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MAPS TO WEIGHT SPACE IN HIDA FAMILIES

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Abstract. Let $\bar{\rho}$ be a two-dimensional $\mathbb{F}_p$-valued representation of the absolute Galois group of the rationals. Suppose $\bar{\rho}$ is odd, absolutely irreducible and ordinary at p. Then we show that $\bar{\rho}$ arises from the irreducible component of a Hida family (of necessarily greater level than that of $\bar{\rho}$) whose map to weight space, including conjugate forms, has degree at least 4.

Key words : Galois representation; modular form; Hida theory.

1. Introduction

Let $p \geq 5$ be an odd prime and $f \in S_2(\Gamma_0(Np))$ a weight two eigenform that is new of level Np where $p \nmid N$. Let ρ_f and $\bar{\rho}_f$ be the p-adic and mod p representations of $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ associated to f. We assume $\bar{\rho}_f$ is absolutely irreducible so it is well-defined. As p is in the level of f, the eigenvalue of U_p is ± 1 and ρ_f is ordinary at p, in particular $\rho_f|_{G_p = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \psi\epsilon & * \\ 0 & \psi^{-1} \end{pmatrix}$ where ϵ is the p-adic cyclotomic character and ψ is unramified of order 1 or 2. We know that f belongs to a Hida family, by which in this paper we mean the irreducible component (of the spectrum) of the ordinary (arbitrary weight) Hecke algebra of tame level N containing f. We will abuse the term 'Hida family' to refer to both the ring $\mathbb{T}$ and $\mathcal{T} = Spec(\mathbb{T})$. When we say the Hida family contains a point, we are referring to a map $Spec(R) \to \mathcal{T}$ for a suitable extension R of $\mathbb{Z}_p$. Dimitrov and Ghate refer to the collection of all components at a fixed tame

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level having residual global representation $\bar{\rho}_f$ as its *Hida community.* It is well-known that the Hida family containing f is a finite flat $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$-algebra $\mathbb{T}$ that is an integral domain possessing homomorphisms $\mathbb{T} \to \bar{\mathbb{Z}}_p$ that correspond to classical eigenforms of weights $k \geq 2$ (and sometimes weight $k = 1$). If the reader prefers, rather than f, she may keep in mind the example of an elliptic curve with semistable reduction at p.

While it is known that f determines $\mathbb{T}$, little is known about how to recover the explicit information of the family $\mathbb{T}$ from $f = \sum_{n=1}^{\infty} a_n q^n$. C. Franc has pointed out that given two eigenforms g and h of level M that are congruent mod p, there seems to be no known algorithm to determine whether g and h are in the same family!

For f with split multiplicative reduction, the p-adic L-function $L_p(f,s)$ has a trivial zero at $s = 1$. In the elliptic curve case Mazur, Tate and Teitelbaum conjectured a relation between the classical L-function $L(E,1)$ and $L_p'(E,1)$. Recall the $\mathcal{L}$-invariant of a semistable at p elliptic curve E with Tate period q_E is

$$\mathcal{L}_E = \frac{\log q_E}{v_p(q_E)}. \tag{1.1}$$

In [MTT] it was conjectured

$$L_p(E,1) = \mathcal{L}_E \frac{L(E,1)}{\Omega_E} \tag{1.2}$$

where Ω_E is the real period of E. Greenberg and Stevens proved this conjecture in [GS] using Hida theory and by relating $\mathcal{L}_E$ to the derivative of the $Frob_p$-eigenvalue in the Hida family. Indeed, their result applies to f split multiplicative but in this more general case computing $\mathcal{L}_f$ explicitly is very involved. See for instance [CST]. The $\mathcal{L}$-invariant contains other interesting information. In [GS2] Greenberg and Stevens gave a simple criterion in terms of $\mathcal{L}_f$ that guarantees the existence of another weight 2 point in $\mathcal{T}$ whose level is prime to p. Below is a slight generalization of their result, Proposition 5.1 of [GS2].

