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ON THE THEORY OF CONTINUOUS TIME SERIES

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This paper considers spectral estimation for a zero-mean strictly stationary r-vector valued continuous time series. The case of interest is when some of observations are missing due to some random failure. Spectral estimation procedures are developed in disjoint segments of observations. Expanded finite Fourier transform, modified poriodogram and spectral density statistics are constructed. The theoretical properties of these estimators are developed. Asymptotic distributions are discussed

Key words : Disjoint segments of observations; modified periodogram; spectral density matrix and Wishart matrix.

1. INTRODUCTION

The importance of spectral densities for stochastic process and the usefulness of their estimation is well established in time series analysis literature. The question of spectral estimation has been studied extensively, see [5, 7, 8, 13, 22, 24, 25, 32, 35]. Methods for autocovariance and spectral estimation for stochastic processes sampled at irregular locations have been developed [4, 21]. Some spectral estimation techniques involve mapping the inherent irregular structure of time series so that regularly-spaced spectral analysis can be performed, for example through sampling [9, 12]. For an overview of preprocessing methods for spectral estimation of irregular

time series that parallel approaches built for regularly-spaced data, see [34]. Models specifically developed for time series with missing data and their use for spectral estimation have been discussed in the literature [10, 11]. Mondal and Percival [28] formulate unbiased spectral estimators assuming wavelet models of stationary time series and also investigate their asymptotic properties. If the missing observations occur with a periodic structure, Jones [24] provides a development for spectral estimation of a stationary time series. Parzen [31] developed the theory of amplitude-modulated stationary processes, and applied this theory to missing data problems [30], considering in detail the case when observations are missing in some periodic way. The amplitude-modulated series is constructed by replacing missing observations in the original series by their mean value. Schenok [33] considered the case when an observation is made or not according to the out come of a Bernoulli trial. Bloomfield [3] considered the case when a more general random mechanism is involved, see also [1, 6, 14, 15-20, 23, 26, 27, 29].

In this paper, we define the expanded finite Fourier transform in disjoint segments of observations in the case when there are some randomly missing observations. It is used to construct the modified periodogram. An estimator of the spectral density is constructed. Statistical properties of each statistic are developed. Asymptotic distributions for modified periodogram and spectral density estimators are derived. The paper is organized as follows. Section 2 introduces the basic definitions and used notations. The modified series is defined in Section 3. Section 4 considers the expanded finite Fourier transform and its properties. The modified periodogram, the spectral density estimator and its properties are given in Section 5.

2. Observed Series

Let $X(t)(t \in R)$ be a zero mean r-vector valued strictly stationary continuous time series, and all of whose moments exist. We set

$$E\{X(t+u)\overline{X}^{'}(t)\} = C_{XX}(u), (t, u \in R), \tag{2.1}$$

where $\overline{X}^{'}(t)$ is the complex conjugate transpose of $X(t)$. Suppose

$$\int_{-\infty}^{\infty} |C_{XX}(u)|\, du < \infty, \tag{2.2}$$

where $|C_{XX}(u)|$denotes the matrix of absolute values. We may then define $f_{XX}(\lambda)$ the $r \times r$ matrix of second order spectral densities by

$$f_{XX}(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} C_{XX}(u) \exp(-i\lambda u) du, (\lambda \in R). \tag{2.3}$$

Let the r-vector valued continuous time series $X(t)$ have real components $X_a(t)$, $(a = 1, 2, .., r)$. We set

$$C_{a_1,a_2,...a_k}(u_1, u_2, ..., u_{k-1}) = cum\{X_{a_1}(t + u_1), X_{a_2}(t + u_2), ..., X_{a_k}(t)\}, \tag{2.4}$$

$$(a_1, a_2, ..., a_k = 1, 2, ..., r; u_1, u_2, ..., u_{k-1}, t \in R; k = 2, ...).$$

