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ON THE WEIGHTS OF SIMPLE PATHS IN WEIGHTED COMPLETE GRAPHS

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Consider a weighted simple graph G on the vertex set $\{1, ..., n\}$. For any path p in G, we call $w_G(p)$ the sum of the weights of the edges of the path, and, for any $\{i, j\} \subset \{1, ..., n\}$, we define the multiset

$$\mathcal{D}_{\{i,j\}}(G) = \{w_G(p) |\ p \text{ a simple path between } i \text{ and } j\}.$$

We establish a criterion for a multisubset of $\mathbb{R}$ to be of the form $\mathcal{D}_{\{i,j\}}(G)$ for some weighted complete graph G and for some i, j vertices of G.

Besides we establish a criterion for a family $\{\mathcal{D}_{\{i,j\}}\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$ of multisubsets of $\mathbb{R}$ to be of the form $\{\mathcal{D}_{\{i,j\}}(G)\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$ for some weighted complete graph G with vertex set $\{1, ..., n\}$.

Key words : Graphs; weights of graphs.

1. INTRODUCTION

The problem of the realization of metrics or, more generally, positive symmetric matrices by graphs has a very rich literature. The problem can be described shortly

as follows. Given a weighted simple graph G (that is a simple graph such that every edge is endowed with a real number, which we call the weight of the edge), for any path p in G, we call $w_G(p)$ the sum of the weights of the edges of the path, and, for any $\{i,j\}$ 2-subset of the vertex set of G, we define

$$D_{\{i,j\}}(G) = min\{w_G(p)|\ \text{a simple path between } i \text{ and } j\}.$$

More simply, we will often denote it by $D_{i,j}(G)$. Given positive real numbers $D_{\{i,j\}}$ (or simply $D_{i,j}$) for any $\{i,j\} \subset \{1,...,n\}$, we can wonder whether there exists a positive-weighted graph G whose vertex set contains $\{1,...,n\}$ and such that $D_{i,j}(G) = D_{i,j}$ for any $\{i,j\} \subset \{1,...,n\}$.

In 1964 Hakimi and Yau proved that a family of positive real numbers $\{D_{i,j}\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$ is realizable by a positive-weighted graph if and only if it is a metric on $\{1,...,n\}$ (see [7]).

In the same years, also a criterion for a metric on $\{1,...,n\}$ to be realized by a nonnegative-weighted tree with leaf set $\{1,...,n\}$ was established, see [3, 14, 15]: given a metric on $\{1,...,n\}$, $\{D_{i,j}\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$, there exists a nonnegative-weighted tree T with leaf set $\{1,...,n\}$ such that $D_{i,j} = D_{i,j}(T)$ for any $\{i,j\} \subset \{1,...,n\}$ if and only if, for all $i,j,k,h \in \{1,...,n\}$, the maximum of $\{D_{i,j} + D_{k,h}, D_{i,k} + D_{j,h}, D_{i,h} + D_{k,j}\}$ is attained at least twice.

In [2] Bandelt and Steel proved a result, analogous to Buneman's one, for general weighted trees: for any family of real numbers $\{D_{i,j}\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$, there exists a weighted tree T with leaf set $\{1,...,n\}$ such that $D_{i,j}(T) = D_{i,j}$ for any $\{i,j\} \subset \{1,...,n\}$ if and only if, for any $a,b,c,d \in \{1,...,n\}$, we have that at least two among $D_{a,b} + D_{c,d}$, $D_{a,c} + D_{b,d}$, $D_{a,d} + D_{b,c}$ are equal.

Some references on the problem of the realization of metric spaces by trees or, more generally, by graphs can be found for instance in [9], in [14], just to quote two among many possible papers, and in the book [6].

