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WEAKLY S-SUPPLEMENTED SUBGROUPS AND P-NILPOTENCY OF FINITE GROUPS

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A subgroup H of a group G is said to be weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. In this paper, we investigate the influence of weakly s-supplemented subgroups on the structure of finite groups. Some recent results are generalized.

Key words : Finite groups; p-nilpotent groups; weakly s-supplemented subgroups.

1. INTRODUCTION

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [5]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$, $\mathbb{Z}_l$ a cyclic group of order l. A semidirect product of a group A by a group B is denoted by $A : B$. We use $\mathcal{U}$ to denote the class of supersolvable groups. The $\mathcal{U}$-hypercenter of a group G will be denoted by $Z_{\mathcal{U}}(G)$ which is the product of all normal subgroups H of G such that every chief factor of G below H has prime order.

A subgroup H of G is called *complemented* in G if there is a subgroup K such that $G = HK$ and $H \cap K = 1$. In this case, K is called a complement of H in G. A group G is said to be *p-nilpotent* if a Sylow p-subgroup of G has a normal complement. There are many theorems that characterize the structure of finite group by using of complementation of some suitable subgroups. A celebrated theorem of Hall said that a group of G is solvable if and only if each Sylow subgroup of G is complemented. In [4], Hall also proved that each subgroup of G is complemented if and only if G is supersolvable with each Sylow subgroup being elementary abelian. Ballester-Bolinches, Guo proved that if each minimal subgroup of G is complemented, then G is supersolvable [3]. In [9], Miao used complemented subgroups to characterize p-nilpotency of finite groups, and got some interesting results. Asaad [1] got a nice result regarding p-nilpotency of finite groups which generalized and unified some known related results. Asaad's result was recently improved in [11] by Isaacs, Li and the author.

There is a generalization of complemented subgroup, called *c-supplemented subgroup* [17]. A subgroup H is called *c-supplemented* in G if there is a subgroup K such that $G = HK$ and $H \cap K \leq H_G$, the core of H in G. More recently, Asaad proved the following result.

***Theorem* 1.1** [2, Theorem 3.5] — *Let G be a finite group and let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups H of P with $|H| = p^n|D|$(n=0,1) are c-supplemented in G. If P is a non-abelian 2-group and $|D| = 1$, suppose further that each cyclic subgroup of P of order 4 is c-supplemented in G. Then G is p-nilpotent.*

The main purpose of this paper is to investigate the structure of G through assuming that some families of subgroups of G are *weakly s-supplemented* [15] in

G. To be precise, we prove the following results:

***Theorem* 1.2** — *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups $|H|$ of $|P|$ with $|H| = p^n|D|$ $(n = 0, 1)$ are weakly s-supplemented in G. Also, when P is a non-abelian 2-subgroup and $|D| = 1$, we suppose further that each cyclic subgroup of P of order 4 is weakly s-supplemented in G. Then G is p-nilpotent.*

***Theorem* 1.3** — *Let $P \in Syl_p(G)$, where G is a finite group of odd order and p is the smallest prime divisor of $|G|$. Let $D \leq P$ such that $1 < |D| < |P|$, and assume that every subgroup of P of order $|D|$ is weakly s-supplemented in G. Suppose also that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.*

Remark : generally, Theorem 1.3 does not hold for $p = 2$. Let's see the following example:

Let $G = SL(2, 3) \cong Q_8 : \mathbb{Z}_3 : \mathbb{Z}_2$. It is easy to check each minimal subgroup of G of order 2 is weakly s-supplemented in G, and $N_G(G_2) = G_2$ which is of course 2-nilpotent. However, G is not 2-nilpotent.

2. Preliminaries

Lemma 2.1 — Let G be a group.

(1) An s-permutable subgroup of G is subnormal in G;

(2) If $H \leq K \leq G$ and H is s-permutable in G, then H is s-permutable in K;

(3) If H is s-permutable Hall subgroup of G, then $H \lhd G$;

(4) Let $K \lhd G$ and $K \leq H$. Then H is s-permutable in G if and only if H/K is s-permutable in G/K.

