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GLOBAL ASYMPTOTIC STABILITY OF A SECOND-ORDER SYSTEM OF DIFFERENCE EQUATIONS

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In this paper a sufficient condition is obtained for the global asymptotic stability of the following system of difference equations

$$x_{n+1} = \frac{x_n y_{n-1}^b + 1}{x_n + y_{n-1}^b}, \quad y_{n+1} = \frac{y_n x_{n-1}^b + 1}{y_n + x_{n-1}^b} \; n = 0, 1, 2, ...$$

where the parameter $b \in [0, \infty)$ and the initial values $(x_k, y_k) \in (0, \infty)$ (for k = -1, 0).

Key words : Rational difference equations; system; global asymptotic stability; equilibrium point; semicycle.

1. Introduction

Recently, there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so forth. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the global behaviors of their solutions (see [1-8] and the references cited therein).

In [2] De Vault *et al*. proved that the unique equilibrium of the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, n = 0, 1, 2, ...$$

where $A \in (0, \infty)$, is globally asymptotically stable and proved the oscillatory behavior of the positive solutions of this difference equation.

From on, Papaschinopoluos and Schinas [5] extended the results obtained in [2] to the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-p}}, y_{n+1} = A + \frac{x_n}{x_{n-p}}, n = 0, 1, 2, ...$$

where $A \in (0, \infty)$, p, q are positive integers and $x_{-q}, x_{-q+1}, ...x_0, y_{-p}, y_{-p+1}, ...y_0$, are positive initial values.

Li and Zhu [4] proved that the unique positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, n = 0, 1, 2, ...$$

where $a \in [0, \infty)$ and x_1, x_0 are positive, is globally asymptotically stable.

In [1] Abu-Saris *et al*. extended the results obtained in [4] to the following difference equation

$$x_{n+1} = \frac{x_n x_{n-k} + a}{x_n + x_{n-k}}, n = 0, 1, 2, ...$$

where k is nonnegative integer, $a \in [0, \infty)$ and $x_{-k}, ..., x_0$ are positive, is globally asymptotically stable.

Also, in [8] Yalcinkaya *et al*. extended the results obtained in [4] to the following system of difference equations

$$z_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \quad t_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}} \; n = 0, 1, 2, ...$$

where $a \in (0, \infty)$ and the initial values $(z_k, t_k) \in (0, \infty)$ (for k = -1, 0).

In this paper, we consider the following system of the difference equations

$$x_{n+1} = \frac{x_n y_{n-1}^b + 1}{x_n + y_{n-1}^b}, \quad y_{n+1} = \frac{y_n x_{n-1}^b + 1}{y_n + x_{n-1}^b} \; n = 0, 1, 2, ... \tag{1.1}$$

where the parameter $b \in [0, \infty)$ and the initial values $(x_k, y_k) \in (0, \infty)$ (for k = -1, 0). Our main aim is to investigate the global asymptotic behavior of its solutions.

We need the following denitions and theorem [3]:

Let *I* be some interval of real numbers and let

$$f, g : I \times I \to I$$

be continuously differentiable functions. Then for all initial values $(x_k, y_k) \in I, k = -1, 0$, the system of difference equations

$$x_{n+1} = f(x_n, y_{n-1}), y_{n+1} = g(y_n, x_{n-1}), n = 0, 1, 2, ... \tag{1.2}$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.

Definition 1.1 — A point $(\overline{x}, \overline{y})$ called an equilibrium point of the system (1.2) if

$$\overline{x} = f(\overline{x}, \overline{y}) \text{ and } \overline{y} = g(\overline{y}, \overline{x})$$

It is easy to see that the system (1.1) has the unique positive equilibrium $(\overline{x}, \overline{y}) = (1, 1)$.

Definition 1.2 — Let $(\overline{x}, \overline{y})$ be an equilibrium point of the system (1.2).

(a) An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every initial points (x_{-1}, y_{-1}) and (x_0, y_0) for which $||(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})|| + ||(x_0, y_0) - (\overline{x}, \overline{y})|| < \delta$, the iterates (x_n, y_n) of (x_{-1}, y_{-1}) and (x_0, y_0) satisfy $||(x_n, y_n) - (\overline{x}, \overline{y})|| < \varepsilon$ for all $n > 0$. An equilibrium point $(\overline{x}, \overline{y})$ is said to be unstable if it is not stable. (By $||.||$ we denote the Euclidean norm in $\mathbb{R}^2$ given by $||(x, y)|| = \sqrt{x^2 + y^2}$).