More recently, analogous problems have been investigated for k-weights of weighted graphs with $k \geq 3$: for any nonnegative-weighted graph G and any distinct vertices of G, $i_1,...,i_k$, we define $D_{\{i_1,...,i_k\}}(G)$ to be the minimum of the

weights of the connected subgraphs of G with vertex set containing $i_1, ..., i_k$; the $D_{\{i_1,...,i_k\}}(G)$ are called "k-weights" of G. See the papers [11, 8, 12, 13, 1].

The problem of reconstructing weighted trees from data involving the distances between the leaves has several applications, such as phylogenetics and some algorithms to reconstruct trees from the data $\{D_{i,j}\}$ have been proposed (among them we quote neighbour-joining method, invented by Saitou and Nei in 1987, see [10, 16]).

Obviously the problems of realization of symmetric matrices by graphs and of reconstructing the weighted graphs from the "distances" between the vertices may have some applications, also in the case the weights are not all positive or all negative. Imagine that a particle, by going through an edge of a graph, gets or looses some substance (as much as the weight of the edge). If we know how much the substance of this particle varies by going from a vertex i of the graph to another vertex j (the value $D_{i,j}$) for any i and j, we can try to reconstruct the weighted graph (which can represent a graph in the human body, a hydraulic web...).

In this short note, we consider a basic graph theory problem, which is, in some way, linked to the quoted ones. Consider a weighted simple graph G on the vertex set $\{1, ..., n\}$. For any i, j distinct vertices of G, we define the multiset

$$\mathcal{D}_{\{i,j\}}(G) = \{w_G(p) |\ p \text{ a simple path between } i \text{ and } j\}$$

($\mathcal{D}_{i,j}(G)$ for short). In §3 we establish a criterion for a multisubset of $\mathbb{R}$ to be of the form $\mathcal{D}_{i,j}(G)$ for some weighted complete graph G and for some i, j vertices of G (see Theorem 8).

Besides, in §4, we establish a criterion for a family $\{\mathcal{D}_{i,j}\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$ of multisubsets of $\mathbb{R}$ to be of the form $\{\mathcal{D}_{i,j}(G)\}_{\{i,j\}\in\binom{\{1,...,n\}}{2}}$ for some weighted complete graph G with vertex set $\{1, ..., n\}$ (see Theorem 9).

2. Notation and Remarks

Notation 1 : By simple path in a graph, we will mean an unoriented path with distinct vertices. If the graph is simple, we will denote a path by the sequence of

their vertices. Obviously $(v_1, .., v_k)$ and $(v_k, ..., v_1)$ are the same path. A simple path between i and j (i and j vertices of the graph) will denote a simple path whose ends are i and j.

Let $[n] = \{1, ..., n\}$. Let K_n denote, as usual, the complete graph on the vertex set $[n]$.

Remark 2 : The number of the simple paths between two vertices in K_n is

$$N_n := \sum_{k=1,...,n-1} \frac{(n-2)!}{(n-k-1)!} = 1+(n-2)+(n-2)(n-3)+....+(n-2)!+(n-2)!$$

PROOF : Let i and j be two vertices of K_n. Obviously there is only one simple path between i and j with only one edge and there are $n-2$ paths between i and j with 2 edges.

There are $\binom{n-2}{2}2!$ simple paths between i and j with 3 edges (we have to choose the two vertices of the path besides i and j and their order in the path).

More generally there are $\binom{n-2}{k-1}(k-1)!$ simple paths between i and j with k edges (we have to choose the $k-1$ vertices of the path besides i and j and their order in the path).

Then, in all, they are

$$1+(n-2)+\binom{n-2}{2}2!+\binom{n-2}{3}3!+.....+\binom{n-2}{n-3}(n-3)!+\binom{n-2}{n-2}(n-2)! =$$

$$= 1 + (n-2) + \frac{(n-2)!}{(n-4)!} + \frac{(n-2)!}{(n-5)!} + ... + (n-2)! + (n-2)! =$$

$$= 1 + (n-2) + (n-2)(n-3) + ... + (n-2)! + (n-2)!$$

Q.e.d.

