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# SOLITARY SOLUTION OF MODIFIED BAD AND GOOD BOUSSINESQ EQUATION BY USING OF TANH AND EXTENDED TANH METHODS

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In this work, a mathematical approach based on the reduction of order for solving ordinary differential equations, the standard tanh method and the extended tanh method has been used to obtain solutions of modified bad Boussinesq and modified good Boussinesq equations.

**Key words** : Tanh method; solitary solution; modified Boussinesq equation.

## 1. INTRODUCTION

The nonlinear phenomena are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, nonlinear optics and etc [1]. A variety of powerful method such as the inverse scattering [2], the Backlund transformation [3,4], sine-cos method[5], tanh-sech method [1,6], the extended tanh method [7], Hirota's bilinear technique[12] and the homogeneous balance method[13] were used to solve nonlinear equations. The bad and good Boussinesq (Bq) equations are as follows:

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0, \tag{1.1}$$

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} = 0. \tag{1.2}$$

Where the French scientist Josseph Boussinesq (1842-1929) described in the 1870's model equation for the propagation of long waves on the surface of water with a small amplitude [8,20]. These equations has been solved using the VIM, HPM, ADM, Exp-function method, F-expansion method [14-17], the standard tanh method and the extended tanh method [8,19]. In this paper solitary solution of theirs special modified states will be obtained, by the standard tanh, the extended tanh and mathematical analysis method. By adding a nonlinear term of the form $3(u^2 u_x)_x$ to Boussinesq equations, the new modified theirs form is obtained as follows: bad modified Bq. equation:

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} + 3(u^2 u_x)_x = 0, \tag{1.3}$$

good modified Bq equation:

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} + 3(u^2 u_x)_x = 0, \tag{1.4}$$

## 2. THE STANDARD TANH AND EXTENDED TANH METHOD

The tanh method is a reliable and accurate algebraic method to obtain exact solution of nonlinear equations. All derivatives of it are represented by a tanh itself. First, a power series in tanh was used as an ansatz to obtain analytical solutions of travelling wave. To avoid complexity, Malfliet [10, 11] had customized the tanh technique by introducing tanh as a new variable. Various forms of the generalized tanh method have been developed for obtaining multi-traveling wave solution. Wazwaz also used this method for several forms of nonlinear partial differential equation[1, 6, 19]. A partial differential equation (PDE) can be converted to an ordinary differential equation (ODE) upon using a wave variable:

$$z = x - ct \tag{2.1}$$

Introducing a new independent variable $y = \tanh \mu z$ that leads to change of derivatives [7]:

$$\frac{d}{dz} = (1 - y^2)\frac{d}{dy}$$

$$\frac{d^2}{dz^2} = (1-y^2)[-2y\frac{d}{dy} + (1-y^2)\frac{d^2}{dy^2}]$$

$$\frac{d^3}{dz^3} = (1-y^2)[(6y^2-2)\frac{d}{dy} - 6y(1-y^2)\frac{d^2}{dy^2} + (1-y^2)^2\frac{d^3}{dy^3}] \qquad (2.2)$$

The solution for the standard tanh method can be proposed as a finite power series in y in the form

$$u(\mu z) = S(y) = \sum_{(k=0)}^{m} a_k y^k \qquad (2.3)$$

The extended tanh method admits the use of the finite expansion

$$u(\mu z) = S(y) = \sum_{(k=0)}^{m} a_k y^k + \sum_{(k=1)}^{m} b_k y^k \qquad (2.4)$$

Where the parameter *m* in most cases, is a positive integer that will be determined. To determine the parameter *m* we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With m determined, collect all coefficients of power of $y$ in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k$,$(k = 0, , m), \mu$ and $c$ for the standard tanh method and the parameters $a_k$, $(k = 0, ...., m), b_k, (k = 1, ...., m), \mu$ and $c$ for the extended tanh method [18].

## 3. The Mathematical Analysis Method

This method is a mathematical approach based on the reduction of order for solving differential equations [9].

We first use the independent variable $z = x - ct$ to convert PDE to ODE, and using transformation

$$\frac{du}{dz} = p \qquad (3.1)$$

So that:

$$\frac{d^2u}{dz^2} = p\frac{dp}{du} \qquad (3.2)$$

Substituting Eq. (5) in Eq. (1) we have a first order ODE[9].

