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STARSHAPEDNESS IN THE OBSTACLE PROBLEM

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We establish the starshapedness (with respect to the origin) of coincidence set in the obstacle problem for second order elliptic equations.

Key words : Obstacle problem; coincidence set; starshapedness.

1. INTRODUCTION

In this paper we consider the obstacle problem for second order elliptic equations associated with the operator

$$Au = -\text{div}\, a(\nabla u) \quad \text{in } \mathcal{D}^{'}(\Omega),$$

where Ω is an open bounded domain in $\mathbb{R}^N(N \geq 2)$ and the function $a = a(\eta) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuously differentiable in $\eta \in \mathbb{R}^N \setminus \{0\}$. Given a function $\psi \in W^{1,p}(\Omega)(1 <p< \infty)$, we define

$$K_\psi = \{v \in W_0^{1,p}(\Omega);\ v \geq \psi,\ \text{a.e. in } \Omega\},$$

which is nonempty provided $\psi^+ \in W_0^{1,p}(\Omega)$.

A function u in K_ψ is a solution to the obstacle problem

$$Au = f \quad \text{in } \{u > \psi\} = \{x \in \Omega; u(x) > \psi(x)\}, \tag{1.1}$$

if

$$\int_\Omega a(\nabla u)\nabla(v-u)\mathrm{d}x \geq \int_\Omega f(v-u)\mathrm{d}x, \quad \forall\, v \in K_\psi,$$

where f is a given function in some $L^q(\Omega)$.

Let $I(\psi)$ be the coincidence set defined by

$$I(\psi) = \{x \in \Omega; u(x) = \psi(x)\}.$$

According to the known results (see [1-8] for instance), any bounded solution u to (1.1) is $C^{1,\tau}(\Omega)$ for some $\tau \in (0,1)$ when $q > N$. Moreover,

$$Au - (A\psi - f)\chi_{I(\psi)} = f \ \text{ a.e. in } \Omega.$$

But there is only little information regarding the coincidence set $I(\psi)$ or the free boundary $\partial I(\psi)$. For $N = 2$, under the hypotheses of convexity of Ω and analyticity and strong concavity of ψ, it was shown in [9] and [10] that $\partial I(\psi)$ is a regular analytic Jordan curve (see also [11]). For $N > 2$, it is not known whether or not the same hypotheses imply the same conclusion. In 1984, Sakaguchi considered the obstacle problem for the harmonic operator (see [12]). Using an idea of Caffarelli and Spruck [13], the author showed that the coincidence set is star-shaped with respect to the origin, and that $\partial I(\psi)$ is a regular analytic hypersurface under certain conditions on the obstacle. Later then, using an idea of Lewis [14], Sakaguchi proved that the solution to the obstacle problem is real analytic in the noncoincidence set. Proceeding as in the case of the harmonic operator, the author obtained the starshapedness of the coincidence set for the $p-$harmonic operator with $p > 1$ (see [15]).

We should note that it is important to assume Ω is convex and ψ is concave to establish the starshapedness of the coincidence set (see [11, 12, 15]). Moreover, we should note that in the earlier year, starshapedness of level sets of the solution to the obstacle problem with $p = 2$ was proved by Kawohl [16].

Thanks to the $C^{1,\beta}-$regularity in the obstacle problem for $p-$Laplacian type equations with $p > 1$

$$-\text{div}\, a(x, \nabla u) = f \quad \text{in } \{u > \psi\},$$

obtained by Rodrigues recently [1], this paper will focus on the starshapedness in the obstacle problem (1.1) by using a similar technique to [12,15]. The result obtained in this paper is naturally an extension of $p-$harmonic obstacle problem.

