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SOLUTIONS OF RICCATI-ABEL EQUATION IN TERMS OF THIRD ORDER TRIGONOMETRIC FUNCTIONS

Robert M. Yamaleev

Joint Institute for Nuclear Research, Laboratory of Informational Technologies, Dubna, Russia

e-mail: yamaleev@jinr.ru

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Solutions of the generalized Riccati equations with third order nonlinearity, named as Riccati-Abel equation, are expressed via third order trigonometric functions. It is shown, as the ordinary Riccati equation, also the Riccati-Abel equation has a relationship with a linear differential equations. A summation formula for solutions of Riccati-Abel equation is established. Possible applications of this formula in the generalized dynamics is outlined. The method admits an extension to the case of generalized Riccati equations with any order of nonlinearity

Key words : Complex algebra; trigonometry; polynomial; summation formula; Riccati-Abel equations.

1. INTRODUCTION

Consider the first order differential equation

$$f(u,x) = \frac{du}{dx}. \tag{1.1}$$

If we approximate $f(u,x)$, while x is kept constant, we will get

$$Q_0(x) + Q_1(x)u + Q_2(x)u^2 + Q_3(x)u^3 + \cdots = \frac{du}{dx}. \tag{1.2}$$

When the series in the left-hand side is restricted with a second order polynomial, the equation is the Riccati equation [1].

The Riccati equation is one of the widely used equations of mathematical physics. The ordinary Riccati equations are closely related to the second order linear differential equations. For the solutions of the ordinary Riccati equations with constant coefficients, a summation formula can be derived. These solutions are presented by trigonometric functions induced by general complex algebra.

In particular, if $f(u,x)$ is a cubic polynomial, then the equation is called Riccati-Abel equation. Abel's original equation was written in the form

$$(y+s)\frac{dy}{dx}+p+qy+ry^2=0. \tag{1.3}$$

This equation is converted into Riccati-Abel equation by transformation $y+s=1/z$, which yields

$$\frac{dz}{dx}=rz+(q-s'-2rs)z^2+(p-qs+rs^2)z^3. \tag{1.4}$$

It is seen that the case $Q_0(u,\phi)=0$ was actually considered by Abel [2].

When the series in the left-hand side of equation (1.2) is given by the n-order polynomial, we deal with the generalized Riccati equations. The solution of the generalized Riccati equation with constant coefficients can be denominated as a *generalized tangent function*. The generalized Riccati equations are used, for example, in various problems of renorm-group theory [5]. The mean field free energy concept and the perturbation renormalization group theory deal with first order differential equations with polynomial non-linearity.

The fact that the solutions of special kind of Abel's nonlinear differential equation can be expressed via third order hyperbolic functions has been found in [3]. The concept of third order hyperbolic functions arose within the framework of third order *multicomplex algebra* with unique generator E defined by the cubic equation $x^3=\pm 1$ (see, for instance, [4] and references therein). In the present paper we also use the third order trigonometric functions. However, our approach quite different of the results of the Vein's paper. Firstly, we use the trigonometric functions which arise within the general complex algebra of third order where the generator is defined by general cubic equation (see, eq.(4.1)), and secondly, we will represent the solutions of Riccati-Abel equation in a completely different form.

The aim of this paper is to explore solutions of the Riccati-Abel equation with constant coefficients and to derive some kind of summation formula for them. Summation (addition) formulae for solutions of linear differential equations are considered as important features of these functions. Let us mention, for example, a summation formulae for the trigonometric sine-cosine functions, the Bessel functions, the hypergeometric functions and their various generalizations. Whereas solutions of the linear differential equations with constant coefficients admit universal methods of obtaining summation formulas (see, for instance, [6, 7]), the solutions of nonlinear equations require special investigations. In this context let us mention the addition formulae for Jacobi and Weierstrass elliptic functions [8].

In general, the solutions of the generalized Riccati equations with cubic and higher polynomials do not admit any summation formula. Nevertheless, by careful analysis we found a new summation law according to which in order to obtain a summation formula for the solutions of the third order Riccati equation two independent variables should be used. In this way we will establish an interconnection between solutions of Riccati-Abel equation and the characteristic functions of generalized complex algebra of third order.

The paper is presented by the following sections. Section 2 deals with solution of ordinary Riccati equation with constant coefficients. Summation formula for the solutions is derived and interrelation with solutions of the linear differential equations is underlined. In Section 3, the Riccati-Abel equation is integrated, a corresponding algebraic equation for solutions is derived, a summation formula for solutions is established. In Section 4, the solutions of Riccati-Abel equation are constructed within generalized complex algebra of third order. In Section 5, it is shown that the Riccati-Abel equation is an evolution equation of the generalized classical dynamics.