Remark 8 : $N_n - 1 = (n-2)[1 + (n-3) + ... + (n-3)! + (n-3)!] = (n-2)N_{n-1}$.

Definition 4 — Given a weighted simple graph G, for any path p in G, we define $w_G(p)$ (or simply $w(p)$) to be the sum of the weights of the edges of p, that

is, we define

$$w(p) = w(v_1, ..., v_k) = \sum_{i=1,...,k-1} w(v_i, v_{i+1})$$

for every path $p = (v_1, ..., v_k)$. For any $\{i, j\}$ 2-subset of the vertex set of G, we define the multiset

$$\mathcal{D}_{\{i,j\}}(G) = \{w_G(p)|\ p \text{ a simple path between } i \text{ and } j\}.$$

More simply, we will denote it by $\mathcal{D}_{i,j}(G)$ (or $\mathcal{D}_{j,i}(G)$).

Definition 5 — Given two sets $S, T \subset \mathbb{R}$ of the same cardinality, we define a "reciprocal order" for S and T a bijection $f : S \to T$. We define the difference of S and T by f to be the multiset

$$\{s - f(s)|\ s \in S\}.$$

Definition 6 — Let Y be a multisubset of $\mathbb{R}$ whose elements $y_{l,m}$ are indexed by the 2-subsets $\{l, m\}$ of $[n]$ (we write $y_{l,m}$ instead of $y_{\{l,m\}}$). For any $i, j \in [n]$ with $i \neq j$, let $h_{i,j}(Y)$ be the multisubset of $\mathbb{R}$ given by the elements

$$y_{i,i_1} + y_{i_1,i_2} + ... + y_{i_{r-1},i_r} + y_{i_r,j}$$

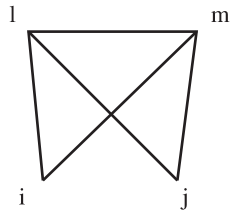
for $r \in \mathbb{N}$, $i_1, ..., i_r \in [n] - \{i, j\}$ distinct.

3. THEOREM 8

Remark 7 : Let G be a weighted complete graph on the vertex set $[n]$ and let $i, j \in [n]$ with $i \neq j$. Then

1) for any distinct $l, m \in [n] - \{i, j\}$,

$$w(l, m) = \frac{1}{2}\Big(w(i, l, m, j) + w(i, m, l, j) - w(i, l, j) - w(i, m, j)\Big)$$



2) for any distinct $l, m \in [n] - \{i, j\}$,

$$\begin{cases} w(i,l) + w(j,l) = w(i,l,j) \\ w(i,m) + w(j,m) = w(i,m,j) \\ w(i,m) + w(j,l) = w(i,m,l,j) - w(l,m) \\ w(i,l) + w(j,m) = w(i,l,m,j) - w(l,m) \end{cases}$$

3) for any distinct $i_1, ..., i_r \in [n]-\{i, j\}$, we have that $w(i, i_1, ..., i_r, j)$ is equal to

$$\frac{1}{2}\left(w(i,i_1,i_r,j) - w(i,i_r,i_1,j) + \sum_{s=1,...,r-1}\left(w(i,i_s,i_{s+1},j) + w(i,i_{s+1},i_s,j)\right)\right) - \sum_{s=2,...,r-1} w(i,i_s,j)$$

4) for any distinct $l, o, m \in [n] - \{i, j\}$,

$$w(i,m,l,j)+w(i,o,m,j)+w(i,l,o,j) = w(i,l,m,j)+w(i,m,o,j)+w(i,o,l,j)$$

PROOF : The only statement that needs a calculation is 3:

$$\begin{aligned} w(i,i_1,...,i_r,j) &= w(i,i_1) + w(i_1,i_2) + ... + w(i_{r-1},i_r) + w(i_r,j) \\ &= w(i,i_1,i_r,j) - w(i_1,i_r) + w(i_1,i_2) + ... + w(i_{r-1},i_r) \\ &= \frac{1}{2}\Big(w(i,i_1,i_r,j) - w(i,i_r,i_1,j) \\ &\quad + \sum_{s=1,...,r-1}\left(w(i,i_s,i_{s+1},j) + w(i,i_{s+1},i_s,j)\right)\Big) \\ &\quad - \sum_{s=2,...,r-1} w(i,i_s,j), \end{aligned}$$