## 4. Using the Standard Tanh and Extended Tanh Methods

The new modified bad Bq equation given by:

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0 \tag{4.1}$$

Substituting Eq. (5) into Eq. (11) gives:

$$-u'''' + (c^2 - 1)u'' - 3(u^2)'' + 3(u^2u')' = 0 \tag{4.2}$$

Twice integrating (12), setting the constant of integrating to zero, we obtain

$$-u'' + (c^2 - 1)u - 3u^2 + u^3 = 0 \tag{4.3}$$

Balancing $u^3$ with $u''$ gives

$$m + 2 = 3m, \tag{4.4}$$

$$m = 1 \tag{4.5}$$

the standard tanh method assumes that finite expansion

$$u(\mu z) = a_0 + a_1 y \tag{4.6}$$

substituting (16) into (13) and using the set of equations (6), collecting the coefficients of $y^j, 0 \leq j \leq 3$, and equating this coefficients to zero, we find the system of algebraic equations for $a_0, a_1, c$ and $\mu$:

$$\begin{gathered} y^0 : (c^2 - 1)a_0 - 3a_0^2 + a_0^3 = 0 \\ y^1 : (c^2 - 1)a_1 + 2a_1\mu^2 - 6a_0a_1 + 3a_0^2a_1 = 0 \\ y^2 : -3{a_1}^2 + 3a_0{a_1}^2 = 0 \\ y^3 : -2\mu^2a_1 + {a_1}^3 = 0 \end{gathered} \tag{4.7}$$

Solving the system of equations (17), we find the following set of solution:

$$a_0 = 1$$
$$a_1 = \pm\sqrt{(4-c^2)}$$
$$\mu = \sqrt{\frac{4-c^2}{2}} \tag{4.8}$$

and $c$ is any real number.

Substituting (18) into (16), for $4 - c^2 > 0$ we obtain the kink and anti kink solution.

$$u = 1 + \sqrt{4-c^2}\tanh[\sqrt{\frac{4-c^2}{2}}(x-ct)] \tag{4.9}$$

$$u = 1 - \sqrt{4-c^2}\tanh[\sqrt{\frac{4-c^2}{2}}(x-ct)] \tag{4.10}$$

The plot of equation (19) with $c = \sqrt{3}$ and $t = 0$ is shown in figure1, and as can be seen it represent a kink.

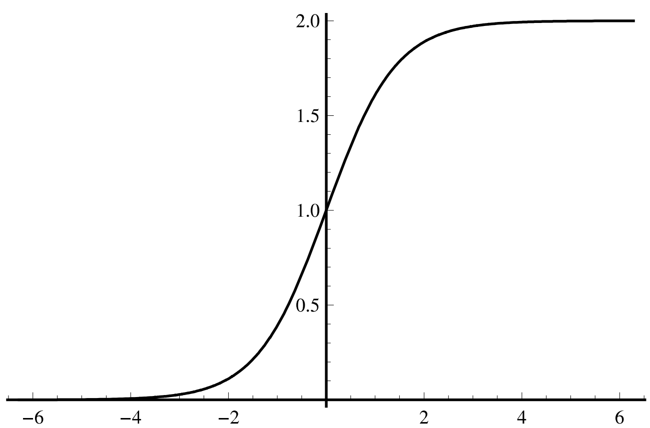


Figure 1: Two dimensional plot of Eq. (19) for kink with $c = \sqrt{3}$ and t=0

For $4 - c^2 > 0$ we obtain the kink and anti kink solution. For $4 - c^2 < 0$ we find the complex solution:

$$u = 1 + i\sqrt{4-c^2}\tan[\sqrt{\frac{4-c^2}{2}}(x-ct)] \tag{4.11}$$

$$u = 1 + i\sqrt{4-c^2}\tan[\sqrt{\frac{4-c^2}{2}}(x-ct)] \tag{4.12}$$

The extended tanh method assumes that finite expansion

$$u(\mu z) = a_0 + a_1 y + b_1 y^{-1} \tag{4.13}$$

substituting (13) into (23), collecting the coefficient of $y^j, -3 \leq j \leq 3$ and equating these coefficients to zero and solving the system of algebraic for $a_0, a_1, c, b_1$ and $\mu$ we find the following set of solution:

$$\begin{aligned} a_1 &= \pm\frac{\sqrt{(4-c^2)}}{2} \\ b_1 &= \pm\frac{\sqrt{(4-c^2)}}{2} \\ a_0 &= 1 \\ \mu &= \frac{\sqrt{4-c^2}}{2\sqrt{2}} \end{aligned} \tag{4.14}$$

substituting (24) into (23), for $4 - c^2 > 0$ we find

$$\begin{aligned} u &= 1 + \frac{\sqrt{4-c^2}}{2}\tanh\left[\sqrt{\frac{4-c^2}{8}}(x-ct)\right] + \frac{\sqrt{4-c^2}}{2} \\ &\quad \tanh^{-1}\left[\sqrt{\frac{4-c^2)}{8}}(x-ct)\right] \end{aligned} \tag{4.15}$$

$$\begin{aligned} u &= 1 - \frac{\sqrt{4-c^2}}{2}\tanh\left[\sqrt{\frac{4-c^2}{8}}(x-ct)\right] - \frac{\sqrt{4-c^2}}{2} \\ &\quad \tanh^{-1}\left[\sqrt{\frac{4-c^2)}{8}}(x-ct)\right] \end{aligned} \tag{4.16}$$

The diagram of equation (25)is shown in figure 2 for $c = \sqrt{3}, t = 0$.

For $(4 - c^2) < 0$ we find complex solution

$$\begin{aligned} u &= 1 + i\frac{\sqrt{4-c^2}}{2}\tan\left[\sqrt{\frac{4-c^2}{8}}(x-ct)\right] + i\frac{\sqrt{4-c^2}}{2} \\ &\quad \tan^{-1}[\sqrt{\frac{(4-c^2)}{8}}(x-ct)] \end{aligned} \tag{4.17}$$

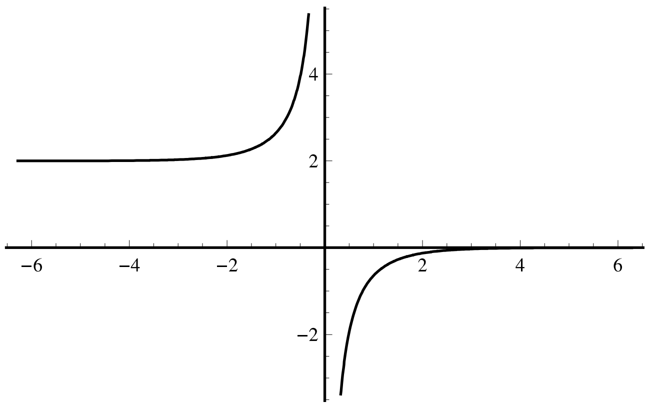


Figure 2: Two dimensional plot of Eq. (25) with $c = \sqrt{3}$ and t=0.

$$\begin{aligned} u &= 1 - i\frac{\sqrt{4-c^2}}{2}\tan\left[\sqrt{\frac{4-c^2}{8}}(x-ct)\right] - i\frac{\sqrt{4-c^2}}{2} \\ &\quad \tan^{-1}\left[\sqrt{\frac{(4-c^2)}{8}}(x-ct)\right] \end{aligned} \tag{4.18}$$

The modified good Bq. equation given by:

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0 \tag{4.19}$$

Using the standard tanh method gives

$$a_0 = 1$$

$$a_1 = \mp\sqrt{(4-c^2)}$$

$$\mu = i\sqrt{(4-c^2/2)} \tag{4.20}$$

For $(4-c^2) > 0$ we find periodic solution

$$u = 1 - \sqrt{4-c^2}\tan[\sqrt{\frac{(4-c^2)}{2}}(x-ct)] \tag{4.21}$$

$$u = 1 + \sqrt{4-c^2}\tan[\sqrt{\frac{(4-c^2)}{2}}(x-ct)] \tag{4.22}$$

The diagram of equation (31) is shown in figure 3.