We use the standard structural assumptions on the operator A(see [1,17,18]), namely

$$a^i(0) = 0, \tag{1.2}$$

$$\sum_{i,j=1}^{N} \frac{\partial a^i}{\partial \eta_j}(\eta)\xi_i\xi_j \geq \gamma_0|\eta|^{p-2}|\xi|^2, \tag{1.3}$$

$$|\frac{\partial a^i}{\partial \eta_j}(\eta)| \leq \gamma_1|\eta|^{p-2}, \tag{1.4}$$

for some positive constants $\gamma_0, \gamma_1 > 0$, all $\eta \in \mathbb{R}^N\backslash\{0\}$, and all $\xi \in \mathbb{R}^N, i, j = 1, ..., N$.

Under the assumptions on the operator A, one may get the following weak comparison principle for general elliptic equations (see [19]).

Proposition 1.1 (Weak Comparison Principle) — Suppose A satisfies the structural conditions (1.2)-(1.4). Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfy

$$-\text{div}\, a(\nabla u) \leq -\text{div}\, a(\nabla v) \quad \text{in } \Omega.$$

If $\Omega' \subseteq \Omega$ is open and $u \le v$ on $\partial\Omega'$, then $u \le v$ in Ω'.

For the existence of a solution to (1.1) with Hölder continuous gradient, we assume that

$$f, A\psi \in L^\infty(\Omega),\ \partial\Omega \in C^{1,\alpha} \text{ for some } \alpha \in (0,1).$$

Remark 1.2 : According to [1], there exists a unique solution u to (1.1). Moreover, $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

In order to obtain starshapedness of coincidence set, we need to make more restrictive assumptions on A, i.e.

(A_1) $a(\eta)$ is C^2-continuous in $\eta \in \mathbb{R}^N \setminus \{0\}$.

(A_2) For any $\eta \in \mathbb{R}^N$ satisfying $|\eta| \le M_0$, there exists a constant $C_0 = C_0(M_0)$ such that $\sum_m^N a^i_{\eta_j\eta_m}(\eta)\eta_m = C_0 a^i_{\eta_j}(\eta)$ holds for all $i, j = 1, ..., N$.

Remark 1.3 : One may verify easily that the $p-$Laplace operator satisfies (A_1) and (A_2).

2. Starshapedness of Coincidence Set

In this paper, as the previous work done by Sakaguchi, we assume Ω is a convex domain in $\mathbb{R}^N$ with the origin $0 \in \Omega$ and $f \equiv 0$ (we state Remark 2.6 for $f \neq 0$ in the end of this paper). Let $h \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$ be a nonnegative convex function which is positive on $\partial\Omega$ and homogeneous of degree $s > 1$ in Ω. Give the certain obstacle $\psi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$, which is negative on $\partial\Omega$, defined by

$$\psi(x) = -h(x) + c, \tag{2.1}$$

where $c > 0$ is a positive constant.

Under the assumptions on the operator A((1.2)-(1.4), (A_1) and (A_2)), let u be the solution to (1.1). The main result in this paper is as follows.

***Theorem* 2.1** — *The coincidence set* $I(\psi)$ *is starshaped with respect to the origin* 0.

The proof of Theorem 2.1 will be given later. Firstly, we claim

Proposition 2.2 — There exists a number $r > 0$ such that $B_r(0) \subset I(\psi)$.

PROOF : Let I_1 be the set of points $y \in \Omega$ for which the tangent plane of the graph $(\cdot, \psi(\cdot))$ at $(y, \psi(y))$,

$$\Pi_y : x_{N+1} = W_y(x) = \nabla\psi(y) \cdot (x - y) + \psi(y),$$

does not meet $\Omega \times \{0\}$. Since h is homogeneous of degree s, so $h(0) = 0, \max_\Omega \psi = \psi(0) = c > 0$, thus $0 \in I_1$. Moreover, since $\psi \in C^1(\overline{\Omega})$, I_1 contains a neighborhood of 0. Now for any $y \in I_1$, we claim $u(x) \leq W_y(x)$ in Ω.