where the last equality holds by part 1. *Q.e.d.*

***Theorem* 8 —** *Let Y be a multisubset of $\mathbb{R}$ of cardinality N_n. There exists a weighted complete graph G on the vertex set $[n]$ such that Y is equal to $\mathcal{D}_{i,j}(G)$ for some $\{i, j\} \subset [n]$ if and only if we can index the elements y of Y by the finite sequences of elements in $[n]$ from i to j without repetitions in such a way that:*

a) for any $r > 2$ *and any distinct* $i_1, ..., i_r \in [n] - \{i, j\}$

$$y_{i,i_1,...,i_r,j} = \frac{1}{2}\left(y_{i,i_1,i_r,j} - y_{i,i_r,i_1,j} + \sum_{s=1,...,r-1} (y_{i,i_s,i_{s+1},j} + y_{i,i_{s+1},i_s,j}) \right) - \sum_{s=2,...,r-1} y_{i,i_s,j}$$

b) for any distinct $l, o, m \in [n] - \{i, j\}$

$$y_{i,m,l,j} + y_{i,l,o,j} + y_{i,o,m,j} = y_{i,l,m,j} + y_{i,m,o,j} + y_{i,o,l,j} \ .$$

PROOF $\Rightarrow$ Follows from Remark 7.

$\Leftarrow$ Let G be a weighted complete graph on the vertex set $[n]$ and whose weights of the edges are defined in the following way:

• we define $w(i, j) = y_{i,j}$

• for any distinct $l, m \in [n] - \{i, j\}$ we define

$$w(l, m) = \frac{1}{2}\left(y_{i,l,m,j} + y_{i,m,l,j} - y_{i,l,j} - y_{i,m,j}\right) \tag{1}$$

• we define $w(i, l)$ and $w(j, l)$ for l varying in $[n] - \{i, j\}$ as solutions of the linear system:

$$\begin{cases} w(i,l) + w(j,l) = y_{i,l,j} \\ w(i,m) + w(j,m) = y_{i,m,j} \\ w(i,m) + w(j,l) = y_{i,m,l,j} - w(l,m) \\ w(i,l) + w(j,m) = y_{i,l,m,j} - w(l,m) \end{cases} \quad l, m \text{ varying in } [n] - \{i, j\} \tag{2}$$

Observe that the linear system is solvable (but the solutions are not unique), in fact:

- the linear system given only by the four equations above with l and m fixed has coefficient matrix

$$\begin{array}{c}
\begin{array}{cccc} \text{il} & \text{jl} & \text{im} & \text{jm} \end{array} \\
\left(\begin{array}{cccc}
1 & 1 & & \\
 & & 1 & 1 \\
 & 1 & 1 & \\
1 & & & 1
\end{array}\right)
\end{array}$$

where all the entries we didn't write are zero and we wrote over the matrix for instance: $il,\ jl,\ im,\ jm$, instead of the variables $w(i,l), w(j,l), w(i,m), w(j,m)$; the rank is 3; precisely the sum of the first two rows is equal to the sum of the last two rows; so it is solvable if and only if

$$y_{i,l,j} + y_{i,m,j} = y_{i,m,l,j} - w(l,m) + y_{i,l,m,j} - w(l,m),$$

which is true by the definition of $w(l,m)$

- the linear system (2) has coefficient matrix

$$\begin{array}{c}
\begin{array}{cccccccc} \text{il} & \text{jl} & \text{im} & \text{jm} & \text{io} & \text{jo} & \text{it} & \text{jt} \end{array} \\
\left(\begin{array}{cccccccc}
1 & 1 & & & & & & \\
 & & 1 & 1 & & & & \\
 & 1 & 1 & & & & & \\
1 & & & 1 & & & & \\
 & & & & 1 & 1 & & \\
 & & & & & 1 & 1 & \\
 & & & & 1 & & & 1 \\
 & & & & & 1 & & 1 \\
 & 1 & & & & & & 1 \\
 & & 1 & & & & 1 & \\
1 & & & & & & & 1 \\
 & & & & & & & \\
 & & & & \ldots\ldots\ldots & & &
\end{array}\right)
\end{array}$$

