Indian J. Pure Appl. Math., **44**(4): 467-472, August 2013

ON SIZE, ORDER, DIAMETER AND MINIMUM DEGREE [1]

Simon Mukwembi

School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Westville Campus, South Africa
e-mail: mukwembi@ukzn.ac.za

(*Received* 15 *April* 2012; *after final revision* 26 *July* 2012;
accepted 8 *November* 2012)

Let G be a finite connected graph. We give an asymptotically tight upper bound on the size of G in terms of order, diameter and minimum degree. Our result is a strengthening of an old classical theorem of Ore [Diameters in graphs, *J. Combin. Theory*, **5** (1968), 75-81] if minimum degree is prescribed and constant.

Key words : Size; diameter; minimum degree.

1. INTRODUCTION

Let $G = (V, E)$ be a finite, connected, simple graph. We denote the order of G by n and the size by m. The degree of a vertex v in G is denoted by $\deg_G(v)$, and the minimum degree of G by δ. For two vertices u, v in G, $d_G(u, v)$ denotes the

[1]This material is based upon work supported financially by the National Research Foundation.

usual distance between u and v in G, i.e., the minimum number of edges on a path from u to v and the *diameter* of G is the number $d = \max\{d_G(u,v) : u, v \in V\}$. The diameter, apart from being an interesting graph-theoretical parameter, plays a significant role in analyzing communication networks (see, for example [2]). In such networks, the time delay or signal disgradation for sending a message from one point to another is often proportional to the distance between the two points. The diameter can be used to indicate the worst-case performance.

Several bounds on the size of a graph in terms of other graph parameters, for example, order and radius [8, 4], order and degree set [7], and order and domination [3] have been investigated. An upper bound on the size in terms of order and diameter was determined by Ore [5] as early as 1968. Recently, several authors, for example [6, 1], have presented simple and short proofs to Ore's theorem. In this work, we present an upper bound on the size in terms of order, diameter and minimum degree. Our bound, for a fixed minimum degree, is a strengthening of Ore's theorem [5] which we state below.

***Theorem* 1** [5] — *Let G be a connected graph of order n, diameter d and size m. Then*

$$m \leq \frac{1}{2}(n-d-1)(n-d+4)+d.$$

We will make use of the well-known handshaking lemma.

Lemma 1 — Let G be a graph of order n and size m. Then $2m = \sum_{x \in V} \deg_G(x)$.

2. Results

The following result is a strengthening of Ore's theorem if minimum degree is prescribed and constant.

***Theorem* 2** — *Let G be a connected graph of order n, diameter d, minimum degree $\delta \geq 2$ and size m. Then*

$$m \leq \frac{1}{2}\left[n - \frac{1}{3}d(\delta+1)\right]^2 + (2\delta+1)\left(n - \frac{1}{6}d(\delta+2)\right),$$

and the bound, for fixed δ, is asymptotically tight.

PROOF : Let $P = v_0, v_1, \ldots, v_d$ be a diametral path of G. Let $P_1 \subseteq V(P)$ be the set

$$P_1 := \left\{ v_{3i+1} \mid i = 0, 1, \ldots, \left\lfloor \frac{d-1}{3} \right\rfloor \right\}.$$

For each vertex $v \in P_1$, choose any δ neighbours $u_1, u_2, \ldots, u_\delta$ of v and denote the set $\{v, u_1, u_2, \ldots, u_\delta\}$ by $M[v]$ and set $P' := \cup_{v \in P_1} M[v]$. Then

$$|P'| = (\delta + 1)\left(\left\lfloor \frac{d-1}{3} \right\rfloor + 1\right). \tag{1}$$

Claim 1 : $\sum_{x \in V(P')} \deg_G(x) \leq 2(\delta + 1)n - (3\delta + 2)\left(\left\lfloor \frac{d-1}{3} \right\rfloor + 1\right)$.

PROOF OF CLAIM 1 : Partition P_1 as $P_1 = P_2 \cup P_3$, so that for each $u, v \in P_i$, $i = 2, 3$, $d_G(u, v) \geq 6$. Precisely, let $P_2 \subset P_1$ be the set $P_2 = \{v_j \in P_1 \mid j \equiv 1 \pmod 6\}$, and $P_3 = P_1 - P_2$. Write the elements of P_2 as $P_2 = \{w_1, w_2, \ldots, w_{|P_2|}\}$. For each $w_j \in P_2$, let $M[w_j] = \{w_j, u_1^j, u_2^j, \ldots, u_\delta^j\}$, where $u_1^j, u_2^j, \ldots, u_\delta^j$ are neighbours of w_j. The fact that for each $u, v \in P_2$, $d_G(u, v) \geq 6$, yields

$$\begin{aligned} n \quad \geq \quad & (\deg_G(w_1) + 1) + (\deg_G(w_2) + 1) + \cdots + (\deg_G(w_{|P_2|}) + 1) \\ & + |P_2| + \sum_{v \in P_3} (\deg_G(v) + 1), \end{aligned}$$

and for $t = 1, 2, \ldots, \delta$,

$$n \geq (\deg_G(u_t^1) + 1) + (\deg_G(u_t^2) + 1) + \cdots + (\deg_G(u_t^{|P_2|}) + 1) + |P_3|.$$