Assumption I. $X(t)$ is strictly stationary continuous time series all of whose moments exist. For each $j = 1, 2, ..., k - 1$ and any k-tuple $a_1, a_2, ..., a_k$, we have

$$\int_{R^{k-1}} \left| u_j C_{a_1,a_2,...a_k}(u_1, u_2, ..., u_{k-1}) du_1 ... du_{k-1} \right| < \infty, (k = 2, 3, ...). \tag{2.5}$$

Because cumulants are measures of the joint dependence of random variables, (2.5) is seen to be a form of mixing or asymptotic independence requirement for values of $X(t)$ well separated in time. If $X(t)$satisfies Assumption I we may define its cumulant spectral densities by

$$\begin{aligned} & f_{a_1 a_2 ... a_k}(\lambda_1, \lambda_2, ..., \lambda_{k-1}) \\ = \; & (2\pi)^{-k+1} \int_{R^{k-1}} C_{a_1,a_2,...a_k}(u_1, u_2, ..., u_{k-1}) \times \exp(-i \sum_{j=1}^{k-1} \lambda_j u_j) \\ & du_1 ... du_{k-1}, (\lambda_j \in R, a_1, a_2, ..., a_k = 1, 2, ...r; k = 2, ...). \end{aligned} \tag{2.6}$$

If $k = 2$ the cross-spectra $f_{a_1 a_2}(\lambda)$ are collected together in the matrix $f_{XX}(\lambda)$ of (2.3).

Assumption II. Let $h_a^{(T)}(t) = h_a(\frac{t}{T}), t \in (0, T)$ is bounded, is of bounded variation and vanishes for all t outside the interval $(0, T)$, that is called data window.

3. Modified Series

Let $N(t) = \{N_a(t)\}_{a=1,2,...,r} (t \in R)$ be a process independent of $X(t)$ such that for every t

$$P\{N_a(t) = 1\} = p, P\{N_a(t) = 0\} = q, \tag{3.1}$$

note that

$$E\{N_a(t)\} = p. \tag{3.2}$$

The success of recording an observation not depend on the fail of another and so it is independent. We may then define the modified series

$$Y(t) = N(t)X(t), \tag{3.3}$$

where

$$Y_a(t) = N_a(t)X_a(t), \tag{3.4}$$

and

$$N_a(t) = \left\{ \begin{array}{ll} 1 & \text{if } X_a(t) \text{ is observed} \\ 0 & \text{if } X_a(t) \text{ is missed} \ , \end{array} \right\} \tag{3.5}$$

the properties of the modified series are discussed in [18].

4. Expanded Finite Fourier Transform

Elhassanein [15] constructed the expanded finite Fourier transform on disjoint segments of observations for a strictly stationary r-vector valued discrete time series in the case when there are some randomly missing observations, see also [17-19].

In this section, the expanded finite Fourier transform for a strictly stationary r-vector valued continuous time series in the case when there are some randomly missing observations will be constructed. Expression for its mean, variance and cumulant will be derived. The results introduced here may be regarded as an extent to Elhassanein [15], and Ghazal and Elhassanein [17-19].

Suppose that the series $X(t)$ is observed over the interval $(0, T)$. We split the interval into L disjoint segments each of length N. For $\lambda \in R$, we define the expanded finite Fourier transform of the modified series

$$\begin{aligned} d_Y^{(lN)}(\lambda) &= \left(2\pi \int_{lN}^{(l+1)N} [h^{(N)}(t - lN)]^2 dt \right)^{-\frac{1}{2}} \\ &\quad \times \int_{lN}^{(l+1)N} h^{(N)}(t - lN) \exp(-i\lambda t) Y(t) dt, \end{aligned} \tag{4.1}$$

where $l = 0, 1, ..., L-1, -\infty < \lambda < \infty$, and $h(t)$ is the data window satisfies *Assumption II*.

The moments of the expanded finite Fourier transform will be given in the following theorem.