(5) If H, K are s-permutable in G, then $H \cap K$ is also s-permutable in G.

(6) Suppose that P is a p-subgroup of G for some prime p. Then P is s-permutable in G if and only if $N_G(P) \geq O^p(G)$.

PROOF : (1)-(5) are from [6], (6) is [13, Lemma A].

Lemma 2.2 [15, Lemma 2.10] — Let G be a group and $H \leq K \leq G$. Then

(1) Suppose that H is normal in G. Then K/H is weakly s-supplemented in G/H if and only if K is weakly s-supplemented in G.

(2) If H is weakly s-supplemented in G, then H is weakly s-supplemented in K.

(3) Suppose that H is normal in G. Then the subgroup HE/H is weakly s-supplemented in G/H for every weakly s- supplemented in G subgroup E satisfying $(|H|, |E|) = 1$.

Lemma 2.3 — Let N be a minimal normal subgroup of a group G contained in $O_p(G)$. Then there is no subgroup D such that $1 < |D| < |N|$ and every subgroup of N of order $|D|$ is weakly s-supplemented in G.

PROOF : Assume that the lemma is false, and let H be any subgroup of N of order $|D|$ which is weakly s-supplemented in G. Then there is a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$. Clearly, $1 \neq N \cap T \lhd G$ since H is properly contained in N. Thus $N \cap T = N$. It follows that $H = H \cap T = H_{sG}$. Then each subgroup of N is s-permutable in G. By [15, Lemma 2.11], some maximal subgroup of N is normal in G. Then N has prime order and $|D| = 1$, a contradiction.

Lemma 2.4 [18, Appendix C, Theorem 6.3] — Let P be a normal p-subgroup of a group G such that $|G/C_G(P)|$ is a power of p. Then $P \leq Z_\infty(G)$, the hypercenter of G.

Lemma 2.5 [18, Chapter 1,Theorem 7.19] — Let H be normal in G. Then $H \leq Z_{\mathcal{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathcal{U}}(G/\Phi(H))$.

Lemma 2.6 [8, Theorem 3.4] — Let P be a normal p-subgroup of a group G,

$p > 2$. If every subgroup of P of order p is weakly s-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$.

The following result regarding minimal non-p-nilpotent groups is well-known (see, for example, [5]).

Lemma 2.7 — Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then

(a) G has a normal Sylow p-sugroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime q other than p;

(b) The exponent of P is p or 4;

(c) $\Phi(P) \leq Z(G)$.

Lemma 2.8 [19, Lemma 2.12] — Let G be a finite group, and let p be the minimal prime divisor of $|G|$. Suppose that P is a Sylow p-subgroup of G such that each maximal subgroup of P has a p-nilpotent supplement in G. Then G is p-nilpotent.

Lemma 2.9 [14, II, 7.9] — Let H be a nilpotent normal subgroup of a group G. If $H \cap \Phi(G) = 1$,then H is the direct product of some minimal normal subgroups of G.

The next result regarding the minimal case of our main result is needed in our proof. For its proof, please refer to [10].

Lemma 2.10 — Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If cyclic subgroups of P of order p and 4 are weakly s-supplemented in G, then G is p-nilpotent.

3. Main Results

The first result is regarding the maximal case of our result.

***Theorem* 3.1** — *Let G be a group and P a Sylow p-subgroup of G, where*

p is the smallest prime dividing $|G|$. If all maximal subgroups of P are weakly s-supplemented in G, then G is p-nilpotent.

PROOF : Assume that the result is false and let G be a counterexample with minimal order. We will conduct a contradiction in several steps.

By Lemma 2.2, Step 1 is obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. $O_p(G) \neq 1$.

If $O_p(G) = 1$, by Lemma 2.1, then all maximal subgroups of P are complemented in G. Lemmas 3 and 3* of [3] imply that G is p-nilpotent, a contradiction.

Step 3. Let N be a minimal normal subgroup of G contained in P. Then G/N is p-nilpotent; furthermore, $\Phi(G) = 1$ and $N = O_p(G)$.