Indeed,

$$\begin{aligned} W_y \geq 0 =& u \text{ on } \partial\Omega, \\ W_y \geq \psi =& u \text{ in } I(\psi). \end{aligned} \tag{2.2}$$

Particularly, due to the closedness of $I(\psi)$, we have

$$W_y \geq \psi = u \text{ on } \partial I(\psi).$$

On the other hand, it is easy to see

$$AW_y = -\text{div } a(\nabla W_y(x)) = -\text{div } a(\nabla\psi(y)) = 0 = Au \text{ in } \Omega \setminus I(\psi).$$

We deduce from Weak Comparison Principle (Proposition 1.1) that $u(x) \leq W_y(x)$ in $\Omega \setminus I(\psi)$. Furthermore, we get $u(x) \leq W_y(x)$ in Ω by (2.2).

Now note that $\psi(y) \leq u(y) \leq W_y(y) = \psi(y)$. Thus $y \in I(\psi)$. It follows that $I_1 \subset I(\psi)$. This completes the proof.

Now, basing on Remark 1.2, we introduce the function $v \in C^0(\overline{\Omega})$ defined by

$$v(x) = x \cdot \nabla(u - \psi)(x) - s(u - \psi)(x). \tag{2.3}$$

It follows from the homogeneity of degree of $h(x)$ that

$$x \cdot \nabla h(x) = sh(x) \quad \text{in } \Omega, \tag{2.4}$$

which and (2.1) and (2.3) imply

$$v(x) = x \cdot \nabla u - su(x) + sc \quad \text{in } \Omega. \tag{2.5}$$

For v defined by (2.3), we claim

Proposition 2.3 — $v > 0$ on $\partial\Omega$.

PROOF : Fix any point $x^0 \in \partial\Omega$. By convexity of Ω and Proposition 2.2, one may find a plane Π_1 through the tangent to $\partial\Omega$ at x^0 which is tangent to the graph $(\cdot, \psi(\cdot))$ at some point. Also, through Π_1 one may find anther plane Π_2 which is tangent to the graph $(\cdot, \psi(\cdot))$ at some point $(z, \psi(z)) \in \Omega \times \mathbb{R}$ such that

$$z \in I_1, \ \psi(z) > 0,$$

and

$$W_z(x^0) = \nabla\psi(z) \cdot (x^0 - z) + \psi(z) = 0.$$

Note that $W_z \geq 0 = u$ on $\partial\Omega$, $W_z \geq \psi = u$ in $I(\psi)$, $I(\psi)$ is closed, and

$$AW_z = 0 = Au \quad \text{in } \Omega \setminus I(\psi).$$

It follows from Weak Comparison Principle that

$$W_z \geq u \quad \text{in } \Omega \setminus I(\psi).$$

Since $W_z(x^0) = 0 = u(x^0)$ and x^0 is regarded as an outward directed vector from Ω at $x^0 \in \partial\Omega$, we have

$$x^0 \cdot \nabla(u(x^0) - 0) \geq x^0 \cdot \nabla(W_z(x^0) - 0).$$

Therefore

$$x^0 \cdot \nabla u(x^0) \geq x^0 \cdot \nabla W_z(x^0) = x^0 \cdot \nabla \psi(z) = -\psi(z) + z \cdot \nabla \psi(z). \qquad (2.6)$$

By (2.1)(2.4) and (2.6), we deduce

$$x^0 \cdot \nabla u(x^0) + sc \geq (s-1)\psi(z),$$

which implies

$$\begin{aligned} v(x^0) &= x^0 \cdot \nabla u(x^0) - su(x^0) + sc \\ &\geq x^0 \cdot \nabla u(x^0) + sc \\ &\geq (s-1)\psi(z) > 0. \end{aligned}$$

This completes the proof of Proposition 2.3.

Now we prove

Lemma 2.4 — $v \geq 0$ in Ω.