where all the entries we didn't write are zero, in fact the linear system, written explicitely, is

$$\begin{cases} w(i,l)+w(j,l)=y_{i,l,j} \\ w(i,m)+w(j,m)=y_{i,m,j} \\ w(i,m)+w(j,l)=y_{i,m,l,j}-w(l,m) \\ w(i,l)+w(j,m)=y_{i,l,m,j}-w(l,m) \\ w(i,o)+w(j,o)=y_{i,o,j} \\ w(i,o)+w(j,m)=y_{i,o,m,j}-w(o,m) \\ w(i,m)+w(j,o)=y_{i,m,o,j}-w(o,m) \\ w(i,o)+w(j,l)=y_{i,o,l,j}-w(o,l) \\ w(j,o)+w(i,l)=y_{i,l,o,j}-w(o,l) \\ w(i,t)+w(j,t)=y_{i,t,j} \\ w(i,t)+w(j,o)=y_{i,t,o,j}-w(t,o) \\ w(i,o)+w(j,t)=y_{i,o,t,j}-w(t,o) \\ w(i,t)+w(j,m)=y_{i,t,m,j}-w(t,m) \\ w(j,t)+w(i,m)=y_{i,m,t,j}-w(t,m) \\ w(i,t)+w(j,l)=y_{i,t,l,j}-w(t,l) \\ w(j,t)+w(i,l)=y_{i,l,t,j}-w(t,l) \\ ..... \end{cases}$$

The coefficient matrix has rank equal to the number of its columns minus 1; in fact observe that the rows from the 7^{th} to the 9^{th}, the rows from the 12^{th} to 16^{th} and so on are linear combinations of the previous ones, for instance the sum of the 3^{rd} and the 5^{th} is equal to the sum of the 7^{nd} and the 8^{th}.

Obviously the linear system is solvable if and only if the same relations hold for the constant terms and this is true if and only if, for any distinct $l, m \in [n]-\{i,j\}$, the equation (1) holds, and, for any distinct $l, o, m \in [n]-\{i,j\}$,

$$y_{i,m,l,j}-w(m,l)+y_{i,o,j}=y_{i,m,o,j}-w(m,o)+y_{i,o,l,j}-w(o,l),$$

which is equivalent to assumption b.

Now we show that $w(i,i_1,....,i_r,j)=y_{i,i_1,....,i_r,j}$ for any $i_1,...,i_r \in [n]-\{i,j\}$.

First suppose that $r=1$. We have that $w(i,i_1,j)=w(i,i_1)+w(i_1,j)$, which

is equal to $y_{i,i_1,j}$ by the definition of the $w(i,l)$ and $w(j,l)$ for $l \in [n] - \{i,j\}$. Analogously if $r = 2$.

If $r > 2$, by assumption a, we have that

$$\begin{aligned} y_{i,i_1,\ldots,i_r,j} &= \frac{1}{2}\left(y_{i,i_1,i_r,j} - y_{i,i_r,i_1,j} + \sum_{s=1,\ldots,r-1} (y_{i,i_s,i_{s+1},j} + y_{i,i_{s+1},i_s,j}) \right) \\ &\quad - \sum_{s=2,\ldots,r-1} y_{i,i_s,j}, \end{aligned}$$