2. Ordinary Riccati Equation, Summation Formula and General Complex Algebra

2.1 The ordinary Riccati equation

Consider the Riccati equation with constant coefficients

$$u^2 - a_1 u + a_0 = \frac{du}{d\phi}. \tag{2.1}$$

If coefficients a_0, a_1 are constants then a great simplification results because it is possible to obtain the complete solution by means of quadratures. Thus, equation (2.1) admits direct integration

$$\int \frac{dx}{x^2 - a_1 x + a_0} = \int d\phi. \tag{2.2}$$

Let $x_1, x_2 \in C$ be roots of the polynomial equation

$$x^2 - a_1 x + a_0 = 0. \tag{2.3}$$

In order to calculate the integral (2.2), the following formula expansion is used:

$$\frac{1}{x^2 - a_1 x + a_0} = \frac{1}{2x_1 - a_1}\frac{1}{x - x_1} + \frac{1}{2x_2 - a_1}\frac{1}{x - x_2}, \tag{2.4}$$

where,

$$2x_1 - a_1 = (x_1 - x_2), \;\; 2x_2 - a_1 = (x_2 - x_1).$$

Then the integral (2.2) is easily calculated and the result is given by the logarithmic functions

$$\int_w^u \frac{dx}{x^2 - a_1 x + a_0} = \frac{1}{m_{12}}(\;\log\frac{u - x_1}{u - x_2} - \;\log\frac{w - x_1}{w - x_2}\;) = \phi(u) - \phi(w), \tag{2.5}$$

where $m_{12} = x_1 - x_2$. Now, let us keep the first logarithm of (2.5) depending on the initial limit of the integral, that is

$$\frac{1}{m_{12}} \log[\frac{u - x_1}{u - x_2}] = \phi(u).$$

By inverting the logarithm function, we come to the algebraic equation for solution of (2.1),

$$\exp(m_{12}\phi) = \frac{u - x_1}{u - x_2}. \tag{2.6}$$

Let $u(\phi_0) = 0$, then

$$\exp(m_{12}\phi_0) = \frac{x_1}{x_2}. \tag{2.7}$$

As soon as the point $\phi = \phi_0$ is determined, one may calculate the function $u(\phi)$ by making use of algebraic equation (2.6). Since $a_1 = x_1 + x_2$, from (2.7) it follows that

$$a_1 = m_{12} \coth(m_{12}\phi_0/2).$$

Consequently, from (2.6) we obtain

$$u(\phi,\phi_0)=\frac{1}{2}m_{12}\coth(m_{12}\phi_0/2)-\frac{1}{2}m_{12}\coth(m_{12}\phi/2).$$

2.2 Summation (addition) formula for function $u=u(\phi,\phi_0)$.

Consider the following integral equation

$$\int^u \frac{dx}{x^2-a_1x+a_0}+\int^v \frac{dx}{x^2-a_1x+a_0}=\int^w \frac{dx}{x^2-a_1x+a_0}. \tag{2.8}$$

The quantity w is a function of u and v. If the function $w=f(u,v)$ is an algebraic function then this function can be considered as a summation formula. Write (2.8) in the following notations $\phi_u+\phi_v=\phi_w$. Then, $w(\phi_w)=w(\phi_u+\phi_v)=f(u(\phi_u),\ v(\phi_v)\ )$.

Calculating the integrals in (2.8), we come to the following algebraic equation

$$\frac{1}{2m}\log\frac{u-x_1}{u-x_2}\frac{v-x_1}{v-x_2}=\frac{1}{2m}\log\frac{w-x_1}{w-x_2}. \tag{2.9}$$

Thus, the function $w(u,v)$ has to satisfy the equation

$$\frac{u-x_1}{u-x_2}\frac{v-x_1}{v-x_2}=\frac{w-x_1}{w-x_2}. \tag{2.10}$$

Multiplying fractions and taking into account the fact that x_1,x_2 obey (2.3), we get

$$\frac{uv-x_1(u+v)+a_1x_1-a_0}{uv-x_2(u+v)+a_1x_2-a_0}=\frac{\frac{uv-a_0}{u+v-a_1}-x_1}{\frac{uv-a_0}{u+v-a_1}-x_2}=\frac{w-x_1}{w-x_2},w=\frac{uv-a_0}{u+v-a_1}. \tag{2.11}$$

This is a summation formula for solutions of the Riccati equation (2.1).

2.3 Relationship with General complex algebra

Like the cotangent function can be defined as a ratio of cosine and sine functions, the solution of the Riccati equation $u(\phi,\phi_0)$ also can be represented as a ratio of modified cosine and sine functions. Firstly, let us construct these functions.

Consider general complex algebra generated by the (2×2) matrix [9]

$$E=\begin{pmatrix} 0 & -a_0 \\ 1 & a_1 \end{pmatrix} \tag{2.12}$$

obeying the quadratic equation (2.3):

$$E^2 - a_1 E + a_0 I = 0, \tag{2.13}$$

with I-unit matrix. Expansion with respect to E of the exponential function $\exp(E\phi)$ leads to the Euler formula [10]

$$\exp(E\phi) = g_1(\phi; a_0, a_1)E + g_0(\phi; a_0, a_1). \tag{2.14}$$

In terms of the roots x_1, x_2 this matrix equation is separated into two equations

$$\exp(x_2\phi) = x_2\ g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1), \quad \exp(x_1\phi) = x_1\ g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1), \tag{2.15}$$

from which an explicit form of g-functions can be obtained. Apparently, g_0 and g_1 are modified (generalized) cosine-sine functions with the following formulas of differentiation

$$\frac{d}{d\phi} g_1(\phi; a_0, a_1) = g_0(\phi; a_0, a_1) + a_1\ g_1(\phi; a_0, a_1), \quad \frac{d}{d\phi} g_0(\phi; a_0, a_1) = -a_0\ g_1(\phi; a_0, a_1). \tag{2.16}$$

Form a ratio of two equations of (2.15) as follows

$$\exp(m_{21}\phi) = \frac{x_2\ g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1)}{x_1\ g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1)}. \tag{2.17}$$

Let $g_1(\phi; a_0, a_1) \neq 0$. Then,

$$\exp(m_{21}\phi) = \frac{x_2 + D}{x_1 + D}, \tag{2.18}$$

where

$$D = \frac{g_0(\phi; a_0, a_1)}{g_1(\phi; a_0, a_1)}.$$

Differential equation for function $D(\phi)$ is obtained by using (2.16):

$$D^2 + a_1 D + a_0 = -\frac{dD}{d\phi}. \tag{2.19}$$

Thus, we have proved that the function

$$u(\phi; a_0, a_1) = -D = -\frac{g_0(\phi; a_0, a_1)}{g_1(\phi; a_0, a_1)} \tag{2.20}$$

obeys the Riccati equation.