PROOF : We use analogous technique as [15] to prove this lemma. Since $Au = 0$ in $\Omega \setminus I(\psi)$, we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial a^i(\nabla u)}{\partial \eta_j} \frac{\partial u_{x_j}}{\partial x_i} = 0 \quad \text{in } \Omega \setminus I(\psi). \qquad (2.7)$$

Applying the differential operator $x \cdot \nabla$ to (2.7) and using (2.5), we get

$$\begin{aligned} \sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u) v_{x_i x_j} &+ \sum_{j=1}^{N}(\sum_{i,m=1}^{N} a^i_{\eta_j \eta_m}(\nabla u) u_{x_m x_i}) v_{x_j} \\ &+ (s-2)\sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u) u_{x_j x_i} \\ &+ (s-1)\sum_{i,j,m=1}^{N} a^i_{\eta_j \eta_m}(\nabla u) u_{x_j x_i} u_{x_m} = 0 \quad \text{in } \Omega \setminus I(\psi). \end{aligned}$$

On the other hand, by (2.7) and A_2, it follows

$$\begin{aligned}(s-1)\sum_{i,j,m=1}^{N} a^i_{\eta_j\eta_m}(\nabla u)u_{x_jx_i}u_{x_m} &= C_0(s-1)\sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u)u_{x_jx_i} \\ &= 0 \quad \text{in } \Omega\setminus I(\psi). \qquad (2.8)\end{aligned}$$

Therefore,

$$\begin{aligned}L(v) &= \text{div}\,[(\nabla v)(a^i_{\eta_j}(\nabla u))^T_{N\times N}] = \sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u)v_{x_ix_j} \\ &\quad + \sum_{j=1}^{N}(\sum_{i,m=1}^{N} a^i_{\eta_j\eta_m}u_{x_mx_i})v_{x_j} \\ &= 0 = L(0) \quad \text{in } \Omega\setminus I(\psi), \qquad (2.9)\end{aligned}$$

where $a^i_{\eta_j}(\nabla u) = a^i_{\eta_j}(\nabla u)(\nabla u),\ (a^i_{\eta_j}(\nabla u))^T_{N\times N}$ is the $N\times N$ matrix

$$(a^i_{\eta_j}(\nabla u))^T_{N\times N} = \begin{pmatrix} a^1_{\eta_1}(\nabla u) & \dots & a^N_{u_{\eta_1}}(\nabla u) \\ a^1_{\eta_2}(\nabla u) & \dots & a^N_{u_{\eta_2}}(\nabla u) \\ \vdots & & \vdots \\ a^1_{\eta_N}(\nabla u) & \dots & a^N_{\eta_N}(\nabla u) \end{pmatrix}_{N\times N}.$$

By Proposition 2.3 and the fact $v = 0$ on $\partial I(\psi)$, we have $v \geq 0$ on $\partial(\Omega\setminus I(\psi))$. We deduce from comparison principle for linear operator that $v \geq 0$ in Ω.

Proof of Theorem 2.1: By Lemma 2.4, we have $v \geq 0$ in $\Omega \setminus I(\psi)$, which implies

$$x\cdot\nabla(u-\psi) \geq s(u-\psi) > 0 \ \text{ in } \Omega\setminus I(\psi). \qquad (2.10)$$

By the definition of $I(\psi)$, we deduce that the coincidence set $I(\psi)$ is star shaped with respect to the origin. Indeed, if this is not true, then there exist a unit vector $\xi \in \mathbb{R}^N$ and two positive constants t_1, t_2 with $t_1 < t_2$ such that $t_1\xi \in I(\psi), t_2\xi \in I(\psi)$ and $t\xi \in \Omega\setminus I(\psi)$ for all $t \in (t_1, t_2)$. Since $(u-\psi)(t_i\xi) =$

$0 (i = 1, 2)$, we get by the mean value theorem that $t_0\xi \cdot \nabla(u - \psi)(t_0\xi) = 0$ for some $t_0 \in (t_1, t_2)$, which is a contradiction to (2.10).

Remark 2.5 : It is obvious that if $s \geq 2$ and $f(x) \equiv C$ is a nonpositive constant, then Theorem 2.1 holds.

