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A GENERALIZATION OF THE SYMMETRY BETWEEN COMPLETE AND ELEMENTARY SYMMETRIC FUNCTIONS

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A generalization for the symmetry between complete symmetric functions and elementary symmetric functions is given. As corollaries we derive the inverse of a triangular Toeplitz matrix and the expression of the Toeplitz-Hessenberg determinant. A very large variety of identities involving integer partitions and multinomial coefficients can be generated using this generalization. The partitioned binomial theorem and a new formula for the partition function $p(n)$ are obtained in this way.

Key words : Complete symmetric functions; elementary symmetric functions; multinomial coefficients; integer partitions.

1. Introduction

Let k and n be two positive integers. The k-th elementary symmetric function

$$e_k = e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k} \ ,$$

and the k-th complete homogeneous symmetric function

$$h_k = h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}$$

are well-known [5]. In particular, we have $e_0 = 1$ and $h_0 = 1$. For $k > n$ or $k < 0$, we set $e_k = 0$ and $h_k = 0$.

The following relations

$$e_k = \det \left(h_{i-j+1} \right)_{1 \le i,j \le k} \qquad \text{and} \qquad h_k = \det \left(e_{i-j+1} \right)_{1 \le i,j \le k}$$

are well-known. The first is a special case of Jacobi-Trudi identity [4, eq. 0.2] and the second is a special case of Nägelsbach-Kostka identity [4, eq. 0.3]. On the other hand, the relations

$$\begin{aligned} h_k &= \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+\cdots+t_k} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} e_1^{t_1} e_2^{t_2} \cdots e_k^{t_k} , \\ e_k &= \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+\cdots+t_k} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k}, \end{aligned}$$

where

$$\binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} = \frac{(t_1+\cdots+t_k)!}{t_1! \cdots t_k!}$$

is the multinomial coefficient, can be found in [6, pp. 3-4]. Thus, we can write

$$\det \left(f_{i-j+1} \right)_{1 \le i,j \le k} = \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+\cdots+t_k} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} f_1^{t_1} f_2^{t_2} \cdots f_k^{t_k} , \tag{1.1}$$

where f is any of these complete or elementary symmetric functions.

The starting point of this paper is the following generalization for the symmetry between complete symmetric functions and elementary symmetric functions:

***Theorem* 1** — *Let $\{a_n\}_{n \ge 0}$ and $\{b_n\}_{n \ge 0}$ be two sequences such that*

$$\sum_{k=0}^{n} (-1)^k a_k b_{n-k} = \delta_{0,n} , \tag{1.2}$$

where $\delta_{i,j}$ is the Kronecker's delta. Then

$$\frac{a_n}{a_0} = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n+t_1+\cdots+t_n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} \left(\frac{b_1}{b_0} \right)^{t_1} \cdots \left(\frac{b_n}{b_0} \right)^{t_n}$$

and

$$\frac{b_n}{b_0} = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n+t_1+\cdots+t_n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} \left(\frac{a_1}{a_0}\right)^{t_1} \cdots \left(\frac{a_n}{a_0}\right)^{t_n} .$$

The inverse of a triangular Toeplitz matrix is given by

Corollary 1 — Let A be a square matrix defined by

$$A = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & \ddots & \ddots & \\ a_n & \ldots & a_1 & a_0 \end{bmatrix}, \text{ with } a_0 \neq 0. \text{ Then } \quad A^{-1} = \begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_n & \ldots & b_1 & b_0 \end{bmatrix},$$

where $b_0 = 1/a_0$ and

$$b_k = \frac{1}{a_0} \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{t_1+\cdots+t_k} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} \left(\frac{a_1}{a_0}\right)^{t_1} \cdots \left(\frac{a_k}{a_0}\right)^{t_k} ,$$

for $k \in \{1,\ldots,n\}$.