Summing, we get

$$(\delta + 1)n \geq \sum_{v \in (\cup_{y \in P_2} M[y])} \deg_G(v) + (\delta + 1)|P_2| + (2\delta + 1)|P_3|. \tag{2}$$

Similarly,

$$(\delta + 1)n \geq \sum_{v \in (\cup_{y \in P_3} M[y])} \deg_G(v) + (\delta + 1)|P_3| + (2\delta + 1)|P_2|.$$

Combining this with (2), we get

$$\begin{aligned}\sum_{x\in V(P')} \deg_G(x) &= \sum_{v\in(\cup_{y\in P_2} M[y])} \deg_G(v) + \sum_{v\in(\cup_{y\in P_3} M[y])} \deg_G(v)\\ &\leq 2(\delta+1)n-(\delta+1)(|P_2|+|P_3|)-(2\delta+1)(|P_2|+|P_3|)\\ &= 2(\delta+1)n-(3\delta+2)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right),\end{aligned}$$

and the claim is proven.

Now let $Q := V - P'$. Then from (1),

$$|Q| = n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right). \tag{3}$$

Claim 2 : Let $v \in Q$. Then $\deg_G(v) \leq n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)+2\delta$.

PROOF OF CLAIM 2 : Assume that $v \in Q$. Since P is a shortest path, v can only be adjacent to at most $2\delta+1$ vertices in P'. Hence

$$\deg_G(v) \leq n-|P'|-|\{v\}|+2\delta+1 = n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)-1+2\delta+1$$

and so the claim is proven.

By Claim 2, and from (3), we have

$$\begin{aligned}\sum_{x\in Q} \deg_G(x) &\leq |Q|\left(n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)+2\delta\right)\\ &= \left[n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)\right]\\ &\quad\left[n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)+2\delta\right].\end{aligned}$$

Combining this with Claim 1, we get

$$\begin{aligned}\sum_{x\in V}\deg_G(x) &= \sum_{x\in V(P')}\deg_G(x)+\sum_{x\in Q}\deg_G(x)\\ &\le 2(\delta+1)n-(3\delta+2)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)\\ &\quad+\left[n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)\right]\\ &\quad\left[n-(\delta+1)\left(\left\lfloor\frac{d-1}{3}\right\rfloor+1\right)+2\delta\right].\end{aligned}$$

Using the fact that $\left\lfloor\frac{d-1}{3}\right\rfloor+1\ge\frac{d}{3}$, we get

$$\begin{aligned}\sum_{x\in V}\deg_G(x) &\le 2(\delta+1)n-(3\delta+2)\frac{d}{3}\\ &\quad+\left[n-\frac{d}{3}(\delta+1)\right]\left[n-\frac{d}{3}(\delta+1)+2\delta\right]\\ &= \left[n-\frac{1}{3}d(\delta+1)\right]^2+(2\delta+1)\left(2n-\frac{1}{3}d(\delta+2)\right).\end{aligned}$$

An application of Lemma 1 yields

$$\begin{aligned}m &\le \frac{1}{2}\left(\left[n-\frac{1}{3}d(\delta+1)\right]^2+(2\delta+1)\left(2n-\frac{1}{3}d(\delta+2)\right)\right)\\ &= \frac{1}{2}\left[n-\frac{1}{3}d(\delta+1)\right]^2+(2\delta+1)\left(n-\frac{1}{6}d(\delta+2)\right),\end{aligned}$$

as desired.

To see that the bound is asymptotically tight, consider the graph G with diameter d, $d\equiv 2 \bmod 3$, constructed as follows: $V(G)=V_0\cup V_1\cup\cdots\cup V_d$, where

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 2 \bmod 3, \\ n - \frac{1}{3}d(\delta+1) + \frac{2}{3}\delta - \frac{7}{3} & \text{if } i = 1, \\ \delta & \text{if } i = d-1, \\ \delta - 1 & \text{otherwise} \end{cases}$$

and two distinct vertices $v \in V_i, v' \in V_j$ are joined by an edge if and only if $|j - i| \leq 1$. Then

$$m(G) > \binom{n - \frac{1}{3}d(\delta+1) + \frac{2}{3}\delta - \frac{7}{3}}{2}. \qquad \square$$

References

1. D. Auger, I. Charon, O. Hudry and A. Lobstein, On the sizes of graphs and their powers: The undirected case, *Discrete Appl. Math.*, **159** (2011), 1666-1675.

2. F. R. K. Chung, The average distance and the independence number. *J. Graph Theory*, **12** (1988), 229-235.

3. P. Dankelmann, G. S. Domke, W. Goddard, P. Grobler, J. H. Hattingh and H. C. Swart, Maximum sizes of graphs with given domination parameters, *Discrete Math.*, **281** (2004), 137-148.

4. P. Dankelmann and L. Volkmann, Minimum size of a graph or digraph of given radius, *Inform. Process. Lett.*, **109** (2009), 971-973.

5. O. Ore, Diameters in graphs, *J. Combin. Theory*, **5** (1968), 7581.

6. Z. Tao, X. Junming and L. Jun, On diameter and average distance of graphs, *OR Transactions*, **8** (2004), 1-6.

7. A. Tripathi and S. Vijay, On the least size of a graph with a given degree set, *Discrete Appl. Math.*, **154** (2006), 25302536.

8. V. Vizing, The number of edges in a graph of given radius, *Soviet Math. Dokl.* **8** (1967), 535536.