Summation formulae for g-functions are well defined (see, for example, Ref. [9]). They are

$$g_0(a+b) = g_0(a)g_0(b) - a_0 g_1(a)g_1(b),$$

$$g_1(a+b) = g_1(a)g_0(b) + g_0(a)g_1(b) + a_1 g_1(a)g_1(b).$$

$$\frac{g_0(a+b)}{g_1(a+b)} = \frac{g_0(a)g_0(b) - a_0 g_1(a)g_1(b)}{g_1(a)g_0(b) + g_0(a)g_1(b) + a_1 g_1(a)g_1(b)}. \tag{2.21}$$

By taking into account (2.20) we get

$$u(a+b) = -\frac{g_0(a+b)}{g_1(a+b)} = \frac{u(a)u(b) - a_0}{u(a) + u(b) - a_1}. \tag{2.22}$$

which coincides with (2.11).

3. Generalized Riccati Equation with Cubic Order Polynomial

3.1 The Riccati-Abel equation

Consider the following non-linear differential equation with constant coefficients

$$u^3 - a_2 u^2 + a_1 u - a_0 = \frac{du}{d\phi}, \tag{3.1}$$

which admits direct integration by

$$\int_w^u \frac{dx}{x^3 - a_2 x^2 + a_1 x - a_0} = \phi(w) - \phi(u). \tag{3.2}$$

This integral is calculated by making use of the well-known method of the partial fractional decomposition [11].

$$\frac{1}{x^3 - a_2 x^2 + a_1 x - a_0} = \frac{1}{(x-x_3)(x-x_2)(x-x_1)}$$

$$= \frac{(x_3 - x_2)}{V}\frac{1}{x-x_1} + \frac{(x_1 - x_3)}{V}\frac{1}{x-x_2} + \frac{(x_2 - x_1)}{V}\frac{1}{x-x_3}, \tag{3.3}$$

where V is the Vandermonde's determinant [12]

$$V = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), \tag{3.4}$$

and the distinct constants $x_1, x_2, x_3 \in C$ are roots of the cubic polynomial

$$f(x) = x^3 - a_2x^2 + a_1x - a_0 = 0. \tag{3.5}$$

By using expansion (3.3) the integral (3.2) is easily calculated

$$\int_w^u \frac{dx}{x^3 - a_2x^2 + a_1x - a_0} =$$

$$\frac{(x_3 - x_2)}{V}\log\frac{u - x_1}{w - x_1} + \frac{(x_1 - x_3)}{V}\log\frac{u - x_2}{w - x_2} + \frac{(x_2 - x_1)}{V}\log\frac{u - x_3}{w - x_3} = \phi(u) - \phi(w). \tag{3.6}$$

Let us introduce the following notations:

$$m_{ij} = (x_i - x_j),\ i, j = 1, 2, 3,\ \text{with } m_{21} + m_{32} + m_{13} = 0, \tag{3.7}$$

and write equation (3.6) as follows:

$$\int^u \frac{dx}{x^3 - a_2x^2 + a_1x - a_0} = \log\ (u - x_1)^{m_{32}}(u - x_2)^{m_{13}}(u - x_3)^{m_{21}} = V\phi(u) \tag{3.8}$$

and invert the logarithm. This leads to the following algebraic equation

$$[u - x_1]^{m_{32}}[u - x_2]^{m_{13}}[u - x_3]^{m_{21}} = \exp(V\phi). \tag{3.9}$$

This equation can also be written in the fractional form

$$[\frac{u - x_1}{u - x_3}]^{m_{32}}[\frac{u - x_2}{u - x_3}]^{m_{13}} = \exp(V\phi). \tag{3.10}$$

Thus, the problem of solution of differential equation (3.1) is reduced to the problem of solution of the algebraic equation (3.10). Notice, if the roots of cubic equation and function u are defined in the field of real numbers, then this equation is meaningful only for a certain domain of definition of $u(\phi)$.

3.2 Semigroup property of fractions of n-order monic polynomials on the set of roots of $n+1$-order polynomial

In this section let us recall a semigroup property of the fractions of n-order polynomials defined on the set of roots of $n+1$-order polynomial. Let $F(x, n+1)$ be $(n+1)$ order polynomial with $(n+1)$ distinct roots $x_i, i = 1, \ldots, n+1$. Denote this set of roots by $FX(n+1)$.

Lemma 3.1 — Let $P_a(x_i, n)$ be the n-order polynomial on $x_i \in FX(n+1)$. The product of two n-order polynomials

$$P_a(x_i, n) * P_b(x_i, n)$$

is also an n-order polynomial $P_c(x_i, n)$.