This corollary is immediate from Theorem 1 because in accordance with the relation (1.2), we can write

$$\begin{pmatrix} a'_0 & & & \\ a'_1 & a'_0 & & \\ \vdots & \ddots & \ddots & \\ a'_n & \ldots & a'_1 & a'_0 \end{pmatrix} \begin{pmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_n & \ldots & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_n & \ldots & b_1 & b_0 \end{pmatrix} \begin{pmatrix} a'_0 & & & \\ a'_1 & a'_0 & & \\ \vdots & \ddots & \ddots & \\ a'_n & \ldots & a'_1 & a'_0 \end{pmatrix} = I,$$

where $a'_k = (-1)^k a_k$ for $0 \leq k \leq n$ and I is the identity matrix. On the other hand, the relation (1.2) can be rewritten as

$$\begin{pmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_n & \ldots & b_1 & b_0 \end{pmatrix} \begin{pmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Applying Cramer's rule to this matrix equation, we obtain

$$a'_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & & & 1 \\ b_1 & b_0 & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ b_n & \dots & b_1 & 0 \end{vmatrix} = (-1)^n \frac{1}{b_0^{n+1}} \begin{vmatrix} b_1 & b_0 & & \\ b_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & b_0 \\ b_n & \dots & b_2 & b_1 \end{vmatrix}.$$

Thus, by Theorem 1, we deduce the expression of the Toeplitz-Hessenberg determinant:

Corollary 2 — For $n > 0$

$$\begin{vmatrix} a_1 & a_0 & & \\ a_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & a_0 \\ a_n & \dots & a_2 & a_1 \end{vmatrix} = \sum_{t_1+2t_2+\cdots+nt_n=n} (-a_0)^{n-t_1-\cdots-t_n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}.$$

This corollary is known as Trudi's formula and the case $a_0 = 1$ of this formula is known as Briochi's formula [7, Vol. 3, Ch. VII]. We remark that the relation (1.1) is a special case of Corollary 2. Moreover, the next corollary follows immediately from Theorem 1 and Corollary 2.

Corollary 3 — Let $\{a_n\}_{n\geq 0}$ be a sequences such that $a_0 \neq 0$. If the sequence $\{b_n\}_{n\geq 0}$ is defined by

$$b_0 = \frac{1}{a_0} \quad \text{and} \quad b_n = \frac{1}{a_0^{n+1}} \begin{vmatrix} a_1 & a_0 & & \\ a_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & a_0 \\ a_n & \dots & a_2 & a_1 \end{vmatrix} \quad \text{then} \quad a_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_1 & b_0 & & \\ b_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & b_0 \\ b_n & \dots & b_2 & b_1 \end{vmatrix}.$$

Recall that the invert transform of a sequence $\{a_n\}_{n>0}$ is a sequence $\{b_n\}_{n>0}$ defined by

$$1 + \sum_{n>0} b_n x^n = \frac{1}{1 - \sum_{n>0} a_n x^n}$$

and its combinatorial interpretations can be seen in [2]. We remark that the invert transform of the sequence $\{(-1)^{n+1}a_n\}_{n>0}$ is the case $a_0 = 1$ in Theorem 1.

Theorem 1 generates a very large variety of identities involving integer partitions and multinomial coefficients. Some examples are presented in the last section of this paper.

2. Proof of Theorem 1

We note that $a_0 b_0 = 1$ and then we consider

$$A(x) = \frac{1}{a_0}\sum_{n\geq 0} a_n x^n \qquad \text{and} \qquad B(x) = \frac{1}{b_0}\sum_{n\geq 0} b_n x^n \ .$$

Clearly

$$A(-x)B(x) = 1 \ .$$

We have

$$\begin{aligned} A(-x) &= \frac{1}{1 - \dfrac{1}{b_0}\displaystyle\sum_{n\geq 1}(-b_n)x^n} \\ &= 1 + \sum_{j\geq 1}\left(\frac{1}{b_0}\sum_{n\geq 1}(-b_n)x^n\right)^j . \end{aligned} \tag{2.1}$$

For each j, the coefficient of x^n in

$$\left(\frac{1}{b_0}\sum_{n\geq 1}(-b_n)x^n\right)^j$$

is

$$\sum_{\substack{t_1+\cdots+t_n=j \\ t_1+2t_2+\cdots+nt_n=n}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}\left(-\frac{b_1}{b_0}\right)^{t_1}\cdots\left(-\frac{b_n}{b_0}\right)^{t_n}$$