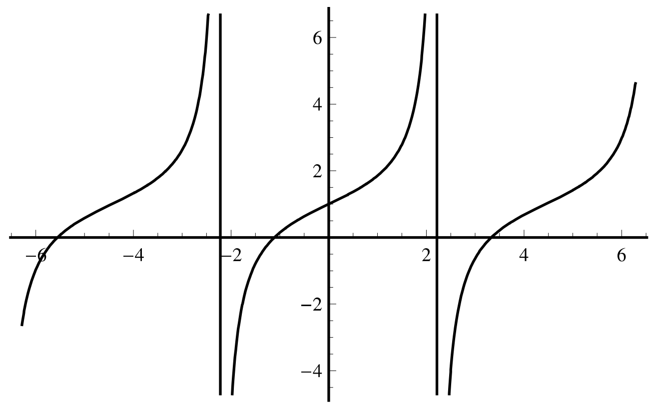


Figure 3: Two dimensional plot of Eq. (31) with $c = \sqrt{3}$ and t=0

For$(4 - c^2) < 0$ we find complex solution

$$u = 1 - i\sqrt{4 - c^2}\,\tanh[\sqrt{\frac{4 - c^2}{2}}(x - ct)] \tag{4.23}$$

$$u = 1 + i\sqrt{4 - c^2}\,\tanh[\sqrt{\frac{4 - c^2}{2}}(x - ct)] \tag{4.24}$$

Using the extended tanh method for modified good Bq. equation gives

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \mp\frac{\sqrt{4 - c^2}}{2} \\ b_1 &= \mp\frac{\sqrt{4 - c^2}}{2} \\ \mu &= i\sqrt{\frac{4 - c^2}{8}}) \end{aligned} \tag{4.25}$$

substituting (35) into (13) we obtain:

$$\begin{aligned} u \;\; = \;\; & 1 - \frac{\sqrt{4 - c^2}}{2}\tan\left[\frac{\sqrt{4 - c^2}}{8}(x - ct)\right] - \frac{\sqrt{4 - c^2}}{2} \\ & \tan^{-1}\left[\frac{\sqrt{4 - c^2}}{8}(x - ct)\right] \end{aligned} \tag{4.26}$$

$$u = 1 + \frac{\sqrt{4-c^2}}{2}\tan\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] + \frac{\sqrt{4-c^2}}{2} \tan^{-1}\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] \quad (4.27)$$

The diagram of equation (36) is shown in figure 4

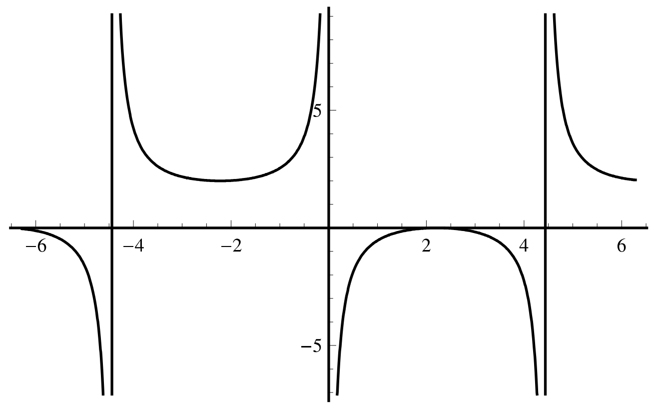


Figure 4: Two dimensional plot of Eq. (36) with $c = \sqrt{3}$ and t=0

For $(4-c^2) < 0$ we find complex solution

$$u = 1 - i\frac{\sqrt{4-c^2}}{2}\tanh\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] - i\frac{\sqrt{4-c^2}}{2} \tanh^{-1}\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] \quad (4.28)$$

$$u = 1 + i\frac{\sqrt{4-c^2}}{2}\tanh\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] + i\frac{\sqrt{4-c^2}}{2} \tanh^{-1}\left[\frac{\sqrt{4-c^2}}{8}(x-ct)\right] \quad (4.29)$$

## 5. Using the Mathematical Analysis Method

The modified bad Bq. equation given by:

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} + 3(u^2u_x)_x = 0 \tag{5.1}$$