Let N be a minimal normal subgroup of G contained in P. By Lemma 2.2, we have G/N satisfies the hypotheses of the theorem. By induction, G/N is p-nilpotent. It follows that N is the unique minimal normal subgroup of G contained in P, $\Phi(G) = 1$ and $O_p(G)$ is abelian. Thus there is a maximal subgroup M of G such that $G = N : M$, and so $O_p(G) \cap M$ is normalized by N and M, thus by G. Then $O_p(G) \cap M = 1$ since N is unique. This implies $N = O_p(G)$.

Step 4. Finishing the proof.

By Step 3, we have G is solvable. Then N is self-centralized in G. Choose a suitable non-trivial Sylow q-subgroup Q of G such that $PQ \leq G$, q is a prime distinct with p. If PQ is a proper subgroup of G, by induction, $PQ = Q : P$, which implies that Q centralizes N, contrary to that N is self-centralized in G. By Step 3, each maximal subgroup of P containing N has a p-nilpotent supplement. By Lemma 2.8, we have P has a maximal subgroup P_1 which has no p-nilpotent supplement in G. Clearly, P_1 does not contain N, and so $P = NP_1$. By hypotheses, there is a subgroup T such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. By Lemma 2.1 and Step 3, there holds $(P_1)_{sG} \leq P_1 \cap N \leq N = O_p(G)$, and

$(P_1)_{sG}^G = (P_1)_{sG}^{O^p(G)P} = (P_1)_{sG}^P \leq (P_1 \cap N)^P \leq N$. Since $N = O_p(G)$ is minimal normal in G, we have $(P_1)_{sG}^G = 1$ or N. If $(P_1)_{sG}^G = 1$, then $P_1 \cap T \leq (P_1)_{sG} \leq (P_1)_{sG}^G = 1$. This implies that T is p-nilpotent, contrary to the choice of P_1. Thus $(P_1)_{sG}^G = P_1 \cap N = N$. This implies that $N \leq P_1$, a final contradiction.

This finishes the proof of the theorem. □

***Theorem* 3.2** — *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups $|H|$ of $|P|$ with $|H| = p^n|D|$ $(n = 0, 1)$ are weakly s-supplemented in G. Also, when P is a non-abelian 2-subgroup and $|D| = 1$, we suppose further that each cyclic subgroup of P of order 4 is weakly s-supplemented in G. Then G is p-nilpotent.*

PROOF : Assume that the result is false and let G be a counterexample with minimal order. We will conduct a contradiction in several steps.

By Lemmas 2.2 and 2.10 and Theorem 3.1, we have Steps 1-2.

Step 1. $O_{p'}(G) = 1$ and $O_p(G) \neq 1$.

Step 2. $|D| > p$ and $|P : D| > p$.

By Step 2 and Lemma 2.2, we have Step 3.

Step 3. $Z(G) = 1$, and thus $Z_\infty(G) = 1$.

Step 4. If N is a minimal normal subgroup of G contained in P, then $|N| \leq |D|$.

This step follows from Lemma 2.3.

Step 5. $O_p(G) \neq P$.

Suppose that $P = O_p(G)$. By induction, $G = O_p(G) : Q = P : Q$ for some q-subgroup of G, $q \neq p$. Then by induction, we have $Q \cong \mathbb{Z}_q$. By Step 1, $\Phi(G)$ is a p-subgroup. If $|\Phi(G)| > |D|$, by minimal choice of G, then $\Phi(G)Q$ is p-nilpotent.

It follows that $G/C_G(\Phi(G))$ is a p-group. By Lemma 2.4, $\Phi(G) \leq Z_\infty(G)$, contrary to Step 3. Then, by Lemma 2.2, we have $\Phi(G) = 1$ or $|\Phi(G)| = |D|$. Otherwise, $G/\Phi(G)$ satisfies the hypotheses of the theorem, then $G/\Phi(G)$, and so G, is p-nilpotent.