So the coefficient of x^n in the right side of (2.1) is

$$\sum_{j\geq 1}\sum_{\substack{t_1+\cdots+t_n=j \\ t_1+2t_2+\cdots+nt_n=n}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}\left(-\frac{b_1}{b_0}\right)^{t_1}\cdots\left(-\frac{b_n}{b_0}\right)^{t_n}$$

or

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} \left(-\frac{b_1}{b_0}\right)^{t_1} \cdots \left(-\frac{b_n}{b_0}\right)^{t_n} .$$

The first identity is proved. The second identity follows easily from the relation

$$\sum_{k=0}^{n} (-1)^k b_k (-1)^n a_{n-k} = \delta_{0,n}$$

and from the first identity.

3. Some Applications

In this section, we use Theorem 1 to prove some identities involving integer partitions and multinomial coefficients. We start with an interesting version of the well-known binomial theorem.

3.1 Partitioned binomial theorem

Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences defined by

$$a_n = \begin{cases} -y, & \text{if } n=0, \\ x, & \text{otherwise} \end{cases}$$

and

$$b_n = \begin{cases} -\frac{1}{y}, & \text{if } n=0, \\ (-\frac{1}{y})^{n+1} x(x+y)^{n-1}, & \text{otherwise,} \end{cases}$$

with $y \neq 0$. It is an easy exercise to show that

$$\sum_{k=0}^{n} (-1)^k a_k b_{n-k} = \delta_{0,n}$$

According to Theorem 1, we obtain.

Corollary 4 — [The partitioned binomial theorem] For $n > 0$

$$(x+y)^{n-1} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} x^{t_1+\cdots+t_n-1} y^{n-t_1-\cdots-t_n} .$$

In the case $x = -1$ and $y = 1$ by Corollary 1, we have

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & 1 & & \\ -1 & -1 & -1 & 1 & \\ -1 & -1 & -1 & -1 & 1 \\ \cdots & & & & \end{bmatrix}_{n\times n}^{-1} = \begin{bmatrix} 1 & & & & \\ 2^0 & 1 & & & \\ 2^1 & 2^0 & 1 & & \\ 2^2 & 2^1 & 2^0 & 1 & \\ 2^3 & 2^2 & 2^1 & 2^0 & 1 \\ \cdots & & & & \end{bmatrix}_{n\times n} .$$

By Corollary 3, we obtain

$$\begin{vmatrix} x & -y & & \\ x & x & -y & \\ x & x & x & -y \\ x & x & x & x \\ \cdots & & & \end{vmatrix}_{n\times n} = x(x+y)^{n-1}.$$

By Corollary 4, we easily deduce

Corollary 5 — For $n > 0$

$$(x+y)^n = y^n + \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{n}{t_1+\cdots+t_n}\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} x^{t_1+\cdots+t_n} y^{n-t_1-\cdots-t_n} .$$

3.2 Sequence of natural numbers

Let $\{a_n\}_{n\geq 0}$ be a positive integer sequence defined by

$$a_n = \begin{cases} 1, & \text{if } n = 0 \\ n, & \text{otherwise.} \end{cases}$$

Because

$$a_n - a_{n-1} - a_{n-2} + a_{n-4} + a_{n-5} - a_{n-7} - a_{n-8} + \cdots = \delta_{0,n} ,$$

we consider the sequence $\{b_n\}_{n\geq 0}$ defined by

$$b_n = \begin{cases} 1, & \text{if } n = 0 \\ n - 3\left\lfloor \frac{n+1}{3} \right\rfloor, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Applying Theorem 1, we obtain the identity

$$\sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_{3k}=0}} (-1)^{n+t_1+t_4+t_7+\cdots} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = n$$

that can be written as

Corollary 6 — For $n > 0$

$$\sum_{t_1q_1+t_2q_2+\cdots+t_nq_n=n} (-1)^{n+t_1+t_3+t_5+\cdots} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = n \, ,$$