(b) An equilibriumpoint $(\overline{x}, \overline{y})$ is said to be asymptotically stable if there exists $r > 0$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ for all (x_{-1}, y_{-1}) and (x_0, y_0) that satisfy $||(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})|| + ||(x_0, y_0) - (\overline{x}, \overline{y})|| < r$.

Definition 1.3 — Let $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f, g)$, where f and g are continuously differentiable functions at $(\overline{x}, \overline{y})$. The Jacobian matrix of F at $(\overline{x}, \overline{y})$ is the matrix

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{bmatrix}.$$

The linear map $J_F(\overline{x},\overline{y}) : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J_F(\overline{x},\overline{y})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x}(\overline{x},\overline{y})x + \dfrac{\partial f}{\partial y}(\overline{x},\overline{y})y \\ \dfrac{\partial g}{\partial x}(\overline{x},\overline{y})x + \dfrac{\partial g}{\partial y}(\overline{x},\overline{y})y \end{bmatrix} \tag{1.3}$$

is called the linearization of the map F at $(\overline{x},\overline{y})$.

***Theorem* 1.4** — *(Linearized Stability Theorem)*

Let $F = (f, g)$ be a continuously differentiable function defined on an open set I in $\mathbb{R}^2$, and let $(\overline{x},\overline{y})$ in I be an equilibrium point of the map $F = (f, g)$.

(a) If all the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ have modulus less than one, then the equilibrium point $(\overline{x},\overline{y})$ is asymptotically stable.

(b) If at least one of the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ has modulus greater than one, then the equilibrium point $(\overline{x},\overline{y})$ is unstable.

(c) An equilibrium point $(\overline{x},\overline{y})$ of the map $F = (f, g)$ is locally asymptotically stable if and only if every solution of the characteristic equation

$$\lambda^2 - trJ_F(\overline{x},\overline{y}) + detJ_F(\overline{x},\overline{y}) = 0$$

lies inside the unit circle, that is, if and only if

$$|trJ_F(\overline{x},\overline{y})| < 1 + detJ_F(\overline{x},\overline{y}) < 2.$$

Definition 1.5 — Let $(\overline{x},\overline{y})$ be an equilibrium point of the system (1.2). [see 6]

A "string" of consecutive terms $\{x_s, ..., x_m\}$ (resp. $\{y_s, ..., y_m\}$), $s \geq -1, m \leq \infty$ is said to be a positive semicycle if $x_i \geq \overline{x}$ (resp. $y_i \geq \overline{y}$), $i \in \{s, ..., m\}, x_{s-1} < \overline{x}$ (resp. $y_{s-1} < \overline{y}$), and $x_{m+1} < \overline{x}$ (resp. $y_{m+1} < \overline{y}$).

A "string" of consecutive terms $\{x_s, ..., x_m\}$ (resp. $\{y_s, ..., y_m\}$), $s \geq -1, m \leq \infty$ is said to be a negative semicycle if $x_i < \overline{x}$ (resp. $y_i < \overline{y}$), $i \in \{s, ..., m\}, x_{s-1} \geq \overline{x}$ (resp. $y_{s-1} \geq \overline{y}$), and $x_{m+1} \geq \overline{x}$ (resp. $y_{m+1} \geq \overline{y}$).

A “string” of consecutive terms $\{(x_s, y_s), ..., (x_m, y_m)\}$ is said to be a positive (resp. negative) semicycle if $\{x_s, ..., x_m\}, \{y_s, ..., y_m\}$ are positive (resp. negative) semicycles. Finally a ”string” of consecutive terms $\{(x_s, y_s), ..., (x_m, y_m)\}$ is said to be a semicycle positive (resp. negative) with respect to x_n and negative (resp. positive) with respect to y_n if $\{x_s, ..., x_m\}$ is a positive (resp. negative) semicycle and $\{y_s, ..., y_m\}$ is a negative (resp. positive) semicycle.

2. Some Auxiliary Results

In this section, we give the following lemmas which show us the behavior of semicycles of positive solutions of system (1.1). Proofs of Lemmas 2.1 are clear from (1.1). So, they will be omitted.