PROOF : The product $P_{ab}(x_i, 2n) := P_a(x_i, n) * P_b(x_i, n)$ is a polynomial of $2n$-degree with respect to variable x_i. Since x_i obeys $n+1$-order polynomial equation, all monomials with degrees higher than n can be expressed via polynomials of n-degree. Consequently, the polynomial $P_{ab}(x_i, 2n)$ with $x_i \in FX(n+1)$ is reduced into n degree polynomial.

End of proof. Consider two monic polynomials of n-degree $P_a(x_i, n),\ P_b(x_k, n)$ with $x_i \neq x_k \in FX(n+1)$. Form a rational algebraic fraction

$$\frac{P_a(x_i, n)}{P_a(x_k, n)}.$$

The following *Corollary* 3.2 holds true:

The product of two fractions formed by two n-order monic polynomials on the roots of $(n+1)$-order polynomial is a fraction of the same order monic polynomials on the variables,

$$\frac{P_a(x_i, n)}{P_a(x_k, n)} \frac{P_b(x_i, n)}{P_b(x_k, n)} = \frac{P_c(x_i, n)}{P_c(x_k, n)}.$$

3.3 Addition formula for $u(\phi)$

Let $\phi = \phi_0$ be a point where $u(\phi_0) = 0$. Then, (3.10) is reduced to

$$[\frac{x_1}{x_3}]^{m_{32}}[\frac{x_2}{x_3}]^{m_{13}} = \exp(V\phi_0). \tag{3.11}$$

Now, let us make simultaneous translations of the roots $x_k, k = 1, 2, 3$ by some value u. Since the Vandermonde's determinant remains invariant under these translations, the parameter ϕ_0 will undergo some translation by $\phi = \phi_0 + \delta$. In this way one may construct the solution of Riccati-Abel equation (3.1) with initial condition $u(\phi_0) = 0$.

Let the triple u, v, w form a set of solutions of equation (3.1) calculated for three variables $\phi_u, \phi_v, \phi_w = \phi_u + \phi_v$, correspondingly. Then, in accordance with (3.10) we write

$$\exp(V\phi_u)\exp(V\phi_v) = \left\{ \ \left[\frac{u-x_1}{u-x_3}\frac{v-x_1}{v-x_3}\right]^{m_{32}} \left[\frac{u-x_2}{u-x_3}\frac{v-x_2}{v-x_3}\right]^{m_{21}} \ \right\}$$

$$= \left\{ \left(\frac{w-x_1}{w-x_3}\right)^{m_{32}} \left[\frac{w-x_2}{w-x_3}\right]^{m_{21}} \right\} = \exp(V(\phi_u+\phi_v)). \tag{3.12}$$

The problem is to find some rational function expressing w via the pair (u, v), i.e., the function $w = w(u, v)$ has to be a rational function.

Evidently, the method used in the previous section for the ordinary Riccati equation now is not applicable. According to Lemma 3.1 we are able to transform a product of ratios of n-order polynomials into the ratio of n-order polynomials if these polynomials are defined on roots of $n+1$-order polynomial. Thus, we have to seek another way of construction of a summation formula.

Let us present the integral (3.8) as a sum of two integrals by

$$\int^w \frac{dx}{x^3-a_2x^2+a_1x-a_0} = \int^u \frac{dx}{x^3-a_2x^2+a_1x-a_0} + \int^v \frac{dx}{x^3-a_2x^2+a_1x-a_0}$$

$$= \phi = V\log(\,(\,\frac{u-x_1}{u-x_3}\,\frac{v-x_1}{v-x_3}\,)^{m_{32}}\,(\,\frac{u-x_2}{u-x_3}\,\frac{v-x_2}{v-x_3}\,)^{m_{13}}\,). \tag{3.13}$$

In this way we arrive to the following algebraic equation

$$(\,\frac{u-x_1}{u-x_3}\,\frac{v-x_1}{v-x_3}\,)^{m_{32}}(\,\frac{u-x_2}{u-x_3}\,\frac{v-x_2}{v-x_3}\,)^{m_{13}}\,) = \exp(V\phi(u,v)) = \exp(V\phi_u)\exp(V\phi_v). \tag{3.14}$$

Let u, v be solutions of the quadratic equation

$$x^2+tx+s=0,\ t=-(u+v),\ s=uv. \tag{3.15}$$

Then, equation (3.14) is written as

$$[\,\frac{x_1^2+tx_1+s}{x_3^2+tx_3+s}\,]^{m_{32}} * (\,\frac{x_2^2+tx_2+s}{x_3^2+tx_3+s}\,)^{m_{13}}\,) = \exp(V\phi(t,s)). \tag{3.16}$$

Thus from the pair of functions (u, v) we come to another pair (t, s). This pair of functions, in fact, admits a summation rule because the problem is reduced to the task of transformation four-degree polynomial into quadratic polynomial at the solutions of the cubic equation (3.5). Evidently, this task can be easily performed by simple algebraic operations.