which (by cases $r = 1, 2$) is equal to

$$\frac{1}{2}\left(w(i,i_1,i_r,j) - w(i,i_r,i_1,j) + \sum_{s=1,\ldots,r-1} (w(i,i_s,i_{s+1},j) + w(i,i_{s+1},i_s,j)) \right) - \sum_{s=2,\ldots,r-1} w(i,i_s,j),$$

which is equal to $w(i,i_1,\ldots,i_r,j)$ by Remark 7, part 3. *Q.e.d.*

4. THEOREM 9

Roughly speaking, the following theorem says that, given multisubsets of $\mathbb{R}$, $\mathcal{D}_{i,j}$, of cardinality N_n, for $i,j \in [n]$, there exists a weighted complete graph G such that $\mathcal{D}_{i,j}(G) = \mathcal{D}_{i,j}$ for all i,j if and only if we can order the $\mathcal{D}_{i,j}$ reciprocally in such a way that the difference of each pair of them can be divided into $n-2$ multisubsets, one of cardinality $N_{n-1}+1$, the others of cardinality N_{n-1}, such that all the elements of each of these subsets have all the same absolute value and one of the $\mathcal{D}_{i,j}$ is in the image of h.

Theorem 9 *— For any $i,j \in [n]$, let $\mathcal{D}_{i,j}$ be a multisubset of $\mathbb{R}$ of cardinality N_n. There exists a weighted complete graph G with vertex set $[n]$ and such that*

$$\mathcal{D}_{i,j}(G) = \mathcal{D}_{i,j} \quad \forall i,j \in [n]$$

if and only if, for any $i,j,k \in [n]$, there exists a reciprocal order $f^{i,k}_{j,k}$ for $\mathcal{D}_{i,k}$ and $\mathcal{D}_{j,k}$, and, for any $i,k \in [n]$, there exists an element $y_{i,k} \in \mathcal{D}_{i,k}$ such that

A) $f_{j,k}^{i,k}(y_{i,k}) = y_{j,k}$

B) the difference of $\mathcal{D}_{i,k}$ *and* $\mathcal{D}_{j,k}$ *by* $f_{j,k}^{i,k}$ *can be divided into* $n-2$ *multisubsets, which we call*

$$\mathcal{L}_0(i,k|j,k), \quad \mathcal{L}_r(i,k|j,k)$$

for $r \in [n] - \{i,j,k\}$*, the first of cardinality* $N_{n-1}+1$*, the others of cardinality* N_{n-1}*, such that* $\mathcal{L}_0(i,k|j,k)$ *contains one element equal to* $y_{i,k} - y_{j,k}$ *and the other elements are equal to its opposite and* $\mathcal{L}_r(i,k|j,k)$*, for* $r \in [n] - \{i,j,k\}$*, contains* N_{n-2} *elements equal to* $y_{r,i} - y_{r,j}$ *and the others are equal to its opposite.*

C) if $Y := \{y_{l,m}\}_{l,m\in[n]}$*, there exist* $u, v \in [n]$ *such that* $\mathcal{D}_{u,v} = h_{u,v}(Y)$*.*

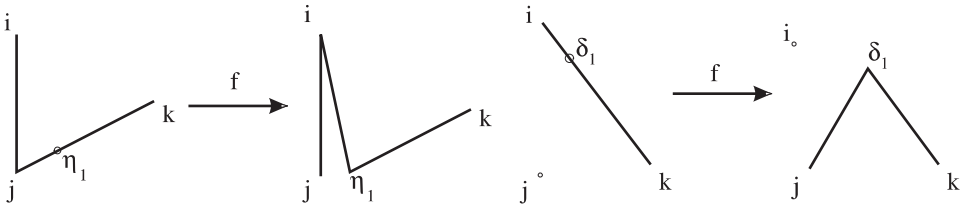
PROOF : $\Rightarrow$ Let $y_{i,j} = w_G(i,j)$. We can divide the paths between i and k into two kinds: the ones passing through j, which we can write as (i,γ,j,η,k) for some γ and η disjoint subsets of $[n]-\{i,j,k\}$, and the others, which we can write as (i,δ,k) with δ subset of $[n]-\{i,j,k\}$.