Remark 2.6 : Suppose $g \in C^1(\overline{\Omega})$ is homogeneous of degree $t \geq 0$ in Ω, and $\widetilde{c}$ is a nonnegative constant. Let f be nonpositive in Ω and given by $f = -g - \widetilde{c}$. If $C_0 \geq 0, s \geq 2$ and $t \geq 0$, or $C_0 \geq 0, s > 1$ and $t \geq \max\{2 - s, 0\}$, then Theorem 2.1 holds. Indeed, since f is nonpositive, one can apply comparison principle in Proposition 2.3 and Theorem 2.2 as well. Now we need to show the process of Lemma 2.4 is valid. It suffices to note that (2.8) becomes

$$\begin{aligned}(s-1)\sum_{i,j,m=1}^{N} a^i_{\eta_j\eta_m}(\nabla u)u_{x_jx_i}u_{x_m} &= C_0(s-1)\sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u)u_{x_jx_i} \\ &= -C_0(s-1)f \quad \text{in } \Omega \setminus I(\psi).\end{aligned}$$

and (2.9) becomes

$$\begin{aligned}L(v) &= \text{div}\,[(\nabla v)(a^i_{\eta_j}(\nabla u))^T_{N\times N}] = \sum_{i,j=1}^{N} a^i_{\eta_j}(\nabla u)v_{x_ix_j} \\ &\quad + \sum_{j=1}^{N}\left(\sum_{i,m=1}^{N} a^i_{\eta_j\eta_m}u_{x_mx_i}\right)v_{x_j} \\ &= f[t + C_0(s-1) + s - 2] - t\widetilde{c} \leq 0 = L(0) \quad \text{in } \Omega \setminus I(\psi).\end{aligned}$$

Remark 2.7 : If the obstacle function ψ is cone-like (“cone” with smooth vertex), i.e., h is given as below

(H_1) $h(kx) = kh(x)$ in $\overline{\Omega} \setminus B_{\epsilon_0}(0),\ \forall\, k \in \mathbb{R}$ s.t. $kx \in \Omega \setminus B_{\epsilon_0}(0)$,

where ϵ_0 is a positive constant and small enough.

$$(H_2)\ \ h_0 = \sup_{B_{\epsilon_0}(0)} h < c.$$

Then one can obtain the desired result as well.

Indeed one may prove the following proposition.

Proposition 2.2 — There exists a number $r > 0$ such that $B_{\epsilon_0}(0) \subset B_r(0) \subset I(\psi)$.

PROOF : Let I_1 be the set of points $y \in \Omega$ for which the tangent plane of the graph $(\cdot, \psi(\cdot))$ at $(y, \psi(y))$,

$$\Pi_y : x_{N+1} = W_y(x) = \nabla\psi(y) \cdot (x - y) + \psi(y),$$

does not meet $\Omega \times \{0\}$. Since h satisfies (H_2), by the definition of ψ, it follows $\inf_{B_{\epsilon_0}} \psi > 0$. Thus $B_{\epsilon_0} \subset I_1$. Moreover, since $\psi \in C^1(\overline{\Omega})$, I_1 contains a neighborhood of B_{ϵ_0}. As Proposition 2.2, one may get the desired result.

Now define $v = x \cdot \nabla(u - \psi) - (u - \psi)$ in $\overline{\Omega}$. By (2.9) we have $x \cdot \nabla h(x) = h(x)$ in $\overline{\Omega} \setminus I(\psi)$ and $v = x \cdot u - u + c$ in $\overline{\Omega} \setminus I(\psi)$. Due to Proposition 2.2', one may use the same method as before to prove $v \geq 0$ on $\partial\Omega$ and $v \geq 0$ in $\overline{\Omega} \setminus I(\psi)$ by replacing $s > 1$ with $s = 1$ in $\overline{\Omega} \setminus I(\psi)$. Then one may obtain the starshapedness of the coincidence set.

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