Theorem 3.3 — *The following summation formula for solutions of Riccati-Abel equation holds true*

$$(t,s)\bigoplus(v,u)=(r,w),$$

where

$$r=\frac{(a_0-2a_2a_1)-a_1(v+t)+(tu+sv)}{(\ 3a_2^2-a_1)+a_2(v+t)+(s+u+tv)}\quad w=\frac{a_2a_0+(v+t)a_0+su}{(\ 3a_2^2-a_1)+a_2(v+t)+(s+u+tv)}. \tag{3.17}$$

PROOF : Consider product of two monic polynomials

$$(x^2+tx+s)(x^2+vx+u)=x^4+x^3(v+t)+x^2(s+u+tv)+x(tu+vs)+su,$$

where x is one of the roots of cubic equation

$$x^3-a_2x^2+a_1x-a_0=0. \tag{3.18}$$

From the cubic equation (3.18) we are able to express x^3 and x^4 as polynomials of second order as follows

$$x^3=a_2x^2-a_1\ x+a_0,\ \ x^4=(\ 3a_2^2-a_1)x^2+(a_0-a_1a_2)\ x+a_2a_0.$$

Then, the four-degree polynomial on roots of the cubic polynomial is reduced into a polynomial of second order

$$x^4+x^3(v+t)+x^2(s+u+tv)+x(tu+vs)+su=Ax^2+Bx+C, \tag{3.19}$$

where A,B,C do not depend of x.

Since we deal with the ratios of polynomials the coefficients of the quadratic polynomial in (3.19) and polynomials in denominator and in numerator have the same leading coefficient, we are able to return to the ratio of monic polynomials. In this way we come to the relations

$$r=\frac{B}{A},\ \ w=\frac{C}{A}, \tag{3.20}$$

where

$$A=(3a_2^2-a_1)+a_2(v+t)+(s+u+tv),B=(a_0-2a_2a_1)-a_1(v+t)+(tu+sv),$$
$$C=a_2a_0+(v+t)a_0+su. \tag{3.21}$$

End of Proof.

4. Generalized Complex Algebra of Third Order and Solutions of Riccati-Abel Equation

In this section we will establish a relationship between characteristic functions of general complex algebra of third order and solutions of Riccati-Abel equation.

The unique generator E of general complex algebra of third-order, CG_3, is defined by cubic equation [13].

$$E^3 - a_2E^2 + a_1E - a_0 = 0. \tag{4.1}$$

The companion matrix E of the cubic equation (4.1) is given by (3×3) matrix

$$E := \begin{pmatrix} 0 & 0 & a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & a_2 \end{pmatrix}. \tag{4.2}$$

The structure of the companion matrix is derived directly from (4.1). For that purpose present the cubic equation

$$x^3 - a_2x^2 + a_1x - a_0 = 0,$$

as follows

$$x\phi_1 - a_0 = 0,\ x(x - a_2) + a_1 = \phi_1,\ x\phi_2 + a_1 = \phi_1,\ x - a_2 = \phi_2.$$

Define the following vector

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 1 \end{pmatrix},$$

Then the bilinear system of equations is equivalently presented in the matrix form

$$x\Phi = E\Phi,$$

where x now is the eigenvalue of the matrix E, which satisfies the cubic equation (4.1).

Consider the expansion

$$\exp(E\phi_1 + E^2\phi_2) = g_0(\phi_1, \phi_2) + E\ g_1(\phi_1, \phi_2) + E^2\ g_2(\phi_1, \phi_2). \tag{4.3}$$

This is an analogue of the Euler formula for exponential function, the function $g_0(\phi_1, \phi_2)$ is an analogue of cosine function, and $g_k(\phi_1, \phi_2), k = 1, 2$ are extensions of the sine function. It is seen, the characteristic functions of GC_3 algebra depend on the pair of "angles". Correspondingly, for each of them we have a *formulae of differentiation*.

$$\frac{\partial}{\partial \phi_1} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & a_2 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, \tag{4.4}$$

$$\frac{\partial}{\partial \phi_2} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & a_0 & a_0 a_2 \\ 0 & -a_1 & a_0 - a_1 a_2 \\ 1 & a_2 & -a_1 + a_2^2 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{4.5}$$

The semigroup of multiplications of the exponential functions leads to the following *addition formulae for g- functions* [14]

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}_{(\psi_c = \psi_a + \psi_b)} = \begin{pmatrix} g_0 & g_2 a_0 & g_1 a_0 + g_2 a_0 a_2 \\ g_1 & g_0 - g_2 a_1 & -g_1 a_1 + g_2(a_0 - a_1 a_2) \\ g_2 & g_1 + g_2 a_2 & g_0 + g_1 a_2 + g_2(-a_1 + a_2^2) \end{pmatrix}_{\psi_a} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}_{\psi_b}, \tag{4.6}$$

where the sub-indices of the brackets indicate dependence of the g-functions of the pair of variables $\psi_i = (\phi_{1i}, \phi_{2i}), i = a, b, c$.