Let $\Phi(G) = 1$. Then P is elementary abelian. It follows that P has a decomposition that $P = N_1 \times N_2 \times \cdots \times N_t$ by Maschke's theorem [16, P. 39], each N_i is minimal normal in G. By [12, Theorems 2.3.2, 2.3.3], all N_i have same order. If $t \geq 2$, then $|N_1| = |N_2| = \cdots = |N_t| = |D|$. Then consider the factor group $G/N_1 \cong (N_2 \times \cdots \times N_t) : Q \cong (N_2 \times \cdots \times N_t) : \mathbb{Z}_q$. By hypotheses and Lemma 2.2, each subgroup of P/N_1 of order p is weakly s-supplemented in G/N_1. By induction, G/N_1 is p-nilpotent since P/N_1 is abelian. Similarly, we have G/N_2 is p-nilpotent, and thus G is p-nilpotent, a contradiction. If $t = 1$, then $P = O_p(G) = N_1$, this is contrary to Lemma 2.3, a contradiction.

Let $|\Phi(G)| = |D|$. Then each minimal subgroup $\overline{H}$ of $\overline{P} = P/\Phi(G)$ of order p is weakly s-supplemented in $\overline{G} = G/\Phi(G)$. Clearly, $\overline{P}$ is abelian. By induction, $\overline{G}$, also G, is p-nilpotent. Thus Step 5 holds.

Step 6. A final contradiction.

By Step 5, $O_p(G)$ is a proper subgroup of P. If $|O_p(G)| > |D|$, since $O_p(G)Q$ is proper in G, by induction, $O_p(G)Q = O_p(G) \times Q$, $Q \in Syl_q(G)$, $q \neq p$. By Lemma 2.4, $O_p(G) \leq Z_\infty(G)$, contrary to Step 3. Thus $|O_p(G)| \leq |D|$. Induction forces $\Phi(O_p(G)) = 1$.

Consider the factor group $G/O_p(G)$. Suppose that $|O_p(G)| = |D|$. Then by hypotheses, each minimal subgroup of $P/O_p(G)$ is weakly s-supplemented in $G/O_p(G)$. Since $O_p(G/O_p(G)) = 1$, each minimal subgroup of $P/O_p(G)$ is complemented in $G/O_p(G)$. Then $G/O_p(G)$ is p-nilpotent, and so $\Phi(G) = 1$ and $O_p(G)$ is minimal normal in G. It follows that $G = O_p(G) : Q : L$, where L is some suitable p-subgroup of G. If $|L| > p$, let L_1 be a maximal subgroup of L, then $O_p(G) : Q : L_1$ is p-nilpotent by induction. We know that p-nilpotency implies p-solvability. Since $O_{p'}(G) = 1$, we have $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $L \cong \mathbb{Z}_p$. In this case, $|P : D| = p$ since $|O_p(G)| = |D|$, contrary to Step 2, a contradiction. Similarly, we also have a contradiction if $|O_p(G)| < |D|$.

This finishes the proof. □

Theorem 3.2 will yield a result of Asaad [2, Theorem 3.5].

Corollary 3.3 [2] — Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups $|H|$ of $|P|$ with $|H| = p^n|D|$ $(n = 0, 1)$ are c-supplemented in G. Also, when P is a non-abelian 2-subgroup and $|D| = 1$, we suppose further that each cyclic subgroup of P of order 4 is c-supplemented in G. Then G is p-nilpotent.

***Theorem* 3.4** — *Let p be a prime and P a normal Sylow p-subgroup of a group G. Suppose that P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups H of P with $|H| = p^n|D|$(n = 0,1) are weakly s-supplemented in G. If P is a non-abelian 2-subgroup and $|D| = 1$, we suppose further that each cyclic subgroup of P of order 4 is weakly s-supplemented in G. Then $P \leq Z_{\mathcal{U}}(G)$.*

PROOF : Let p be the minimal prime divisor of $|G|$. Since $P \lhd G$, PQ is a subgroup for any $Q \in Syl_p(G)$ with $q \neq p$. By Theorem 3.2, PQ is p-nilpotent, then P is centralized by Q. It follows that $G/C_G(P)$ is a p-group, by Lemma 2.4, $P \leq Z_\infty(G)$, and thus $P \leq Z_{\mathcal{U}}(G)$.