Lemma 2.1 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a solution of the system (1.1) and consider the cases:

$$Case\ a : x_0 = x_{-1} = 1 \qquad Case\ b : y_0 = y_{-1} = 1,$$
$$Case\ c : x_0 = y_0 = 1 \qquad Case\ d : x_{-1} = y_{-1} = 1.$$

If one of the above cases occurs, then every positive solution of system (1.1) equal to (1, 1).

Lemma 2.2 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a positive solution of the system (1.1) which is not eventually equal to (1, 1). Then the following statements are true:

(i) $(x_{n+1} - x_n)(x_n - 1) < 0$ and $(y_{n+1} - y_n)(y_n - 1) < 0$ for all $n \geq 0$,

(ii) $(x_{n+1} - 1)(x_n - 1)(y_{n-1}^b - 1) > 0$ and $(y_{n+1} - 1)(y_n - 1)(x_{n-1}^b - 1) > 0$ for all $n \geq 0$.

Proof : In view of system (1.1), we obtain

$$x_{n+1} - x_n = \frac{(1 - x_n)(1 + x_n)}{x_n + y_{n-1}^b}$$

$$y_{n+1} - y_n = \frac{(1 - y_n)(1 + y_n)}{y_n + x_{n-1}^b}$$

$$x_{n+1} - 1 = \frac{(x_n - 1)(y_{n-1}^b - 1)}{x_n + y_{n-1}^b}$$
$$y_{n+1} - 1 = \frac{(y_n - 1)(x_{n-1}^b - 1)}{y_n + x_{n-1}^b}$$

for $n = 0, 1, 2, ...$, from which the inequalities in (i) and (ii) follow. □

Lemma 2.3 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a solution of system (1.1) and suppose that the case, *Case* 1: $x_k, y_k > 1$ (for $k = -1, 0$), holds. Then (x_n, y_n) is a positive semicycle of system (1.1) with an infinite number of terms and it monotonically tends to the positive equilibrium $(\overline{x}, \overline{y}) = (1, 1)$.

PROOF : If $x_k, y_k > 1$ (for $k = -1, 0$), then by Lemma 2.2 (ii), it follows that $x_n, y_n > 1$ for $n \geq 1$, i.e. this positive semicycle has an infinite number of terms. Furthermore, according to Lemma 2 (i), we know that (x_n, y_n) is strictly descreasing for $n \geq 0$. So, the limits

$$\lim_{n\to\infty} x_n = l_1 \text{ and } \lim_{n\to\infty} y_n = l_2$$

exist and are finite. Taking limits on both sides of system (1.1), we have

$$l_1 = \frac{l_1 l_2^b + 1}{l_1 + l_2^b} \; l_2 = \frac{l_2 l_1^b + 1}{l_2 + l_1^b}$$

and thus $l_1 = l_2 = 1$. Therefore, the proof is complete.

Lemma 2.4 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a solution of system (1.1), and consider the cases:

Case 2: $x_{-1}, y_{-1} > 1$ and $x_0, y_0 < 1$;

Case 3: $x_{-1}, y_{-1} < 1$ and $x_0, y_0 > 1$;

Case 4: $x_{-1}, y_{-1} < 1$ and $x_0, y_0 < 1$.

If one of the above cases occurs, then:

(i) Every positive semicycle of system (1.1) consists of one term;

(ii) Every negative semicycle of system (1.1) consists of two terms;

(iii) Every positive semicycle of length one is followed by a negative semicycle of length two;

(iv) Every negative semicycle of length two is followed by a positive semicycle of length one.

PROOF : If Case 2 occurs, then in view of inequality (ii) of Lemma 2.2 we have: $x_1, y_1 < 1;\ x_2, y_2 < 1;\ x_3, y_3 > 1$ and

$$x_{3n+1}, y_{3n+1} < 1;\ x_{3n+2}, y_{3n+2} < 1;\ x_{3n+3}, y_{3n+3} > 1 \quad \forall n \geq 0$$

which imply that a positive semicycle of length one is followed by a negative semicycle of length two which it turn is followed by a positive semicycle of length one.

Proofs of the other cases are similar, so they will be omitted. Therefore, the proof is complete. □

We omit the proofs of the following two results since they can easily be obtained in a way similar to the proof of Lemma 2.4.