***Theorem* 4.1** — *Let $X(t)(t \in (0,T))$ be a strictly stationary r-vector valued continuous time series with mean zero, and satisfy Assumption I. Let $d_a^{(lN)}(\lambda)$ be defined as (4.1), and $h_a(t)$ satisfy Assumption II, for $a = 1, 2, ..., r$, then*

$$E\{d_a^{(lN)}(\lambda)\} = 0 \tag{4.2}$$

$$\begin{aligned} & Cov\{d_a^{(lN)}(\lambda_1), d_b^{(lN)}(-\lambda_2)\} \\ = \; & p^2 \exp(-i(\lambda_1 - \lambda_2)lN) \int_{-N}^{N} C_{ab}(u) \exp(-i\lambda_1 u) H_{ab}^{(N)}(u, \lambda_1 - \lambda_2) du \\ = \; & p^2 \exp(-i(\lambda_1 - \lambda_2)lN) \int_{R} f_{ab}(\nu) \Phi_{ab}^{(N)}(\lambda_1 - \nu, \lambda_2 - \nu) d\nu \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} H_{ab}^{(N)}(u, \lambda_1 - \lambda_2) \;=\; & (2\pi)^{-1} \left[\int_0^N \int_0^N (\,\mathrm{h_a^{(N)}(t_1)})^2 (\,\mathrm{h_b^{(N)}(t_2)})^2 \mathrm{dt_1\, dt_2} \right]^{-\frac{1}{2}} \\ & \times \int_0^N h_a^{(N)}(u+t) h_b^{(N)}(t) \exp(-it(\lambda_1 - \lambda_2)) dt, \end{aligned}$$

and

$$\Phi_{ab}^{(N)}(\lambda_1, \lambda_2) = (2\pi)^{-1} \left[\int_0^N \int_0^N (h_a^{(N)}(t_1))^2 (h_b^{(N)}(t_2))^2 dt_1 dt_2 \right]^{-\frac{1}{2}} \times H_a^{(N)}(\lambda_1) \overline{H_b^{(N)}(\lambda_2)}$$

where

$$H_a^{(N)}(\lambda) = \int_0^N h_a^{(N)}(t) \exp(-i\lambda t) dt$$

for $\lambda_1 = \lambda_2 = \lambda, a = b$ then

$$Var\{d_a^{(lN)}(\lambda)\} = p \int_{-\infty}^{\infty} f_{aa}(\lambda - \nu) \Phi_{aa}^{(N)}(\nu) d\nu, \tag{4.4}$$

and

$$Cum\{d_{a_1}^{(lN)}(\lambda_1),...,d_{a_k}^{(lN)}(\lambda_k)\} = \tag{4.5}$$
$$(2\pi)^{\frac{k}{2}-1}p^k\left(\prod_{j=1}^{k}\int_0^N (h_{a_j}^{(N)}(t_j))^2dt_j\right)^{-\frac{1}{2}} f_{a_1a_2...a_k}(\lambda_1,\lambda_2,...,\lambda_{k-1})G_{a_1...a_k}^{(N)}(\sum_{j=1}^{k}\lambda_j)$$
$$+O(T^{-\frac{k}{2}}),$$

where

$$G_{a_1...a_k}^{(N)} = \int_0^N \left(\prod_{j=1}^{k} h_{aj}^{(N)}(t_j)\right)\exp(-i\lambda t)dt, \lambda\neq 0, \lambda, t\in R,$$

and $O(T^{-\frac{k}{2}})$ *is uniform in* $\lambda_1,\lambda_2,...,\lambda_{k-1}$ *as* $T\to\infty$.