We establish a bijection $f = f_{j,k}^{i,k}$ between the paths between i and k and the paths between j and k, which will define a reciprocal order of $\mathcal{D}_{i,k}$ and $\mathcal{D}_{j,k}$:

$$(i,\gamma,j,\eta,k) \overset{f}{\longmapsto} (j,\gamma^{-1},i,\eta,k)$$

$$(i,\delta,k) \overset{f}{\longmapsto} (j,\delta,k)$$

for γ and η disjoint subsets of $[n]-\{i,j,k\}$, δ subset of $[n]-\{i,j,k\}$ (if $\gamma = (\gamma_1,...,\gamma_t)$, we denote $\gamma^{-1} = (\gamma_t,...,\gamma_1)$).



The paths between i and k can be divided into $n-2$ subsets:

• a subset $\mathcal{P}_0$ of cardinality $N_{n-1}+1$ whose elements are:

- the path (i,k)

- the paths of the kind (i,γ,j,k) with $\gamma \subset [n]-\{i,j,k\}$

(that is the path with no vertices between i and k and the paths passing through j and with no vertices between j and k)

• $n-3$ subsets $\mathcal{P}_r$, for $r \in [n]-\{i,j,k\}$, each of them defined as follows: fix $r \in [n]-\{i,j,k\}$ and consider

- the paths of kind (i,η,k) with $\eta \subset [n]-\{i,j,k\}$, $\eta \neq \emptyset$ and $\eta_1 = r$

- the paths of kind (i,γ,j,η,k) with $\gamma,\eta \subset [n]-\{i,j,k\}$, $\eta \neq \emptyset$ and $\eta_1 = r$

(that is the paths not passing through j and with some vertices between i and k and the paths passing through j and with some vertices between j and k).

Obviously all the subsets $\mathcal{P}_r$, for $r \in [n]-\{i,j,k\}$, have the same cardinality, so the cardinality of each of them is

$$\frac{N_n - 1 - N_{n-1}}{n-3} = \frac{(n-2)N_{n-1} - N_{n-1}}{n-3} = N_{n-1},$$

where the first equality holds by Remark 3.

Observe that for any $p \in \mathcal{P}_0$ we have

$$|w(p) - w(f(p))| = |w(i,k) - w(j,k)| = |y_{i,k} - y_{j,k}|,$$

in particular

$w(i,k) - w(f(i,k)) = w(i,k) - w(j,k) = y_{i,k} - y_{j,k}$

$w(i,\gamma,j,k) - w(f(i,\gamma,j,k)) = w(i,\gamma,j,k) - w(j,\gamma^{-1},i,k)) = w(j,k) - w(i,k) = -y_{i,k} + y_{j,k}$.

Besides, for any $p \in \mathcal{P}_r$ and for any $r \in [n]-\{i,j,k\}$, we have

$$|w(p) - w(f(p))| = |w(i,r) - w(j,r)| = |y_{i,r} - y_{j,r}|$$

in particular

$w(i,\eta,k) - w(f(i,\eta,k)) = w(i,\eta,k) - w(j,\eta,k) = w(i,\eta_1) - w(j,\eta_1) = w(i,r) - w(j,r) = y_{i,r} - y_{j,r}$

$w(i,\gamma,j,\eta,k) - w(f(i,\gamma,j,\eta,k)) = w(i,\gamma,j,\eta,k) - w(j,\gamma^{-1},i,\eta,k)) = -w(i,\eta_1) + w(j,\eta_1) = -w(i,r) + w(j,r) = -y_{i,r} + y_{j,r}$.

So the sets $\mathcal{P}_0$ and $\mathcal{P}_r$ give the subsets $\mathcal{L}_0$ and $\mathcal{L}_r$ in the difference of $\mathcal{D}_{i,k}$ and $\mathcal{D}_{j,k}$ by $f_{j,k}^{i,k}$ and B holds (A and C are obvious).