Lemma 2.5 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a solution of system (1.1), and consider the cases:

Case 5: $x_{-1}, y_{-1}, y_0 > 1$ and $x_0 < 1$;

Case 6: $x_{-1}, x_0, y_0 > 1$ and $y_{-1} < 1$;

Case 7: $x_{-1}, y_0 < 1$ and $y_{-1}, x_0 > 1$;

Case 8: $x_{-1}, x_0 < 1$ and $y_{-1}, y_0 > 1$;

Case 9: $x_{-1}, x_0, y_0 < 1$ and $y_{-1} > 1$;

Case 10: $x_{-1}, y_{-1}, x_0 < 1$ and $y_0 > 1$.

If one of the above cases occurs, then the following hold:

(i) Every positive semicycle associated with $\{x_n\}$ of system (1.1) consists of two terms;

(ii) Every negative semicycle associated with $\{x_n\}$ of system (1.1) consists of four terms;

(iii) Every positive semicycle associated with $\{x_n\}$ of length two is followed by a negative semicycle of length four;

(iv) Every negative semicycle associated with $\{x_n\}$ of length four is followed by a positive semicycle of length two;

(v) Every positive semicycle associated with $\{y_n\}$ of system (1.1) consists of three or one term;

(vi) Every negative semicycle associated with $\{y_n\}$ of system (1.1) consists of one term;

(vii) Every positive semicycle associated with $\{y_n\}$ of length three or one is followed by a negative semicycle of length one;

(viii) Every negative semicycle associated with $\{y_n\}$ of length one is followed by a positive semicycle of length three or one.

Lemma 2.6 — Assume that $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is a solution of system (1.1), and consider the cases:

Case 11: $x_{-1}, y_{-1}, x_0 > 1$ and $y_0 < 1$;

Case 12: $x_{-1} > 1$ and $y_{-1}, x_0, y_0 < 1$;

Case 13: $x_{-1}, y_0 > 1$ and $y_{-1}, x_0 < 1$;

Case 14: $x_{-1}, x_0 > 1$ and $y_{-1}, y_0 < 1$;

Case 15: $x_{-1} < 1$ and $y_{-1}, x_0, y_0 > 1$;

Case 16: $x_{-1}, y_{-1}, y_0 < 1$ and $x_0 > 1$.

If one of the above cases occurs, then the following hold:

(i) Every positive semicycle associated with $\{x_n\}$ of system (1.1) consists of three or one term;

(ii) Every negative semicycle associated with $\{x_n\}$ of system (1.1) consists of one term;

(iii) Every positive semicycle associated with $\{x_n\}$ of length three or one is followed by a negative semicycle of length one;

(iv) Every negative semicycle associated with $\{x_n\}$ of length one is followed by a positive semicycle of length three or one;

(v) Every positive semicycle associated with $\{y_n\}$ of system (1.1) consists of two terms;

(vi) Every negative semicycle associated with $\{y_n\}$ of system (1.1) consists of four terms;

(vii) Every positive semicycle associated with $\{y_n\}$ of length two is followed by a negative semicycle of length four;

(viii) Every negative semicycle associated with $\{y_n\}$ of length four is followed by a positive semicycle of length two.

3. Main Result

***Theorem* 3.1** — *Assume that* $b \in [0, \infty)$. *Then the positive equilibrium point* $(\overline{x}, \overline{y}) = (1, 1)$ *of the system (1.1) is globally asymptotically stable.*

Proof : When $b = 0$, system (1.1) is trivial. So, we only consider the case $b > 0$, and prove the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.1) is both locally asymptotically stable and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$. The characteristic equation of the system (1.1) about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ is

$$\lambda^2 - 0.\lambda + 0 = 0$$

and so it is clear from Theorem 1.4 that positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.1) is locally asymptotically stable. It remains to verify that every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (1.1) converges to $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \overline{x} = 1, \ \lim_{n\to\infty} y_n = \overline{y} = 1 \tag{3.1}$$

If the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (1.1) is nonoscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.1), then according to Lemmas 2.1 and 2.3 respectively, we know that the solution is either eventually equal to (1, 1) or an eventually positive one which has an infinite number of terms and monotonically tends the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.1) and so equation (3.1) holds. Therefore, it suffices to prove that equation (3.1) holds for strictly oscillatory solutions. Now let $(x_n, y_n) \to (\overline{x}, \overline{y})$ be strictly oscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.1). By virtue of Lemmas 2.2 (ii) and 2.4 one can see that every positive semicycle of this solution has one term and every negative semicycle except perhaps for the rst has exactly two terms. Every positive semicycle of length one is followed by a negative semicycle of length two.