PROOF : We will prove (4.5), by (4.1) we get

$$Cum\{d_{a_1}^{(lN)}(\lambda_1),...,d_{a_k}^{(lN)}(\lambda_k)\}$$
$$= (2\pi)^{\frac{k}{2}}p^k\left(\prod_{j=1}^{k}\int_0^N (h_{a_j}^{(N)}(t_j))^2dt_j\right)^{-\frac{1}{2}}\int_0^N...\int_0^N\left(\prod_{j=1}^{k}h_{a_j}^{(N)}(t_j)\right)\exp(-i\sum_{j=1}^{k}\lambda_jt_j)$$
$$\times C_{a_1,a_2,...a_k}(t_1-t_k,...,t_{k-1}-t_k)dt_1...dt_k,$$

let $t_j-t_k=u_j, t_k=t, j=1,2,...,k-1$, and since

$$\left|\int_0^N\left(\prod_{j=1}^{k-1}h_{aj}^{(N)}(u_j+t)\right)h_{a_k}^{(N)}(t)\exp(-i\lambda t)dt-\int_0^N\left(\prod_{j=1}^{k}h_{aj}^{(N)}(t_j)\right)\exp(-i\lambda t)dt\right|$$
$$\leq \quad M^{k-1}C\left(\sum_{j=1}^{k-1}|u_j|\right)$$

for some constants M, C and $(t_j,u_j,\lambda\in R, j=1,...,k)$ see [21], we get

$$Cum\{d_{a_1}^{(lN)}(\lambda_1),...,d_{a_k}^{(lN)}(\lambda_k)\} =$$
$$(2\pi)^{\frac{k}{2}}p^k\left(\prod_{j=1}^{k}\int_0^N (h_{a_j}^{(N)}(t_j))^2dt_j\right)^{-\frac{1}{2}}\int_0^N\left(\prod_{j=1}^{k}h_{a_j}^{(N)}(t_j)\right)\exp(-it\sum_{j=1}^{k}\lambda_j)dt$$
$$\times\int_{-N}^{N}...\int_{-N}^{N}C_{a_1,a_2,...a_k}(u_1,...,u_{k-1})\exp(-i\sum_{j=1}^{k-1}\lambda_ju_j)du_1...du_{k-1}+\varepsilon_T,$$

where

$$|\varepsilon_T| \leq (2\pi)^{\frac{k}{2}} p^k \left(\prod_{j=1}^{k} \int_0^N (h_{a_j}^{(N)}(t_j))^2 dt_j \right)^{-\frac{1}{2}} \times \int_{-N}^{N} ... \int_{-N}^{N} M^{k-1} C \left(\sum_{j=1}^{k-1} |u_j| \right) \left| C_{a_1,a_2,...a_k}(u_1, ..., u_{k-1}) \right| du_1 ... du_{k-1}$$

since $h_{a_j}^{(N)}(t_j)$ satisfy Assumption II for $j = 1, ..., k$ then

$$\prod_{j=1}^{k} \int_0^{N-1} (h_{a_j}^{(N)}(t_j))^2 dt_j \sim T^k \prod_{j=1}^{k} \int_0^1 (h(u_j))^2 du_j$$

which implies to $\varepsilon_T = O(T^{-\frac{k}{2}})$, using (2.6) the proof is completed. □

The expanded finite Fourier transform (4.1), will be used to construct the modified periodogram in the following section.

5. Spectral Density Estimate

In this section, the modified periodogram and the smoothed spectral density estimator will be constructed. Statistical properties of each estimator will be discussed by deriving mean, covariance and cumulant. The asymptotic distribution for each estimator will be derived.

Using expanded finite Fourier transform (4.1), we construct the modified periodogram as

$$I_{ab}^{(lN)}(\lambda) = \left(2\pi p^2 \int_{lN}^{(l+1)N} h_a^{(N)}(t) h_b^{(N)}(t) dt) \right)^{-1} z_a^{(lN)}(\lambda) \overline{z_b^{\prime(lN)}(\lambda)}, \tag{5.1}$$

where

$$z_a^{(lN)}(\lambda) = \sqrt{2\pi \int_{lN}^{(l+1)N} \left[h_a^{(N)}(t) \right]^2 dt} d_a^{(lN)}(\lambda) \tag{5.2}$$

where the bar denotes the complex conjugate. The smoothed spectral density estimate is constructed as

$$f_{ab}^{(T)}(\lambda) = \frac{1}{L} \int_0^L I_{ab}^{lN}(\lambda) du, a, b = 1, 2, ..., r. \tag{5.3}$$

The moments of the modified periodogram will be given in the following theorem.

