beta(y)\right) dy. \tag{3.5}$$

Next we turn to the first term on r.h.s. of (3.1).

Let $N(\epsilon) = \left[\frac{1}{\epsilon \cdot \epsilon^{3/2}}\right]$, when $0 < \epsilon \leq 1$, we have

$$\begin{aligned} &\sum_{N=1}^\infty \left[P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] \\ = &\sum_{N=1}^{N(\epsilon)} \left[P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] \\ &+ \sum_{N=N(\epsilon)+1}^\infty P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) \\ &- \sum_{N=N(\epsilon)+1}^\infty \left(1 - F_\beta(2\epsilon N^{2/3})\right). \end{aligned}$$

Recall that

$$N^{1/6}(\lambda_{\max}\left(H_\beta) - 2\sqrt{N}\right) \Rightarrow F_\beta$$

in distribution (see (1.3)). From [18], we know that the representation of F_β is the hitting distribution of a certain 1-d diffusion. It follows that F_β solves a parabolic PDE with smooth coefficients, so it itself is smooth.

Then as $N \to \infty$,

$$\Delta_N(H_\beta) =: \sup_{-\infty<x<\infty} \Big| P(N^{1/6}\left(\lambda_{\max}(H_\beta) - 2\sqrt{N}\right) \geq x) - (1 - F_\beta(x)) \Big| \to 0. \tag{3.6}$$

Thus, using the Kronecker's lemma, we have

$$\begin{aligned} & \lim_{\epsilon\to 0}(2\epsilon)^{3/2} \sum_{N=1}^{N(\epsilon)} \left[P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] \\ = & \lim_{\epsilon\to 0}(2\epsilon)^{3/2} \sum_{N=1}^{N(\epsilon)} \left[P\left(N^{1/6}(\lambda_{\max}(H_\beta) - 2\sqrt{N}) \geq 2\epsilon N^{2/3}\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] \\ = & \ 0. \end{aligned} \tag{3.7}$$

On the other hand, using the small deviation inequality (2.3), we have

$$\begin{aligned} \sum_{N=N(\epsilon)+1}^{\infty} P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) &\leq C_1 \sum_{N=N(\epsilon)+1}^{\infty} e^{-\beta N\epsilon^{3/2}/C_1} \\ &\leq C_1 \int_{N(\epsilon)}^{\infty} e^{-\beta\epsilon^{3/2}x/C_1} dx \\ &= \frac{C_1}{\beta\epsilon^{3/2}} \int_{\beta/(C_1\epsilon)}^{\infty} e^{-y} dy. \end{aligned}$$

Thus,

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2} \sum_{N=N(\epsilon)+1}^{\infty} P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) = 0. \tag{3.8}$$

Also,

$$\begin{aligned} \sum_{N=N(\epsilon)+1}^{\infty} \left(1 - F_\beta(2\epsilon N^{2/3})\right) &\leq \int_{N(\epsilon)}^{\infty} \left(1 - F_\beta(2\epsilon x^{2/3})\right) dx \\ &= \int_{2/\epsilon^{2/3}}^{\infty} \frac{3y^{1/2}}{2(2\epsilon)^{3/2}} \left(1 - F_\beta(y)\right) dy. \end{aligned}$$

There holds that

$$(2\epsilon)^{3/2} \sum_{N=N(\epsilon)+1}^{\infty} \left(1 - F_\beta(2\epsilon N^{2/3})\right) \leq \frac{3}{2} \int_{2/\epsilon^{2/3}}^{\infty} y^{1/2} \left(1 - F_\beta(y)\right) dy. \tag{3.9}$$

So,

$$\lim_{\epsilon \to 0} (2\epsilon)^{3/2} \sum_{N=N(\epsilon)+1}^{\infty} \left(1 - F_\beta(2\epsilon N^{2/3})\right) = 0. \tag{3.10}$$

By (3.7), (3.8) and (3.10), it is easy to see that

$$\lim_{\epsilon \to 0} (2\epsilon)^{3/2} \sum_{N=1}^{\infty} \left[P\left(\lambda_{\max}(H_\beta) \geq 2\sqrt{N}(1+\epsilon)\right) - \left(1 - F_\beta(2\epsilon N^{2/3})\right)\right] = 0. \tag{3.11}$$

Together with (3.5), the result (1.7) in Theorem 1.1 is obtained.