where

$$q_k = \left\lfloor \frac{3k-1}{2} \right\rfloor .$$

By Corollary 1, we get

$$\begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 2 & 1 & 1 & & & & & \\ 3 & 2 & 1 & 1 & & & & \\ 4 & 3 & 2 & 1 & 1 & & & \\ 5 & 4 & 3 & 2 & 1 & 1 & & \\ 6 & 5 & 4 & 3 & 2 & 1 & 1 & \\ \cdots & & & & & & & \end{bmatrix}_{n\times n}^{-1} = \begin{bmatrix} 1 & & & & & & & \\ -1 & 1 & & & & & & \\ -1 & -1 & 1 & & & & & \\ 0 & -1 & -1 & 1 & & & & \\ 1 & 0 & -1 & -1 & 1 & & & \\ 1 & 1 & 0 & -1 & -1 & 1 & & \\ 0 & 1 & 1 & 0 & -1 & -1 & 1 & \\ \cdots & & & & & & & \end{bmatrix}_{n\times n} ,$$

where on the diagonals of the right side, the matrix element is

$$(-1)^k \left(k - 3 \left\lfloor \frac{k+1}{3} \right\rfloor \right) , \quad k > 0$$

(the label of the first row is zero).

For $n > 0$ by Corollary 3, we have

$$\begin{vmatrix} 1 & 1 & & & & \\ 2 & 1 & 1 & & & \\ 3 & 2 & 1 & 1 & & \\ 4 & 3 & 2 & 1 & 1 & \\ 5 & 4 & 3 & 2 & 1 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ \cdots & & & & & \end{vmatrix}_{n\times n} = n - 3\left\lfloor \frac{n+1}{3} \right\rfloor$$

and

$$\begin{vmatrix} 1 & 1 & & & & \\ -1 & 1 & 1 & & & \\ 0 & -1 & 1 & 1 & & \\ 1 & 0 & -1 & 1 & 1 & \\ -1 & 1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ \cdots & & & & & \end{vmatrix}_{n\times n} = n.$$

3.3 Square numbers

We consider the sequence $\{a_n\}_{n\geq 0}$ defined by

$$a_n = (n+1)^2.$$

The recurrence relation

$$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$$

is well-known [8, A000290]. It is immediate from this recurrence that

$$a_n - 4a_{n-1} + 7a_{n-2} + 8\sum_{k=3}^{n}(-1)^k a_{n-k} = \delta_{0,n}. \tag{3.1}$$

The following result is a consequence of Theorem 1.

Corollary 7 — Let n be a positive integer. Then

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n+t_1+\cdots+t_n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} 4^{t_1}7^{t_2}8^{t_3+\cdots+t_n} = (n+1)^2.$$

By Corollary 1, we get

$$\begin{bmatrix} 1 \\ 2^2 & 1 \\ 3^2 & 2^2 & 1 \\ 4^2 & 3^2 & 2^2 & 1 \\ 5^2 & 4^2 & 3^2 & 2^2 & 1 \\ 6^2 & 5^2 & 4^2 & 3^2 & 2^2 & 1 \\ \ldots \end{bmatrix}_{n\times n}^{-1} = \begin{bmatrix} 1 \\ -4 & 1 \\ 7 & -4 & 1 \\ -8 & 7 & -4 & 1 \\ 8 & -8 & 7 & -4 & 1 \\ -8 & 8 & -8 & 7 & -4 & 1 \\ \ldots \end{bmatrix}_{n\times n},$$

where on the diagonals of the right side, the matrix element is $(-1)^k \cdot 8$ for $k > 2$ (the label of the first row is zero).

By Corollary 3, we have

$$\begin{vmatrix} 4 & 1 \\ 7 & 4 & 1 \\ 8 & 7 & 4 & 1 \\ 8 & 8 & 7 & 4 & 1 \\ 8 & 8 & 8 & 7 & 4 \\ \ldots \end{vmatrix}_{n\times n} = (n+1)^2 \qquad \text{and} \qquad \begin{vmatrix} 2^2 & 1 \\ 3^2 & 2^2 & 1 \\ 4^2 & 3^2 & 2^2 & 1 \\ 5^2 & 4^2 & 3^2 & 2^2 & 1 \\ 6^2 & 5^2 & 4^2 & 3^2 & 2^2 \\ \ldots \end{vmatrix}_{n\times n} = 8.$$

The last identity is true only if $n > 2$.