***Theorem* 5.1** — *Let $X(t)(t \in R)$ be a strictly stationary r-vector valued continuous time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{(lN)}(\lambda) = \{I_{ab}^{(lN)}(\lambda)\}_{a,b=1,2,...,r}$ be given by (5.1), and $h_a(t)$ satisfy Assumption II for $a = 1, 2, ..., r$, then*

$$E\{I_{ab}^{lN}(\lambda)\} = f_{ab}(\lambda) + O(N^{-1}), p \to 1 \tag{5.4}$$

$$\begin{aligned}
&Cov\{I_{a_1b_1}^{lN}(\lambda_1), I_{a_2b_2}^{lN}(\lambda_2)\} \\
&= (G_{a_1b_1}G_{a_2b_2}\Phi_{a_1b_1}^{N}(0)\Phi_{a_2b_2}^{N}(0))^{-1} \\
&\times[G_{a_1a_2}G_{b_1b_2}\Phi_{a_1a_2}^{N}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{N}(\lambda_1-\lambda_2)}f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \\
&+G_{a_1b_2}G_{b_1a_2}\Phi_{a_1b_2}^{N}(\lambda_1+\lambda_2)\overline{\Phi_{b_1a_2}^{N}(\lambda_1+\lambda_2)}f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1) \\
&+(2\pi)G_{a_1b_1a_2b_2}\Phi_{a_1b_1a_2b_2}^{N}(0)f_{a_1b_1a_2b_2}(\lambda_1,-\lambda_1,\lambda_2)] + O(N^{-1})
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&Cum\{I_{a_1b_1}^{l_1N}(\lambda_1), ..., I_{a_kb_k}^{l_kN}(\lambda_k)\} \\
= \;&(\prod_{i=1}^{k} G_{a_ib_i}\Phi_{a_ib_i}^{(N)}(0))^{-1}\sum\{\prod_{j=1}^{k} G_{a_jb_j}\exp(-il_kN\sum_{j=1}^{k}(\mu_j+\gamma_j)) \\
&\times\Phi_{c_jd_j}^{(N)}(\mu_j+\gamma_j)\}\{\prod_{j=1}^{k} f_{c_jd_j}(\mu_j)\} + O(N^{-1})
\end{aligned} \tag{5.6}$$

where the summation extends over all partitions $\{(c_1,\mu_1),(d_1,\gamma_1)\}, ..., \{(c_k,\mu_k),(d_k,\gamma_k)\}$, into pairs of the quantities $(a_1,\lambda_1),(b_1,-\lambda_1), ..., (a_k,\lambda_k),(b_k,-\lambda_k)$ excluding the case with $\mu_j = -\gamma_j = \lambda_m$ for some j, m,where $O(N^{-1})$ is uniform in $\lambda_1, ..., \lambda_k$.

PROOF : By (5.1), we have

$$\begin{aligned}
E\{I_{ab}^{lN}(\lambda)\} &= (p^2G_{ab}\Phi_{ab}^{N}(0))^{-1}E\{d_a^{lN}(\lambda)\overline{d_b^{lN}(\lambda)}\} \\
&= Cov\{d_a^{lN}(\lambda), d_b^{lN}(\lambda)\}
\end{aligned}$$

then by (4.3) the proof of (5.4) is completed. From (5.1), and by Theorem (2.3.2) in [5], we