Introduce two fractions of g-functions by

$$tg = \frac{g_1}{g_2}, \quad sg = \frac{g_0}{g_2}. \tag{4.7}$$

It is seen, these functions are analogues of tangent-cotangent functions. From the addition formulae for g-functions (4.6), the following summation formulae for the general tangent functions are derived.

$$T_0 = \frac{t_0 r_0 + a_0(r_1 + t_1) + a_0 a_2}{r_0 + (t_1 + a_2) r_1 + t_0 + t_1 a_2 + (-a_1 + a_2^2)}, \tag{4.8}$$

$$T_1 = \frac{t_1 r_0 + t_0 r_1 - a_1(r_1 + t_1) + (a_0 - a_1 a_2)\ )}{r_0 + t_0 + a_2(t_1 + r_1) + t_1 r_1 + (-a_1 + a_2^2)}. \tag{4.9}$$

Here the following notations are used

$$T_0(\psi_c)=\frac{g_0(\psi_c)}{g_2(\psi_c)},\ \ T_1(\psi_c)=\frac{g_1(\psi_c)}{g_2(\psi_c)},$$
$$t_0(\psi_a)=\frac{g_0(\psi_a)}{g_2(\psi_a)},\ r_0(\psi_b)=\frac{g_0(\psi_b)}{g_2(\psi_b)}, \tag{4.10}$$
$$t_1(\psi_a)=\frac{g_1(\psi_a)}{g_2(\psi_a)},\ r_1(\psi_b)=\frac{g_1(\psi_b)}{g_2(\psi_b)},$$

and $\psi_i=(\phi_{1i},\phi_{2i}), i=a,b$, $\psi_c=(\phi_{1c}=\phi_{1a}+\phi_{1b},\phi_{2c}=\phi_{2a}+\phi_{2b})$.

Let $x_1,x_2,x_3\in C$ be eigenvalues of E given by distinct values. Then, the matrix equation (4.3) is represented by three separated series $(k=1,2,3)$:

$$\exp(x_k\phi_1+x_k^2\phi_2)=g_0(\phi_1,\phi_2)+x_k\ g_1(\phi_1,\phi_2)+x_k^2\ g_2(\phi_1,\phi_2), \tag{4.11}$$

Form the following ratios for $i\neq k$:

$$\exp((x_i-x_k)\phi_1+(x_i^2-x_k^2)\phi_2)=\frac{g_0(\phi_1,\phi_2)+x_i\ g_1(\phi_1,\phi_2)+x_i^2\ g_2(\phi_1,\phi_2)}{g_0(\phi_1,\phi_2)+x_k\ g_1(\phi_1,\phi_2)+x_k^2\ g_2(\phi_1,\phi_2)}. \tag{4.12}$$

Consider two of these ratios, namely,

$$\exp(m_{13}\phi_1+(x_1^2-x_3^2)\phi_2)=\frac{g_2x_1^2+g_1x_1+g_0}{g_2x_3^2+g_1x_3+g_0}, \tag{4.13a}$$

$$\exp(m_{23}\phi_1+(x_2^2-x_3^2)\phi_2)=\frac{g_2x_2^2+g_1x_2+g_0}{g_2x_3^2+g_1x_3+g_0}. \tag{4.13b}$$

where $m_{ij}=x_i-x_j$. Both sides of equation (4.13a) raise to power m_{32} and both sides of equation (4.13b) raise to power m_{13} and multiply left and right sides of the obtained equations, correspondingly. And, by taking into account that $m_{13}m_{32}+m_{23}m_{13}=0$, we arrive to the following equation

$$\exp(m_{13}m_{32}\phi_1+(x_1+x_3)m_{13}m_{32}\phi_2)\ \exp(m_{23}m_{13}\phi_1+(x_2+x_3)m_{13}\phi_2)$$
$$=[\frac{g_2x_2^2+g_1x_2+g_0}{g_2x_3^2+g_1x_3+g_0}]^{m_{13}}\ [\frac{g_2x_1^2+g_1x_1+g_0}{g_2x_3^2+g_1x_3+g_0}]^{m_{32}}. \tag{4.14}$$

The left hand side of this equation is equal to $\exp(V\phi_2)$, that is,

$$\exp(V\phi_2)=[\frac{g_2x_2^2+g_1x_2+g_0}{g_2x_3^2+g_1x_3+g_0}]^{m_{13}}\ [\frac{g_2x_1^2+g_1x_1+g_0}{g_2x_3^2+g_1x_3+g_0}]^{m_{32}}. \tag{4.15}$$

Let $g_2 \neq 0$, then by dividing numerator and denominator by g_2 we obtain

$$\exp(V\phi_2) = [\frac{x_2^2 + tg\ x_2 + sg}{x_3^2 + tg\ x_3 + sg}]^{m_{13}}\ [\frac{x_1^2 + tg\ x_1 + sg}{x_3^2 + tg\ x_3 + sg}]^{m_{32}}, \qquad (4.16)$$

where

$$tg = \frac{g_1}{g_2},\ \ sg = \frac{g_0}{g_2}.$$

Let u, v be roots of the quadratic equation

$$g_0(\phi_1, \phi_2) + y\ g_1(\phi_1, \phi_2) + y^2\ g_2(\phi_1, \phi_2) = 0. \qquad (4.17)$$

Then the ratios (4.13a,b) can be re-written as follows

$$\exp((x_k - x_l)\phi_1 + (x_k^2 - x_l^2)\phi_2) = \frac{(u - x_k)}{(u - x_l)}\frac{(v - x_k)}{(v - x_l)}. \qquad (4.18)$$

This equation is true for any $k, l = 1, 2, 3,\ k \neq l$. This is to say, for any index we have the same ϕ_1, ϕ_2 and the same u, v. Here u, v depend on two parameters ϕ_1, ϕ_2.

Inversely, if functions $u = u(\varphi_u),\ v = v(\varphi_v)$ are known, then we can find corresponding g by

$$\frac{g_0}{g_2} = uv,\ \ \frac{g_1}{g_2} = u + v.$$

From these two equations we find ϕ_1 and ϕ_2. We expect that

$$\exp(V(\varphi_u + \varphi_v)) = \exp(V\phi_2),$$

or,

$$\varphi_u + \varphi_v = \phi_2.$$

In this way we have established connection between the characteristics of general complex algebra CG_3 and solutions of the Riccati-Abel equation.