For the convenience of statement, without loss of generality, we use the following notation. We denote by x_p and y_p the terms of a positive semicycle of length one, followed by x_{p+1}, x_{p+2} and y_{p+1}, y_{p+1} which are the terms of a negative semicycle of length two. Afterwards, there is the positive semicycles x_{p+3} and y_{p+3} in turn followed by the negative semicycles x_{p+4}, x_{p+5} and y_{p+4}, y_{p+5} so on.

Therefore, we have the following sequences consisting of positive and negative semicycles (for n = 0, 1, ...):

$$\{x_{p+3n}\}_{n=0}^{\infty}, \quad \{x_{p+3n+1}, x_{p+3n+2}\}_{n=0}^{\infty}, \quad \{y_{p+3n}\}_{n=0}^{\infty}, \quad \{y_{p+3n+1}, y_{p+3n+2}\}_{n=0}^{\infty}$$

We have the following assertions:

(i) $x_{p+3n+1} < x_{p+3n+2}$ and $y_{p+3n+1} < y_{p+3n+2}$;

(ii) $x_{p+3n}x_{p+3n+1} > 1$ and $y_{p+3n}y_{p+3n+1} > 1$;

(iii) $x_{p+3n+2}x_{p+3n+3} < 1$ and $y_{p+3n+2}y_{p+3n+3} < 1$.

In fact, inequality (i) immediately follows from Lemma 2.2 (i). From the observations that

$$x_{p+3n+1} = \frac{x_{p+3n}y^b_{p+3n-1} + 1}{x_{p+3n} + y^b_{p+3n-1}} > \frac{x_{p+3n}y^b_{p+3n-1} + 1}{x_{p+3n} + y^b_{p+3n-1}.x^2_{p+3n}} = \frac{1}{x_{p+3n}}$$

and

$$y_{p+3n+1} = \frac{y_{p+3n}x^b_{p+3n-1} + 1}{y_{p+3n} + x^b_{p+3n-1}} > \frac{y_{p+3n}x^b_{p+3n-1} + 1}{y_{p+3n} + x^b_{p+3n-1}.y^2_{p+3n}} = \frac{1}{y_{p+3n}}$$

one can see that (ii) is valid.

As for (iii), it is immediately obtained from

$$x_{p+3n+3} = \frac{x_{p+3n+2}y^b_{p+3n+1} + 1}{x_{p+3n+2} + y^b_{p+3n+1}} < \frac{x_{p+3n+2}y^b_{p+3n+1} + 1}{x_{p+3n+2} + y^b_{p+3n+1}.x^2_{p+3n+2}} = \frac{1}{x_{p+3n+2}}$$

and

$$y_{p+3n+3} = \frac{y_{p+3n+2}x^b_{p+3n+1}+1}{y_{p+3n+2}+x^b_{p+3n+1}} < \frac{y_{p+3n+2}x^b_{p+3n+1}+1}{y_{p+3n+2}+x^b_{p+3n+1}.y^2_{p+3n+2}}$$
$$= \frac{1}{y_{p+3n+2}}, \quad n = 0, 1, 2, ...$$

Combining the above inequalities, we derive

$$\frac{1}{x_{p+3n}} < x_{p+3n+1} < x_{p+3n+2} < \frac{1}{x_{p+3n+3}} \tag{3.2}$$
$$\frac{1}{y_{p+3n}} < y_{p+3n+1} < y_{p+3n+2} < \frac{1}{y_{p+3n+3}} \tag{3.3}$$

From equation (3.2)-(3.3), one can see that $\{x_{p+3n+1}\}_{n=0}^{\infty}$ and $\{y_{p+3n+1}\}_{n=0}^{\infty}$ are increasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{p+3n+1} = L_1 \quad \text{and} \quad \lim_{n\to\infty} y_{p+3n+1} = L_2$$

exist and finite. Accordingly, in view of equation (3.2) - (3.3), we obtain

$$\lim_{n\to\infty} x_{p+3n+2} = L_1 \quad \text{and} \quad \lim_{n\to\infty} x_{p+3n+3} = \frac{1}{L_1}$$

and

$$\lim_{n\to\infty} y_{p+3n+2} = L_2 \quad \text{and} \quad \lim_{n\to\infty} y_{p+3n+3} = \frac{1}{L_2}.$$