have

$$\begin{aligned} & Cov\{I^{lN}_{a_1b_1}(\lambda_1), I^{lN}_{a_2b_2}(\lambda_2)\} \\ = \; & Cov\{d^{lN}_{a_1}(\lambda_1)d'^{lN}_{b_1}(-\lambda_1), d^{lN}_{a_2}(\lambda_2)d'^{lN}_{b_2}(-\lambda_2)\} \\ & Cum\{d^{lN}_{a_1}(\lambda_1), d^{lN}_{b_1}(-\lambda_1), d^{lN}_{a_2}(\lambda_2), d^{lN}_{b_2}(-\lambda_2)\} \\ & +Cov\{d^{lN}_{a_1}(\lambda_1), d^{lN}_{a_2}(\lambda_2)\}Cov\{d^{lN}_{b_1}(-\lambda_1), d^{lN}_{b_2}(-\lambda_2)\} \\ & +Cov\{d^{lN}_{a_1}(\lambda_1), d^{lN}_{b_2}(-\lambda_2)\}Cov\{d^{lN}_{b_1}(-\lambda_1), d^{lN}_{a_2}(\lambda_2)\}. \end{aligned}$$

By Theorem (4.1) the proof of (5.5) is completed. From (5.1), we have

$$\begin{aligned} & Cum\{I^{l_1N}_{a_1b_1}(\lambda_1), ..., I^{l_kN}_{a_kb_k}(\lambda_k)\} \\ = \; & p^{-2k}(\prod_{i=1}^{k} G_{a_ib_i}\Phi^{(N)}_{a_ib_i}(0))^{-1} \\ & \times Cum\{d^{l_1N}_{a_1}(\lambda_1)d'^{l_1N}_{b_1}(-\lambda_1), ..., d^{l_kN}_{a_k}(\lambda_k)d'^{l_kN}_{b_k}(-\lambda_k)\} \end{aligned}$$

By Theorem (2.3.2) in [5] pp. 21, we get

$$\begin{aligned} & Cum\{d^{l_1N}_{a_1}(\lambda_1)d'^{l_1N}_{b_1}(-\lambda_1), ..., d^{l_kN}_{a_k}(\lambda_k)d'^{l_kN}_{b_k}(-\lambda_k)\} \\ = \; & \sum_{\nu} Cum\{d^{l_iN}(\lambda_i); i \in \nu_1\}...Cum\{d^{l_iN}(\lambda_i); i \in \nu_s\}, \end{aligned}$$

where the summation extends over all indecomposable partitions $\nu = [\cup_{j=1}^{s}\nu_j] \in I$, $I = (a_1, ..., a_k\,;\, b_1, ..., b_k)$, $1 \le s \le k$ of the transformed table

$$\begin{array}{cccc} (a_1,\lambda_1), & (b_1,-\lambda_1) & & \{(c_1,\mu_1), \quad (d_1,\gamma_1)\} \\ (a_2,\lambda_2), & (b_2,-\lambda_2) & & \{(c_2,\mu_2), \quad (d_2,\gamma_2)\} \\ . & . & \rightarrow & . \qquad\quad . \\ . & . & & . \qquad\quad . \\ (a_k,\lambda_k), & (b_k,-\lambda_k) & & \{(c_k,\mu_k), \quad (d_k,\gamma_k)\}. \end{array}$$

Then, by Theorem (4.1), we get the proof of (5.6). □

Using Theorem (4.1) the asymptotic distribution of the modified periodogram will be derived in the following theorem.

***Theorem* 5.2** — *Let $X(t)(t \in R)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $I^{lN}_{YY}(\lambda) = \{I^{lN}_{ab}(\lambda)\}_{a,b=1,2,...,r}$ be given by*

(5.1), $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0$ *(mod* 2π*) for* $1 \leq j < k \leq J$ *and* $\Phi_a(t)$ *satisfy Assumption II for* $a = 1, 2, ..., r,$ *. Then* $I_{YY}^{lN}(\lambda_j)$*,* $j = 1, 2, ..J$ *are asymptotically independent* $W_r^c(1, f_{XX}(\lambda_j))$ *variates. Also if* $\lambda = \pm\pi, \pm 3\pi$*.. then* $I_{YY}^{lN}(\lambda)$ *is asymptotically* $W_r(1, f_{XX}(\lambda))$ *independent of the previous variates. Where,* $W_r(\gamma, \sum)$ *denotes an* $r \times r$ *symmetric matrix-valued Wishart variate with covariance matrix* $\sum$ *and* γ *degree of freedom and* $W_r^c(\gamma, \sum)$ *denotes an* $r \times r$*. Hermitian matrix-valued complex Wishart variate with covariance matrix* $\sum$ *and* γ *degree of freedom.*

PROOF : The proof comes directly from Theorem (4.1), for more details about Wishart distribution see [2]. □

In the following theorem we will prove that the spectral density estimator $f_{ab}^{(T)}(\lambda)$ given by (5.3) is asymptotically unbiased estimator for the spectral density function.