For the other formula (1.8) in Theorem 1.1, we can prove it in a similar way, with the only change of the upper-tail of $\lambda_{\max}(H_\beta)$ on the left of the mean. For $0 < \epsilon < 1$, we use the similar equation as (3.1).

$$\begin{aligned} &\sum_{N=1}^{\infty} P\left(\lambda_{\max}(H_\beta) \leq 2\sqrt{N}(1-\epsilon)\right) \\ &= \sum_{N=1}^{\infty} \left[P\left(\lambda_{\max}(H_\beta) \leq 2\sqrt{N}(1-\epsilon)\right) - F_\beta(-2\epsilon N^{2/3})\right] \\ &+ \sum_{N=1}^{\infty} F_\beta(-2\epsilon N^{2/3}). \end{aligned} \tag{3.12}$$

Using the Euler-Maclaurin formula and making a change of variables, we get

$$\begin{aligned}
&\sum_{N=1}^{\infty} F_\beta(-2\epsilon N^{2/3}) \\
= &\int_1^{\infty} F_\beta(-2\epsilon x^{2/3})dx - \int_1^{\infty}(x-[x]+\frac{1}{2})dF_\beta(-2\epsilon x^{2/3}) \\
= &\int_{-\infty}^{-2\epsilon} \frac{3}{2(2\epsilon)^{3/2}} F_\beta(y)(-y)^{1/2}dy + \int_{-\infty}^{-2\epsilon} \\
&\left((-\frac{y}{2\epsilon})^{3/2} - [(-\frac{y}{2\epsilon})^{3/2}] + \frac{1}{2}\right) dF_\beta(y). \qquad (3.13)
\end{aligned}$$

Multiplying $(2\epsilon)^{3/2}$ on both sides of (3.13), and taking $\epsilon \to 0$ yields

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{\infty} F_\beta(-2\epsilon N^{2/3}) = \frac{3}{2}\int_{-\infty}^{0}(-y)^{1/2}F_\beta(y)dy. \qquad (3.14)$$

Then we turn to the first term on the r.h.s. of (3.12). Let $N(\epsilon) = \left[\frac{1}{\epsilon\cdot\epsilon^{3/2}}\right]$, when $0 < \epsilon \le 1$, we have

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left[P\left(\lambda_{\max}(H_\beta) \le 2\sqrt{N}(1-\epsilon)\right) - F_\beta(-2\epsilon N^{2/3})\right] \\
= &\sum_{N=1}^{N(\epsilon)}\left[P\left(\lambda_{\max}(H_\beta) \le 2\sqrt{N}(1-\epsilon)\right) - F_\beta(-2\epsilon N^{2/3})\right] \\
&+ \sum_{N=N(\epsilon)+1}^{\infty} P\left(\lambda_{\max}(H_\beta) \le 2\sqrt{N}(1-\epsilon)\right) - \sum_{N=N(\epsilon)+1}^{\infty} F_\beta(-2\epsilon N^{2/3}).
\end{aligned}$$

Following the similar argument to that for (3.7), (3.8) and (3.10), we will obtain the following results.

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=1}^{N(\epsilon)}\left[P\left(\lambda_{\max}(H_\beta) \le 2\sqrt{N}(1-\epsilon)\right) - F_\beta(-2\epsilon N^{2/3})\right] = 0. \quad (3.15)$$

Meanwhile, we have

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=N(\epsilon)+1}^{\infty} P\left(\lambda_{\max}(H_\beta)\le 2\sqrt{N}(1-\epsilon)\right)=0 \tag{3.16}$$

and

$$\lim_{\epsilon\to 0}(2\epsilon)^{3/2}\sum_{N=N(\epsilon)+1}^{\infty} F_\beta(-2\epsilon N^{2/3})=0. \tag{3.17}$$

Taking (3.14) into account, we can finish the proof of (1.8).

It is worthwhile to be noted, (3.16) relies on Ledoux and Rider's lower tail for $\lambda_{\max}(H_\beta)$ on the left of the mean (see also (2.4)).

$$P\left(\lambda_{\max}(H_\beta)\le 2\sqrt{N}(1-\epsilon)\right)\le C_1^\beta e^{-\beta N^2\epsilon^3/C_1}.$$

From the above proof, we see that the basic procedure is much similar to that in Su's paper [21]. And for the proofs of the other theorems (Theorem 1.3 and Theorem 1.4), we have to say the process is more or less the same. For the clearness, we provide a sketch of the proof of Theorem 1.3. Also, we would like to highlight the small deviation results for β-ensembles by Ledoux and Rider.

PROOF [Proof of Theorem 1.3] : As described before, we take the following decomposition.

$$\begin{aligned}
&\sum_{N=1}^{\infty} P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\\
=\;&\sum_{N=1}^{\infty}\Big[P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\\
&-\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right)\Big]\\
&+\sum_{N=1}^{\infty}\left[1-F_\beta\left(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon\right)\right],
\end{aligned}\tag{3.18}$$

where $M=[\gamma N]$ and $\gamma\ge 1$.