3.4 Catalan numbers

In terms of binomial coefficients, the nth Catalan number is given directly by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively defined objects [3, Section 1.8]. For instance, Segner developed in 1761 a recursive formula for C_n using the triangulation problem:

$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k},\ n \geq 0.$$

for $n \geq 0$. According to this recurrence relation and Theorem 1, for

$$a_k = \begin{cases} 1, & \text{if } k = 0, \\ (-1)^{k-1} C_{k-1}, & \text{otherwise} \end{cases} \qquad \text{and} \qquad b_k = C_k,$$

we obtain the following convolution identity for Catalan numbers

Corollary 8 — Let n be a positive integer. Then

$$C_{n-1} = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{1+t_1+\cdots+t_n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} C_1^{t_1} C_2^{t_2} \cdots C_n^{t_n}.$$

By Corollary 1, we get

$$\begin{bmatrix} C_0 & & & & \\ C_1 & C_0 & & & \\ C_2 & C_1 & C_0 & & \\ C_3 & C_2 & C_1 & C_0 & \\ C_4 & C_3 & C_2 & C_1 & C_0 \\ \cdots & & & & \end{bmatrix}_{n\times n}^{-1} = \begin{bmatrix} 1 & & & & \\ -C_0 & 1 & & & \\ -C_1 & -C_0 & 1 & & \\ -C_2 & -C_1 & -C_0 & 1 & \\ -C_3 & -C_2 & -C_1 & -C_0 & 1 \\ \cdots & & & & \end{bmatrix}_{n\times n}$$

and by Corollary 3, we can write

$$C_{n-1} = (-1)^{n-1} \begin{vmatrix} C_1 & C_0 & & & \\ C_2 & C_1 & C_0 & & \\ C_3 & C_2 & C_1 & C_0 & \\ C_4 & C_3 & C_2 & C_1 & C_0 \\ C_5 & C_4 & C_3 & C_2 & C_1 \\ \cdots & & & & \end{vmatrix}_{n\times n}.$$

3.5 Fibonacci numbers

Recall that the sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers [3, Section 2.6] is defined by the recurrence relation

$$F_n - F_{n-1} - F_{n-2} = 0\ ,$$

with seed values $F_0 = 0$ and $F_1 = 1$. We consider two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ defined by

$$a_n = \begin{cases} 1, & \text{if } n = 0 \\ n \bmod 2, & \text{otherwise} \end{cases} \qquad \text{and} \qquad b_n = \begin{cases} 1, & \text{if } n = 0 \\ F_n, & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that

$$b_n - b_{n-1} - b_{n-3} - b_{n-5} - \cdots = \delta_{0,n}.$$

According to Theorem 1, the Fibonacci number F_n can be expressed as a sum over integer partitions with odd parts.

Corollary 9 — Let n be a positive integer. Then

$$F_n = \sum_{t_1 q_1 + t_2 q_2 + \cdots + t_{\lceil \frac{n}{2} \rceil} q_{\lceil \frac{n}{2} \rceil} = n} \binom{t_1 + \cdots + t_{\lceil \frac{n}{2} \rceil}}{t_1, \ldots, t_{\lceil \frac{n}{2} \rceil}},$$

where $q_k = 2k - 1$ and $\lceil x \rceil$ denotes the smallest integer not less than x.