It suffices to verify that $L_1 = L_2 = 1$. To this end, note that

$$x_{p+3n+3} = \frac{x_{p+3n+2}y^b_{p+3n+1}+1}{x_{p+3n+2}+y^b_{p+3n+1}} \quad \text{and} \quad y_{p+3n+3} = \frac{y_{p+3n+2}x^b_{p+3n+1}+1}{y_{p+3n+2}+x^b_{p+3n+1}}.$$

Take the limits on both sides of the above equality and obtain

$$L_1 = \frac{L_1.L_2^b+1}{L_1+L_2^b} \quad \text{and} \quad L_2 = \frac{L_2.L_1^b+1}{L_2+L_1^b}$$

which imply that $L_1 = L_2 = 1$.

So, we have shown that

$$\lim_{n\to\infty} x_{p+3n+m} = \lim_{n\to\infty} y_{p+3n+m} = 1 \quad \text{for } m \in \{1, 2, 3\}.$$

Similarly, by virtue of Lemma 2.2 (ii) and 2.5 one can see that every positive semicycle associated with $\{x_n\}$ of system (1.1) has two terms and every negative semicycle associated with $\{x_n\}$ of system (1.1) has four terms. Every positive semicycle of length two is followed by a negative semicycle of length four. For the convenience of statement, without loss of generality, we use the following notation. We denote by x_p, x_{p+1} the terms of a positive semicycle of length two, followed by $x_{p+2}, x_{p+3}, x_{p+3}, x_{p+4}$ which are the terms of a negative semicycle of length four. Afterwards, there is the positive semicycles x_{p+6}, x_{p+7} in turn followed by the negative semicycles $x_{p+8}, x_{p+9}, x_{p+10}, x_{p+11}$ and so on.

Therefore, we have the following sequences consisting of positive and negative semicycles (for n = 0, 1, ...):

$$\{x_{p+6n}, x_{p+6n+1}\}_{n=0}^{\infty}, \quad \{x_{p+6n+2}, x_{p+6n+3}, x_{p+6n+4}, x_{p+6n+5}\}_{n=0}^{\infty}$$

Moreover, every positive semicycle associated with $\{y_n\}$ of system (1.1) has one or three terms and every negative semicycle associated with $\{y_n\}$ of system (1.1) has one term. Every positive semicycle of length three or one is followed by a negative semicycle of length one. For the convenience of statement, without loss of generality, we use the following notation. We denote by y_p, y_{p+1}, y_{p+2} the terms of a positive semicycle of length three, followed by y_{p+3} which is the term of a negative semicycle of length one. Afterwards, there is the positive semicycles y_{p+4} followed by the negative semicycles y_{p+5} and so on.

Therefore, we have the following sequences consisting of positive and negative semicycles (for n = 0, 1, ...):

$$\{y_{p+6n}, y_{p+6n+1}, y_{p+6n+2}\}_{n=0}^{\infty}, \quad \{y_{p+6n+3}\}_{n=0}^{\infty}, \quad \{y_{p+6n+4}\}_{n=0}^{\infty}, \quad \{y_{p+6n+5}\}_{n=0}^{\infty}$$

We have the following assertions:

(i) $x_{p+6n} > x_{p+6n+1}, x_{p+6n+2} < x_{p+6n+3} < x_{p+6n+4} < x_{p+6n+5}$ and $y_{p+6n} > y_{p+6n+1} > y_{p+6n+2}$

(ii) $x_{p+6n+1}.x_{p+6n+2} > 1, x_{p+6n+5}.x_{p+6n+6} < 1$ and $y_{p+6n+2}.y_{p+6n+3} > 1, y_{p+6n+3}.y_{p+6n+4} < 1, y_{p+6n+4}.y_{p+6n+5} > 1, y_{p+6n+5}.y_{p+6n+6} < 1.$