Substituting (5) into (3) gives:

$$-u'''' + (c^2-1)u'' - 3(u^2)'' + 3(u^2u')' = 0 \tag{5.2}$$

Twice integrating (41), setting the constant of integrating to zero, we obtain

$$-u'' + (c^2-1)u - 3u^2 + u^3 = 0 \tag{5.3}$$

Substituting (10) into (42), we change Eq. (42) into first order ODE

$$-p\frac{dp}{du} - 3u^2 + (c^2-1)u + u^3 = 0 \tag{5.4}$$

Integrating (43) and letting the constant of integration to be zero, we have

$$\left(\frac{du}{dz}\right)^2 = -u^3 + \frac{(c^2-1)}{2}u^2 + \frac{1}{4}u^4 \tag{5.5}$$

so that

$$\frac{du}{u\sqrt{\frac{u^2}{4} - u + \frac{(c^2-1)}{2}}} = dz \tag{5.6}$$

If $c = \pm\sqrt{3}$, it follows from (45) that the solution of (40) takes the form

$$u = \frac{2}{1 - 2e^{(x-ct)}} \tag{5.7}$$

If $c \neq \pm\sqrt{3}, c^2-1 > 0$ and $c^2 - 3 > 0$ the solution of (40) have the form

$$u = (c^2-1)\left\{\frac{1}{1 + \sqrt{\frac{2}{(c^2-3)}}\sinh[-\sqrt{\frac{(c^2-1)}{2}}(x-ct)]}\right\} \tag{5.8}$$

The diagram of equation (47) for c=2 , t=0 is shown in figure 5: for $c^2 - 3 < 0$

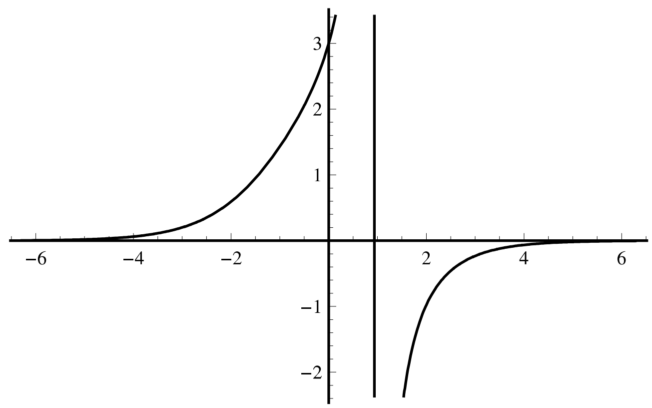


Figure 5: Two dimensional plot of Eq. (47) with c=2 and t=0

$$u = (c^2 - 1)\left\{\frac{1}{1 + i\sqrt{\frac{2}{(c^2-3)}}\sinh[-\sqrt{\frac{(c^2-1)}{2}}(x - ct)]}\right\} \tag{5.9}$$

for $c^2 - 1 < 0$, we have complex solution

$$u = -(c^2 - 1)\left\{\frac{1}{1 + i\sqrt{\frac{2}{(c^2-3)}}\sin[-\sqrt{\frac{(c^2-1)}{2}}(x - ct)]}\right\} \tag{5.10}$$

For the modified good Bq. equation, if $c^2 - 1 > 0$ and $c^2 - 3 > 0$ we obtain periodic solution

$$u = -(1 - c^2)\frac{1}{\sqrt{\frac{2}{c^2-3}}\sin[-\sqrt{\frac{1-c^2}{2}}z - 1]} \tag{5.11}$$

The diagram of equation (50) for $c = 2, t = 0$ is shown in figure 6 complex solution, for $c^2 - 3 < 0$

$$u = -(1 - c^2)\frac{1}{i\sqrt{\frac{2}{c^2-3}}\sin[-\sqrt{\frac{1-c^2}{2}}z - 1]} \tag{5.12}$$

for$c^2 - 1 < 0$

$$u = (1 - c^2)\frac{1}{i\sqrt{\frac{2}{c^2-3}}\sinh[-\sqrt{\frac{1-c^2}{2}}z - 1]} \tag{5.13}$$