For the second sum, using (2.2), making a change of variables and then letting $\epsilon \to 0$, we can easily obtain

$$\lim_{\epsilon\to 0}\epsilon^{3/2}\sum_{N=1}^{\infty}\left[1-F_\beta\left(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon\right)\right] = \frac{3}{2\gamma^{1/4}(1+\sqrt{\gamma})}\int_0^\infty y^{1/2}\left(1-F_\beta(y)\right)dy. \tag{3.19}$$

On the other hand, for the first sum in the r.h.s. of (3.18), we first decompose it at $N(\epsilon)$ with $N(\epsilon)=\left[\frac{1}{\epsilon\cdot\epsilon^{3/2}}\right]$ where $0<\epsilon\le 1$.

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left[P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\right.\\
&\left.-\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right)\right]\\
=\ &\sum_{N=1}^{N(\epsilon)}\left[P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\right.\\
&\left.-\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right)\right]\\
&+\sum_{N=N(\epsilon)+1}^{\infty}P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\\
&-\sum_{N=N(\epsilon)+1}^{\infty}\left(1-F_\beta\left(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon\right)\right).
\end{aligned}\tag{3.20}$$

Based on (1.14), the following limit holds (see also the proof of Theorem 1.1).

$$\begin{aligned}
&\lim_{\epsilon\to 0}\epsilon^{3/2}\sum_{N=1}^{N(\epsilon)}\left[P\left(\rho_{\max}(L_\beta)\ge(\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\right.\\
&\left.-\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right)\right]\\
=\ &\lim_{\epsilon\to 0}\epsilon^{3/2}\sum_{N=1}^{N(\epsilon)}\left[P\left(\frac{\rho_{\max}(L_\beta)-\mu_{M,N}}{\sigma_{M,N}}\ge\theta\right)-(1-F_\beta(\theta))\right]\\
=\ &0,
\end{aligned}\tag{3.21}$$

where $\theta = \left((\sqrt{M}+\sqrt{N})^2\sqrt{MN}\right)^{1/3}\epsilon$.

By (2.7), we have

$$\begin{aligned}&\sum_{N=N(\epsilon)+1}^{\infty} P\left(\rho_{\max}(L_\beta) \geq (\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\\ &\leq C_2 \sum_{N=N(\epsilon)+1}^{\infty} e^{-\beta\sqrt{MN}\epsilon^{3/2}(\frac{1}{\sqrt{\epsilon}}\ \mathrm{V}\ (\frac{M}{N})^{1/4})/C_2}. \end{aligned} \tag{3.22}$$

When $0 < \epsilon \leq 1/\sqrt{\gamma}$, by making a change of variables, (3.22) is reduced to

$$\begin{aligned}&\sum_{N=N(\epsilon)+1}^{\infty} P\left(\rho_{\max}(L_\beta) \geq (\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right)\\ \leq\ & C_2 \int_{N(\epsilon)}^{\infty} e^{-\beta\gamma^{3/4}\epsilon^{3/2}x/C_2}dx\\ =\ & \frac{C_2^2}{\beta\gamma^{3/4}\epsilon^{3/2}} \int_{\beta\gamma^{3/4}/(C_2\epsilon)}^{\infty} e^{-y}dy. \end{aligned}$$

Multiplying $\epsilon^{3/2}$ on the l.h.s. of (3.22), we have

$$\lim_{\epsilon\to 0}\epsilon^{3/2}\sum_{N=N(\epsilon)+1}^{\infty} P\left(\rho_{\max}(L_\beta) \geq (\sqrt{M}+\sqrt{N})^2(1+\epsilon)\right) = 0. \tag{3.23}$$

Also,

$$\begin{aligned}&\sum_{N=N(\epsilon)+1}^{\infty}\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right)\\ &\leq \int_{N(\epsilon)}^{\infty}\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}x^{2/3}\epsilon)\right)dx\\ &= \frac{3}{2\gamma^{1/4}(1+\sqrt{\gamma})\epsilon^{3/2}}\int_{\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}/\epsilon^{2/3}}^{\infty} y^{1/2}\left(1-F_\beta(y)\right)dy. \end{aligned}$$

There holds that

$$\lim_{\epsilon\to 0}\epsilon^{3/2}\sum_{N=N(\epsilon)+1}^{\infty}\left(1-F_\beta(\gamma^{1/6}(1+\sqrt{\gamma})^{2/3}N^{2/3}\epsilon)\right) = 0. \tag{3.24}$$

By (3.21), (3.23), (3.24) together with (3.19), therefore, the formula (1.16) in Theorem 1.3 is proved. For the other limit (1.17), we can take similar procedure only with a change of small deviation result (2.8) in Lemma 2.4. To be noted that the restriction $M = [\gamma N]$ for $\gamma \geq 1$ is included in the condition on M and N of Lemma 2.4.

PROOF [Proof of Theorem 1.4] : Besides the Euler-Maclaurin sum formula and a argument of changing variables, it also depends on formula (1.15) as well as upper tail estimates $\rho_{\min}(L_\beta)$ in Lemma 2.5. The details we omit here.

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