Assume below that p is not the minimal prime divisor of $|G|$. Then $p > 2$. We proceed by doing induction on $|G||P|$.

Step 1. $|D| > p$.

This follows from Lemma 2.6.

Step 2. $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. Let N be a minimal normal subgroup of G contained in $\Phi(P)$. By Lemma 2.3, we have $|N| \leq |D|$. By Lemma 2.2, the hypothesis of the theorem is valid for the pair $(G/N, P/N)$, and so $P/N \leq Z_{\mathcal{U}}(G/N)$ by induction. Then $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ since $P \lhd G$ and $N \leq \Phi(P)$. Hence $P \leq Z_{\mathcal{U}}(G)$ by Lemma 2.5, as desired.

Step 3. P has a decomposition that $P = N_1 \times N_2 \times \cdots \times N_t$, each N_i is minimal normal in G, $t \geq 2$.

Clearly, by Schur-Zassenhaus' theorem, P has a complement $G_{p'}$, a Hall p'-subgroup of G, in G. By Step 2, P is elementary abelian. By Maschke's theorem [16, P.39], P has a decomposition that $P = N_1 \times N_2 \times \cdots \times N_t$, each N_i is minimal normal in G. By Lemma 2.3, we have $t \geq 2$.

Step 4. Finishing the proof.

By Step 3, $P = N_1 \times N_2 \times \cdots \times N_t$ with $t \geq 2$. By Lemma 2.3 and Step 1, $|N_i| \leq |D|$ for each i. By Lemma 2.2, the hypothesis of the theorem is valid for the pair $(G/N_i, P/N_i)$ for each i, and so $P/N_i \leq Z_{\mathcal{U}}(G/N_i)$ by induction. Then each N_i has order p, and so $P \leq Z_{\mathcal{U}}(G)$. This finishes the proof of the theorem. □

Theorem 3.5 — *Let $P \in Syl_p(G)$,where G is a finite group of odd order and p the smallest prime divisor of $|G|$. Let $D \leq P$ such that $1 < |D| < |P|$, and assume that every subgroup of P of order $|D|$ is weakly s-supplemented in G. Suppose also that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.*

PROOF : Let G be a minimal counterexample.

By Lemmas 2.2 and 2.10, Theorem 3.1 and [11, Theorem C], we have Steps 1-3.

Step 1. $O_{p'}(G) = 1$ and $O_p(G) \neq 1$.

Step 2. Every proper subgroup of G containing P is p-nilpotent.

Step 3. $|P : D| > p$ and $|D| > p$.

Step 4. If K is a proper normal subgroup of G, then $1 \neq K_p \leq O_p(G)$. In particular, every minimal normal subgroup of G is contained in P.

By Step 1, we have $K_p \neq 1$. If K_p is normal in G, then $K_p \leq O_p(G)$, as desired. Assume that K_p is not normal in G. If $|K_p| > |D|$, then every subgroup H of K_p with order $|H| = |D|$ is weakly s-supplemented in G, hence in K. Since $P \leq N_G(K_p) < G$, we have $N_K(K_p)$ is p-nilpotent by Step 2. Therefore K is p-nilpotent by the minimal choice of G and so K is a p-group, a contradiction. Now suppose that $|K_p| \leq |D|$. We can pick a normal subgroup H of P such that $K_p \leq H$ and $|H| = p|D|$. Clearly $HK < G$ by Step 3. If H is normal in G, then $K_p \leq H \leq O_p(G)$, as desired. So assume that H is not normal in G. Then $P \leq N_G(H) = N_G((HK)_p) < G$. We can see that HK satisfies the hypotheses of the theorem by Step 2. Hence HK is p-nilpotent by the minimal choice of G. Therefore K is p-nilpotent and then K is a p-group and $K \leq O_p(G)$, as desired.

Step 5. $|O_p(G)| \leq |D|$.