The next task is to prove that the ratio $u = -g_0/g_1|_{g_2=0}$, in fact, satisfies the Riccati-Abel equation.

With this purpose, let us calculate derivatives of g_1, g_0 under the condition

$$g_2(\phi_1, \phi_2) = 0. \qquad (4.19)$$

From this equation it follows that ϕ_1 is an implicit function of ϕ_2, *viz.*, $\phi_1 = \phi_1(\phi_2)$. Thus, we have to prove that the function

$$u(\phi_2) = -\frac{g_0(\phi_1(\phi_2), \phi_2)}{g_1(\phi_1(\phi_2), \phi_2)}, \tag{4.20}$$

obeys the Riccati-Abel equation (3.1) with $\phi = \phi_2$.

Differentiating equation (4.19), we obtain

$$\frac{dg_2}{d\phi_2} + \frac{dg_2}{d\phi_1}\frac{d\phi_1}{d\phi_2} = 0, \quad \frac{d\phi_1}{d\phi_2} = -\frac{\frac{dg_2}{d\phi_2}|_{g_2=0}}{\frac{dg_2}{d\phi_1}|_{g_2=0}}. \tag{4.21}$$

The derivatives of g_2 with respect to ϕ_1, ϕ_2 under the constraint (4.19) we calculate by using (4.4), (4.5):

$$\frac{dg_2}{d\phi_1}|_{g_2=0} = (g_1 + a_1 g_2)|_{g_2=0} = g_1, \quad \frac{dg_2}{d\phi_2}|_{g_2=0} = (g_0 + a_1 g_1 + (a_2^2 - a_1)g_2)|_{g_2=0} = g_0 + a_1 g_1. \tag{4.22}$$

By substituting this result into (4.21), we get

$$\frac{d\phi_1}{d\phi_2} = -\frac{1}{g_1}(g_0 + a_2 g_1). \tag{4.23}$$

Next, we have to calculate derivatives of g_0, g_1 with respect to ϕ_2 under the constraint (4.19). According to formulae (4.4), (4.5) and (4.23) we write

$$\frac{dg_0}{d\phi_2} = \frac{\partial g_0}{\partial \phi_2}|_{g_2=0} + \frac{dg_0}{d\phi_1}|_{g_2=0}\frac{d\phi_1}{d\phi_2} = a_0 g_1, \tag{4.24}$$

$$\frac{dg_1}{d\phi_2} = \frac{\partial g_1}{\partial \phi_2}|_{g_2=0} + \frac{dg_1}{d\phi_1}|_{g_2=0}\frac{d\phi_1}{d\phi_2} = -a_1 g_1 - \frac{g_0}{g_1}(g_0 + a_2 g_1). \tag{4.25}$$

Now we are able to calculate the derivative of the function $u(\phi_2)$, which is defined by the fraction (4.20). Firstly calculate derivative of the fraction:

$$\frac{d}{d\phi_2}\frac{g_0}{g_1} = \frac{1}{g_1^2}(g_1\frac{dg_0}{d\phi_2} - g_0\frac{dg_1}{d\phi_2})$$

$$= \frac{1}{g_1^3}(\, a_0 g_1^3 + a_1 g_1^2 g_0 + g_0^3 + a_2 g_1 g_0^2). \tag{4.26}$$

Now, re-write this equation by taking into account the definition (4.20), where

$$u(\phi_2) = -\frac{g_0(\phi_1(\phi_2), \phi_2)}{g_1(\phi_1(\phi_2), \phi_2)}.$$

In this way we arrive to the Riccati-Abel equation:

$$\frac{du}{d\phi_2} = -a_0 + a_1 u^2 + u^3 - a_2 u^2. \tag{4.27}$$

5. The Riccati-Abel Equation as An Evolution Equation of the Generalized Dynamics

In the relativistic mechanics the evolution of the energy p_0 and the momentum p are performed in such a way that the mass-shell equation remains invariant

$$p_0^2 - p^2 = m^2, \ (c = 1). \tag{5.1}$$

Let us consider a one-parametrical evolution of energy-momentum remaining invariant the (proper)mass m. Introduce a new variable X by $X = p_0 + m$ which obeys the following quadratic equation

$$X^2 - 2p_0 X + p^2 = 0. \tag{5.2}$$

Let x_1, x_2 be two roots of this equation, that is to say

$$x_1 + x_2 = 2p_0, \ x_1 x_2 = p^2, \ x_1 - x_2 = 2m. \tag{5.3}$$

Now consider the evolution generated by the matrix solution of equation (5.2). The matrix obeying (5.2) is defined by (2.12):

$$E = \begin{pmatrix} 0 & -p^2 \\ 1 & 2p_0 \end{pmatrix}. \tag{5.4}$$

Thus the desired evolution equation is the Riccati equation

$$u^2 - 2p_0 u + p^2 = \frac{du}{d\phi}. \tag{5.5}$$

Consequently, from (2.6) we obtain

$$u(\phi,\phi_0) = m\coth(m\phi_0) - m\coth(m\phi) = p_0(\phi_0) - p_0(\phi). \tag{5.6}$$

In the several papers (see refs. [15-19] and references therein) the classical dynamics of third order has been suggested. The evolution in this dynamics is generated by three order polynomial of the form

$$x^3 - 3P_1x^2 + 2P_2x - P^2 = 0. \tag{5.7}$$

From this equation it follows two algebraic equations connecting invariants with momentum P and energy P_1:

$$R_0 = P_1^3 - R_1P_1 - P^2,\;\; R_1 = -2P_2 + 3P_1^2. \tag{5.8}$$

The first of these equations is an analogue of the mass-shell equation (5.1).