***Theorem* 5.3** — *Let* $X(t)(t \in R)$ *be a strictly stationary* r*-vector valued time series with mean zero, and satisfy Assumption I. Let* $f_{ab}^{(T)}(\lambda)$ *be given by (5.3),* $a, b = 1, 2, ..., r,$ *then*

$$E\{f_{ab}^{(T)}(\lambda)\} = f_{ab}(\lambda) + O(N^{-1}) \tag{5.7}$$

$$\begin{aligned}
& Cov\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\} \\
= \; & (L^2\Phi_{a_1b_1}^{(N)}(0)\Phi_{a_2b_2}^{(N)}(0))^{-1} \int_0^L \int_0^L (G_{a_1b_1}(l_1, l_1)G_{a_2b_2}(l_2, l_2))^{-1} \\
& \times [G_{a_1a_2}(l_1, l_2)G_{b_1b_2}(l_1, l_2)\Phi_{a_1a_2}^{(N)}(\lambda_1 - \lambda_2)\overline{\Phi_{b_1b_2}^{(N)}(\lambda_1 - \lambda_2)} \\
& \times \exp(-il_2N(\lambda_1 - \lambda_2))f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \\
& + G_{a_1b_2}(l_1, l_2)G_{b_1a_2}(l_1, l_2)\Phi_{a_1b_2}^{(N)}(\lambda_1 + \lambda_2)\overline{\Phi_{b_1a_2}^{(N)}(\lambda_1 + \lambda_2)} \\
& \times \exp(-il_2N(\lambda_1 + \lambda_2))f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1) \\
& + (2\pi)G_{a_1b_1a_2b_2}(l_1, l_1, l_2, l_2)\Phi_{a_1b_1a_2b_2}^{(N)}(0)f_{a_1b_1a_2b_2}(\lambda_1, -\lambda_1, \lambda_2)]du_1du_2 + O(N^{-1})
\end{aligned} \tag{5.8}$$

PROOF : By (5.3), we have

$$E\{f_{ab}^{(T)}(\lambda)\} = \frac{1}{L}\int_0^L E\{I_{ab}^{lN}(\lambda)\}du$$

then by (5.4) the proof of (5.7) is completed. From (5.3), we get

$$\begin{aligned} & Cov\{f^{(T)}_{a_1b_1}(\lambda_1), f^{(T)}_{a_2b_2}(\lambda_2)\} \\ = \quad & \frac{1}{L^2}\int_0^L\int_0^L Cov\{I^{lN}_{a_1b_1}(\lambda_1), I^{lN}_{a_2b_2}(\lambda_2)\}du_1du_2. \end{aligned}$$

which completes the proof of (5.8). □

Using Theorem (5.2) the asymptotic distribution of the spectral density estimator will be derived in the following theorem.

***Theorem* 5.4** — *Let $X(t)(t \in R)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $f^T_{ab}(\lambda)$ be given by (5.3), $a, b = 1, 2, ..., r$, $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0$ (mod 2π) for $1 \leq j < k \leq J$. Then $Lf^T_{ab}(\lambda_j)$, $j = 1, 2, ..J$ are asymptotically independent $W^c_r(L, f_{ab}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi..$ then $Lf^T_{ab}(\lambda)$ is asymptotically $W_r(L, f_{ab}(\lambda))$ independent of the previous variates.*

PROOF : The proof comes directly by Theorem (5.2) and Theorem (7.3.2) in [2] pp. 162.□

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