Suppose that $|O_p(G)| > |D|$. Then $O_p(G)Q$ is a subgroup for any Sylow q-subgroup of G, $q \neq p$. By minimal choice of G, we have $O_p(G)Q$ is p-nilpotent. Then $G/C_G(O_p(G)$ is a p-group. By Lemma 2.4, $O_p(G) \leq Z_\infty(G)$. Then $Z(G) \neq 1$. Choose a prime-order subgroup N contained in $Z(G)$. By Step 3 and induction, G/N is p-nilpotent, and then G is p-nilpotent, a contradiction. Thus $|O_p(G)| \leq |D|$.

Step 6. Let N be a minimal normal subgroup of G. Then $|N| < |D|$ and G/N is p-nilpotent.

By Steps 1 and 4, $N \leq O_p(G)$. Suppose that $|N| = |D|$. By Step 5, we have $N = O_p(G)$. Since G has odd order, we have that G is solvable, and $O_p(G) < O_{pp'}(G) < G$. By induction and Steps 3-4, there holds $G = O_{pp'p}(G)$. It follows that G has a proper normal subgroup M such that $|G : M| = p$. Clearly, M_p is maximal in P. By Step 3, we have $P \leq N_G(M_p) < G$ since $|D| = |N| = |O_p(G)| < |M_p|$. Then $N_G(M_p)$ is p-nilpotent, and so M satisfies the hypothesis of the theorem. By minimal choice of G, we have M is p-nilpotent, and thus G is p-nilpotent, a contradiction.

Step 7. The final contradiction.

By Step 6, we have that G/N is p-nilpotent for any minimal normal subgroup N of G. It is easy to see that N is a unique minimal normal subgroup of G, and so $\Phi(G) = 1$ and $N = O_p(G)$. Since $\Phi(G) = 1$, there is a maximal subgroup M of G such that $G = N : M$. Clearly, $M = M_pM_{p'}$ which is p-nilpotent. Let H be a maximal subgroup of M_p. Then the subgroup $HM_{p'}N$ has index p in G, thus $HM_{p'}N \trianglelefteq G$. By Step 4, we have $HN \leq O_p(G) = N$. This means that $H = 1$ and $|D| = |N|$, and thus $|P : D| = p$, contrary to Step 3, a final contradiction.

This finishes the proof of the theorem. □

Theorem 3.5 is a generalization of Asaad's result [12, Theorem 3.5] in odd order case.

Corollary 3.6 — Let G be an odd order group and let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup D such that $1 \leq |D| < |P|$ and all subgroups H of P with $|H| = p^n|D|$ (n = 0,1) are c-supplemented in G. Then G is p-nilpotent.

PROOF : Clearly, $O_{p'}(G) = 1$. By Lemma 2.10, we may assume that $|D| > 1$. If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent. Then G is p-nilpotent by Theorem 3.5. Hence we may suppose that P is normal in G. By Shur-Zassenhaus' theorem, there is a Hall p'-subgroup M such that $G = P : M$. Let q be any prime divisor of $|M|$, $Q \in Syl_q(M)$. If $PQ < G$, by induction, we have PQ is p-nilpotent, and thus $PQ = P \times Q$. Consequently, $G = P \times M$ which is p-nilpotent. Thus we may suppose that $G = P : Q$, and $Q \cong \mathbb{Z}_q$ since $O_{p'}(G) = 1$.

On the other hand, suppose that $|\Phi(P)| \geq |D|$. Let N be a minimal normal subgroup of G contained in $\Phi(P)$. By Lemma 2.3 and induction, we have $|N| = |D|$. Then we consider the factor group G/N. Each minimal subgroup of P/N is c-supplemented in G/N. By the minimal choice of G, we have G/N is p-nilpotent, and so G is p-nilpotent, as desired.

Suppose that $|\Phi(P)| < |D|$. Induction forces that $\Phi(P) = 1$. By Maschke's theorem [16, P.39], we can write $P = N_1 \times N_2 \times \cdots \times N_s$, $N_i's$ are some minimal normal subgroups of G, where s is a positive integer. By Lemma 2.3, we have

$s \geq 2$ and $|N_i| \leq |D|$. Clearly, the hypotheses of the corollary are inherited by G/N_i for each i. By induction, we have that G/N_i is p-nilpotent. Since $s \geq 2$, we have G is p-nilpotent.

This finishes the proof of the corollary. □

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