The evolution generated by the polynomial (5.7) is the Riccati-Abel equation

$$\frac{du}{d\phi} = u^3 - 3P_1u^2 + 2P_2u - P^2. \tag{5.9}$$

The solution of the Riccati-Abel equation is related with the evolution of energy P_1 as follows

$$u(\phi,\phi_0) = P_1(\phi_0) - P_1(\phi). \tag{5.10}$$

6. Concluding Remarks

As the ordinary Riccati equation, also the Riccati-Abel equation has a relationship with a linear differential equation. Seeking a summation formula for solutions of Riccati-Abel equation we established a certain relationship of these solutions with multi-trigonometric functions of third order. We have elaborated some rule according to which in order to build a summation formula for solutions of Riccati-Abel equations, it is necessary to consider the pair of solutions, which can be achieved by using an auxiliary variable. This idea can be successfully used for the solutions of generalized Riccati equations of any order with constant coefficients. By increasing the order of the non-linearity, the number of auxiliary variables also will increase. For example, from solutions of generalized Riccati equations of fourth order we have to compose the triplet of solutions with two auxiliary variables, and for n-order generalized Riccati equations it is necessary to compose a set of $(n-1)$ solutions with $(n-2)$ auxiliary variables.

REFERENCES

1. H. T. Davis, *Introduction to nonlinear differential and integral equations*, United States Atomic Energy Commission, U.S. Goverment Printing Office, washington D.C. Reprinted by Dover publicatios, Inc., New York 1960.

2. N. H. Abel, *Oeuvres completes du Niels Henrik Abel*.-Christiana, 1881.

3. P. R. Vein, Functions which satisfy Abel's differential equation, *SIAM J. Appl. Math*., **15**(3) (1967), 618-623.

4. N. Fluery, M. Rauch and R. M. Yamaleev, Commutative extended complex numbers and connected trigonometry, *J. Math. Anal. Appl*., **180**(2) (1993), 123-131.

5. J. M. Yeomans, *Statistical mechanics of phase transitions*, Clarendon Press, Oxford, (1992).
John Cardy, Scaling and renormalization in statistical physics, Cambridge lecture notes in physics, Eds. P. Goddard, J. Yeomans, Cambridge university press 1996.

6. A. Ungar, Addition theorems in ordinary differential equations, *Amer. Math. Monthly*, **94** (1987), 872-875.

7. A. Ungar, Addition theorems for solutions to linear homogeneous constant coefficient differential equations, *Aequations Math*., **26** (1983), 104-112.

8. N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, (1970) Moscow, translated into English as AMS Translations of Mathematical Monographs, Volume 79 (1990) AMS, Rhode Island.

9. R. M. Yamaleev, Geometrical and physical interpretation of evolution governed by general complex algebra, *J. Math. Analysis and Appl*., **340** (2008), 1046-1057.

10. D. Babusci, G. Dattoli, E. Di Palma and E. Sabia, Complex-type numbers and generalization of the Euler identity, *Adv. Appl. Clifford Al*., **22** (2012), 271.

11. Vedic Mathematics: Sixteen Simple Mathematical Formulae from the Vedas, by Swami Sankaracarya (1884-1960), Motilal Banarsidass Indological Publishers and Booksellers, Varnasi, India, 1965;

Stapel, Elizabeth, *Partial-Fraction Decomposition: General Techniques*, Purplemath. Available from http://www.purplemath.com/modules/partfrac.htm.

12. R. Vein and P. Dale, *Determinants and their applications in mathematical physics*, Springer-Verlag, New York, Inc., 1999.
13. R. M. Yamaleev, Multicomplex algebras on polynomials and generalized Hamilton dynamics, *J. Math. Analysis and Appl.*, **322** (2006), 815-824.
14. P. Lankaster, *Theory of matrices*, Academic Press, New York-London (1969), 144.
15. R. M. Yamaleev, Complex algebras on n-order polynomials and generalizations of trigonometry, oscillator model and Hamilton dynamics, *J. Adv. Appl. Clifford Al.*, **15**(2) (2005), 123-150.
16. R. M. Yamaleev, Generalized electrodynamics with ternary internal structure, *J. Adv. Appl. Clifford Al.*, **16**(2) (2006), 167-190.
17. S. Mongkolsakulvong, P. Chaikhan and T. D. Frank, Oscillatory nonequilibrium Nambu systems: the canonical-dissipative Yamaleev oscillator, *Eur. Phys. J.*, **B85** (2012), 90-97.
18. T. D. Frank, Nambu bracket formulation of nonlinear biochemical reactions beyond elementary mass action kinetics, *J. Nonlinear Math. Phys.*, **19** (2012), 125-130.
19. A. Molgado and A. Roriguez-Dominguez, Mapping between the dynamic and mechanical properties of the relativistic oscillator and Euler free rigd body, *J. Nonlinear Math. Phys.*, **14** (2007), 534-542.