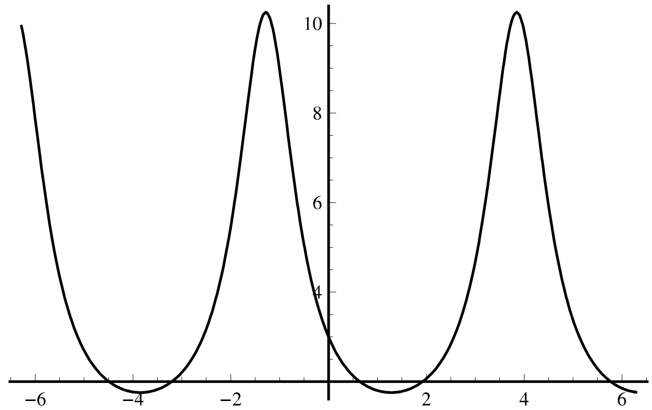


Figure 6: Two dimensional plot of Eq. (50) for compacton with c=2 and t=0

## CONCLUSIONS

The standard tanh, the extended tanh method and a mathematical method based on the reduction of order, are used to study of modified nonlinear bad and good Boussinesq equation. We find variant solution such as kink, anti kink, compacton and periodic solution for these equations in this paper. This work emphasized that tanh and extended tanh methods are powerful technique to solve nonlinear equations and give solitary solution.

## REFERENCES

1. T. A. Abassy, M. A. EL-Tawil and H. EL-Zoheiry, Modified variational iteration method for Boussinesqequation, *Comput Math Appl.*, **123** (2001), 205-209.

2. M. A. Abdou, A. A. Soliman and S. T. EL-Basyony, New applications of Exp-function method for improve Boussinesq equation, *Phys Lett A*, **369** (2007), 469-475.

3. M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, *London Mathematical Society Lecture Note Series*, **149**, Cambridge University Press, Cambridge, 1991.

4. A. Coely, *et al*. (Eds), *Backlund and Darboux Transformations*, American Mathematical Society, Providence, Rhode island, 2001.

5. Z. Dai, J. Huang, M. Jiang and S. Wang, Homoclinic orbits and periodic solutions for Boussinesq equation with even constraint, *Chaos, Solitons Fractals*, **26** (2005), 1189-1194.

6. C. S. Gardner, J. M. Green, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-deVries equation, *Phys. Rev. Lett*., **19**(1967), 1095-1097.

7. R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett*., **27** (1971), 1192-1194.

8. M. Javidi and Y. Jalilian, Exact solitary wave solution of Boussinesq equation by VIM, *Chaos, Solitons Fractals*, **36** (2008), 1256-1260.

9. H. Jafari, A. Borhanifar and S. A. Karimi, *New solitary wave solution for the bad Boussinesq and good Boussinesq equations*, Wiley InterScience, (2008), 1231-1237.

10. H. Jafari, A. Borhanifar and S. A. Karimi, New solitons and periodic solutions for the Kadomtsev-Petvilashili equation, *Nonlinear Sciences and Applications*, **4** (2008), 224-229.

11. S. Lai and Y. Zheng, A study of three types of nonlinear Klein-Gordon equations, DCDIS Series B: *Applications& Algorithms*, **16** (2009), 271-279.

12. W. Malfliet, Solitary wave solutions of nonlinear wave equations, *Am. J. Phys*., **60(7)** (1992), 650-654.

13. W. Malfliet and W. Herman, The tanh method: perturbation technique for conservative systems. *Phys. scr*., **54** (1996), 569-575.

14. M. Rafei, D. D. Ganji, H. R. Mohammadi Daniali and H. Pashaei, Application of homotopy perturbation method ro the RLW and generalized modifieded Boussinesq equations, *Phys Lett A*, **364** (2007), 1-6.

15. M. L. Wang, Exact solution for a compound Kdv-Burgers equation, *phys. Lett. A*, **213** (1996), 279-287.

16. A. M. Wazwaz, The tanh method for compact solutions for variants of the Kdv-Burger and the k(n,n)-Burger equations, *PhysicaD: Nonlinear phenomena*, **213** (2006), 147-151.

17. A. M. Wazwaz, A computational approach to soliton solution of the Kadomtsev-Petviashili equation, *Comput. Math. Appl*., **123** (2001), 205-217.

18. A. M. Wazwaz, The tanh method for traveling wave solutions of nonlinear equation, *Applied Mathematica and Computation*, **154** (2004), 713-723.

19. A. M. Wazwaz, Compactons and solitary wave solutions for the Boussinesq wave equation and its generalized form, *Appl Math Comput*., **182** (2006), 529-535.

20. H. Q. Zhang and Z. Y. Yan, Auto- Darboux Transformation and exact solutions of the Brusselator reaction diffusion model, *Appl. Math. Mech*., **22** (2000), 541-546.