By Corollary 1, we have

$$\begin{bmatrix} 1 & & & & \\ F_1 & 1 & & & \\ F_2 & F_1 & 1 & & \\ F_3 & F_2 & F_1 & 1 & \\ F_4 & F_3 & F_2 & F_1 & 1 \\ \ldots & & & & \end{bmatrix}_{n\times n}^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ -1 & 0 & -1 & 1 & \\ 0 & -1 & 0 & -1 & 1 \\ \ldots & & & & \end{bmatrix}_{n\times n}.$$

In accordance with Corollary 3, we can write

$$\begin{vmatrix} F_1 & 1 & & & \\ F_2 & F_1 & 1 & & \\ F_3 & F_2 & F_1 & 1 & \\ F_4 & F_3 & F_2 & F_1 & 1 \\ F_5 & F_4 & F_3 & F_2 & F_1 \\ \ldots & & & & \end{vmatrix}_{n\times n} = n \bmod 2 \qquad \text{and} \qquad \begin{vmatrix} 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 1 & 0 & 1 & 1 & \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ \ldots & & & & \end{vmatrix}_{n\times n} = F_n.$$

3.6 Euler's partition function

The pentagonal number theorem relates the product and the series representations of the Euler function,

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}.$$

In other words,

$$(1-x)(1-x^2)(1-x^3)\cdots = \sum_{k=0}^{\infty}(-1)^{\lceil \frac{k}{2} \rceil} x^{q_k} ,$$

where the exponents q_k are called generalized pentagonal numbers. By the pentagonal number theorem, we have

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{\prod_{n=1}^{\infty}(1-x^n)} = \frac{1}{\sum_{n=0}^{\infty}(-1)^{\lceil \frac{n}{2} \rceil} x^{q_n}} .$$

We consider the sequence $\{a_n\}_{n\geq 0}$ defined by

$$a_n = p(n)$$

and the sequence $\{b_n\}_{n\geq 0}$ where b_n is the coefficient of x^n in the Euler function

$$(x)_\infty = \prod_{n=1}^{\infty}(1-x^n) ,$$

i.e.,

$$b_n = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2}(3m^2 \pm m), m \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} .$$

It is clear that

$$\sum_{k=0}^{n}(-1)^k a_k b_{n-k} = \delta_{0,n} .$$

Taking into account that

$$b_{q_k} = (-1)^{\lceil \frac{k}{2} \rceil} ,$$

a new formula for $p(n)$ follows easily from Theorem 1.

Corollary 10 — Let n be a positive integer. Then

$$p(n)=\sum_{t_1q_1+t_2q_2+\cdots+t_nq_n=n}(-1)^{t_3+t_4+t_7+t_8+\cdots}\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n},$$

where

$$q_k=\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left\lceil\frac{3k+1}{2}\right\rceil$$

are generalized pentagonal numbers and $\lceil x\rceil$ denotes the smallest integer not less than x.

According to Corollary 1, we can write

$$\begin{bmatrix} 1 \\ p(1) & 1 \\ p(2) & p(1) & 1 \\ p(3) & p(2) & p(1) & 1 \\ p(4) & p(3) & p(2) & p(1) & 1 \\ p(5) & p(4) & p(3) & p(2) & p(1) & 1 \\ \ldots \end{bmatrix}_{n\times n}^{-1}=\begin{bmatrix} 1 \\ -1 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ \ldots \end{bmatrix}_{n\times n},$$

where on these diagonals, the matrix element is $(-1)^{\lceil\frac{k}{2}\rceil}$ or 0. The only non-zero diagonals of this matrix are those which start on a row labeled by a generalized pentagonal number q_k. The label of the first row is 0.

Finally, by Corollary 3, we have

$$\begin{vmatrix} p(1) & 1 \\ p(2) & p(1) & 1 \\ p(3) & p(2) & p(1) & 1 \\ p(4) & p(3) & p(2) & p(1) & 1 \\ p(5) & p(4) & p(3) & p(2) & p(1) \\ \ldots \end{vmatrix}_{n\times n}=(-1)^n\cdot b_n$$

and

$$p(n) = \begin{vmatrix} 1 & 1 & & & & & & \\ -1 & 1 & 1 & & & & & \\ 0 & -1 & 1 & 1 & & & & \\ 0 & 0 & -1 & 1 & 1 & & & \\ -1 & 0 & 0 & -1 & 1 & 1 & & \\ 0 & -1 & 0 & 0 & -1 & 1 & 1 & \\ -1 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 1 \\ \dots & & & & & & & \end{vmatrix}_{n\times n}$$

where on these diagonals, the matrix element is $(-1)^k \cdot b_k$. The label of the first row is 1.

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