$\Leftarrow$ Let G be the weighted complete graph on the vertex set $[n]$ and whose weights are defined by

$$w_G(i,k) = y_{i,k}$$

for any $i,k \in [n]$ with $i \neq k$. For the graph G we can define a set of reciprocal orders ${}^G f_{j,k}^{i,k}$ for the $\mathcal{D}_{i,k}(G)$ as in the proof of the other implication. The difference of $\mathcal{D}_{i,k}(G)$ and $\mathcal{D}_{j,k}(G)$ by such reciprocal orders can be divided into $n-2$ multisubsets, $\mathcal{L}_0^G(i,k|j,k)$ and $\mathcal{L}_r^G(i,k|j,k)$ for $r \in [n] - \{i,j,k\}$, the first of cardinality $N_{n-1}+1$, the others of cardinality N_{n-1}, such that $\mathcal{L}_0^G(i,k|j,k)$ contains one element equal to $w_G(i,k) - w_G(j,k)$ and the other elements are equal to its opposite and $\mathcal{L}_r^G(i,k|j,k)$, for $r \in [n] - \{i,j,k\}$, contains N_{n-2} elements equal to $w_G(r,i) - w_G(r,j)$ and the others are equal to its opposite.

We have to show that

$$\mathcal{D}_{i,k}(G) = \mathcal{D}_{i,k}$$

for any $i,k \in [n]$.

First we want to prove that, for any i,j,k, the difference of $\mathcal{D}_{i,k}(G)$ and $\mathcal{D}_{j,k}(G)$ by ${}^G f_{j,k}^{i,k}$ is equal to the difference of $\mathcal{D}_{i,k}$ and $\mathcal{D}_{j,k}$ by $f_{j,k}^{i,k}$. Obviously $\mathcal{L}_0^G(i,k|j,k) = \mathcal{L}_0(i,k|j,k)$, because $\mathcal{L}_0^G(i,k|j,k)$ will be composed by $w_G(i,k) - w_G(j,k)$ and N_{n-1} opposites of it and $\mathcal{L}_0(i,k|j,k)$ will be composed by $y_{i,k} - y_{j,k}$ and N_{n-1} opposites of it by assumption; so from our definition of the weights of G we can conclude. Also $\mathcal{L}_r^G(i,k|j,k) = \mathcal{L}_r(i,k|j,k)$ for $r \in [n] - \{i,j,k\}$ because $\mathcal{L}_r^G(i,k|j,k)$ is given by N_{n-2} numbers equal to $w_G(i,r) - w_G(j,r) = y_{i,r} - y_{j,r}$ and $N_{n-1} - N_{n-2}$ equal to its opposite and the same $\mathcal{L}_r(i,k|j,k)$.

We have that $\mathcal{D}_{u,v}(G) = h_{u,v}(\{w_G(l,m)\}_{l,m}) = h(\{y_{l,m}\}_{l,m}) = \mathcal{D}_{u,v}$, where

the last equality holds by assumption C. From this and from the fact that the difference of $\mathcal{D}_{k,u}(G)$ and $\mathcal{D}_{u,v}(G)$ by ${}^{G}f^{k,u}_{u,v}$ is equal to the difference of $\mathcal{D}_{k,u}$ and $\mathcal{D}_{u,v}$ by $f^{k,u}_{u,v}$, we get that $\mathcal{D}_{k,u}(G) = \mathcal{D}_{k,u}$ for any $k \in [n] - \{u, v\}$.

From this and from the fact that the difference of $\mathcal{D}_{i,k}(G)$ and $\mathcal{D}_{k,u}(G)$ by ${}^{G}f^{i,k}_{k,u}$ is equal to the difference of $\mathcal{D}_{i,k}$ and $\mathcal{D}_{k,u}$ by $f^{i,k}_{k,u}$, we get that $\mathcal{D}_{i,k}(G) = \mathcal{D}_{i,k}$ for any i, k. *Q.e.d.*

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