Combining the above inequalities, it follows that

$$\frac{1}{x_{p+6n}} < \frac{1}{x_{p+6n+1}} < x_{p+6n+2} < x_{p+6n+3} < x_{p+6n+5} < x_{p+6n+5} < \frac{1}{x_{p+6n+6}} \tag{3.4}$$

$$\frac{1}{y_{p+6n}} < \frac{1}{y_{p+6n+1}} < \frac{1}{y_{p+6n+2}} < y_{p+6n+3} < \frac{1}{y_{p+6n+4}} < y_{p+6n+5} < \frac{1}{y_{p+6n+6}} \tag{3.5}$$

From equation (3.4) - (3.5), one can see that $\{x_{p+6n+2}\}_{n=0}^{\infty}$ and $\{y_{p+6n+3}\}_{n=0}^{\infty}$ are increasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{p+6n+2} = L_3 \quad \text{and} \quad \lim_{n\to\infty} y_{p+6n+3} = L_4$$

exist and finite. Accordingly, in view of equation (3.4) - (3.5), we obtain

$$\begin{aligned} \lim_{n\to\infty} x_{p+6n+3} &= \lim_{n\to\infty} x_{p+6n+4} = \lim_{n\to\infty} x_{p+6n+5} = L_3; \\ \lim_{n\to\infty} x_{p+6n} &= \lim_{n\to\infty} x_{p+6n+1} = \frac{1}{L_3} \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} y_{p+6n+5} &= L_4; \quad \lim_{n\to\infty} y_{p+6n} = \lim_{n\to\infty} y_{p+6n+1} = \lim_{n\to\infty} y_{p+6n+2} \\ &= \lim_{n\to\infty} y_{p+6n+4} = \frac{1}{L_4} \end{aligned}$$

It suffices to verify that $L_3 = L_4 = 1$. To this end, note that

$$x_{p+6n+5} = \frac{x_{p+6n+4} y^b_{p+6n+3} + 1}{x_{p+6n+4} + y^b_{p+6n+3}} \quad \text{and} \quad y_{p+6n+5} = \frac{y_{p+6n+4} x^b_{p+6n+3} + 1}{y_{p+6n+4} + x^b_{p+6n+3}}$$

Take the limits on both sides of the above equality and obtain

$$L_3 = \frac{L_3.L_4^b + 1}{L_3 + L_4^b} \quad \text{and} \quad L_4 = \frac{1/L_4.L_3^b + 1}{1/L_4 + L_3^b}$$

which imply that $L_3 = L_4 = 1$.

So, we have shown that

$$\lim_{n\to\infty} x_{p+6n+m} = \lim_{n\to\infty} y_{p+6n+m} = 1 \quad \text{for } m \in \{1, 2, 3, 4, 5, 6\}.$$

Similarly, by virtue of Lemma 2.2 (ii) and 2.6 one can see that the above equality holds. Therefore, the proof is complete. □

References

1. R. Abu-Saris , C. Cinar and I. Yalcinkaya, On the asymptotic stability of $x_{n+1} = \dfrac{x_n x_{n-k} + a}{x_n + x_{n-k}}$, *Computers Mathematics with Applications*, **56**(5) (2008), 1172-1175.

2. R. De Vault, G. Ladas and S. W. Schultz, On the recursive sequence $x_{n+1} = \dfrac{A}{x_n} + \dfrac{1}{x_{n-2}}$, *Proc. Amer. Math. Soc.*, **126**(11) (1998), 3257-3261.

3. M. R. S. Kulenovic, *Discrete Dynamical Systems and Difference Equations with Mathematica*, ACRC Press Company (2002).

4. X. Li and D. Zhu, Global asymptotic stability in a rational equation, *J. Diff. Equations Appl.*, **9** (2003), 833-839.

5. G. Papaschinopoluos and C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.*, **219** (1998), 415-426.

6. G. Papaschinopoluos and C. J. Schinas, On the behavior of the solutions of a system of two nonlinear difference equations, *Communications on Applied Nonlinear Analysis*, **5** (1998), 47-59.

7. G. Papaschinopoluos, C. J. Schinas and G. Stefanidou, On a k-order system of lyness-type difference equations, *Advances in Difference Equations*, (2007), Article ID 31272, 13 pages.

8. I. Yalcinkaya, C. Cinar and D. Simsek, Global asymptotic stability of a system of difference equations, *Applicable Analysis*, **87**(6) (2008), 689-699.