Theorem 1. *(Greenberg-Stevens) Let $p \geq 5$ be a prime and $f \in S_2(\Gamma_0(Np))$ have multiplicative reduction at p. Suppose $\bar{\rho}_f$ is absolutely irreducible as a $G_{\mathbb{Q}}$-module and that $v_p(\mathcal{L}_f) < 1$. Then the Hida family $\mathcal{T}$ containing f contains another weight 2 form of level N prime to p. In particular the map $\Lambda = \mathbb{Z}_p[[T]] \to \mathbb{T}$ is not an isomorphism.*

The differences between the result of [GS2] and Theorem 1 above are that we state the result along a particular Hida family (irreducible component), work with modular forms rather than elliptic curves and we do not include the hypothesis of *split* multiplicative reduction. If the reduction is nonsplit, one can always twist the curve, and in fact the whole Hida family, by a quadratic character unramified at p to get a split at p elliptic curve. The existence of a weight two form with p not in the level is insensitive to such a twist. Later in the paper we make it a point to avoid nebentype, but that is of order p and independent of this issue.

While a proof of Theorem 1 is included here, we emphasize *it is due to Greenberg and Stevens.*

In this paper the term 'weight space map' refers to the natural map $\mathbb{Z}_p[[T]] \to \mathbb{T}$. In particular by it's degree we mean the degree of the finite map $\mathcal{T} \to Spec(\mathbb{Z}_p[[T]])$. Equivalently, the degree is, for any $k \geq 2$, the $\mathbb{Z}_p$-rank of the weight k quotient of $\mathbb{T}$. In Theorem 1, the degree of the weight space map is at least 2. The Greenberg-Stevens result motivated our interest in proving the result below.

Theorem 2. *Let $p \geq 5$ be a prime and $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ be an odd absolutely irreducible representation such that, up to a quadratic twist, $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ is indecomposable and peu ramifié. There exist infinitely many Hida families associated to $\bar{\rho}$ that are isomorphic to $\mathbb{Z}_p[[U]]$ and whose associated maps to weight space have degree at least* 4. *If $\bar{\rho}$ has Artin conductor N, these families contain weight two points of level NQ and NQp for varying Q.*

A delicate analysis of various Selmer groups and dual Selmer groups associated to deformation problems is the key technical tool for proving Theorem 2 and is used throughout §3. Given an odd ordinary absolutely irreducible residual representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_q)$, it is known in most cases, by Selmer group techniques of [R2] and [T] that there exists an ordinary lift of $\bar{\rho}$ to the Witt vectors, $W(\mathbb{F}_q)$, of $\mathbb{F}_q$ corresponding to a classical ordinary at p modular form g (of almost certainly higher level). With some extra work one can often prove, *though not in the case considered in this paper*, that the map $W(\mathbb{F}_q)[[T]] \to \mathbb{T}$ is an isomorphism. All classical forms in such a family have Fourier coefficients in $W(\mathbb{F}_q)$. See B. Lundell's thesis, [L].

The term peu ramifié is explained in detail in [S]. Briefly, it means that $\bar{\rho}|_{G_p}$ comes from a finite flat group scheme over $\mathbb{Z}_p$. Alternatively, the kernel of $\bar{\rho}|_{G_p}$ fixes the splitting field over $\mathbb{Q}_p$ of $x^p - u$ for some $u \in \mathbb{Z}_p^*$. The hypotheses of Theorem 1 imply the representation there is peu ramifié.

There is the natural question: Are there Hida families where the degree of the weight space map is arbitrarily large? Forthcoming joint work with Khare, [KR], that addresses this. A more basic (and probably more difficult) question is: Given only f classical and ordinary belonging to a unique $\mathbb{T}$, can one determine the degree of the map $\mathbb{Z}_p[[T]] \to \mathbb{T}$?

Below are some notations used in this paper. We will denote by S the union of the set of ramified primes of $\bar{\rho}$ and $\{p, \infty\}$ and by T a finite set containing S. Let $G_{\mathbb{Q}}$ and G_v denote the absolute Galois groups of $\mathbb{Q}$ and $\mathbb{Q}_v$ respectively and for a set X of places G_X will be the Galois group over $\mathbb{Q}$ of the maximal extension of $\mathbb{Q}$ unramified outside X. We have already repeated notation by using T as a set and as the variable of the Iwasawa algebra $\mathbb{Z}_p[[T]]$, but as there will be no ambiguity, we continue with these notations. Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ be an absolutely irreducible Galois representation whose restriction to G_p is multiplicative peu ramifié and indecomposable. Let N be the tame conductor of $\bar{\rho}$. The primes dividing N belong to S. Let ϵ be the p-adic cyclotomic character, $\bar{\epsilon}$ its mod p reduction and ω be the Teichmüller lift of the mod p cyclotomic character $\bar{\epsilon}$. Recall that if, up to an unramified quadratic twist, a representation restricted to G_p is given by $\begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ there are two possibilities for the $*$, flat or semistable. These terms are synonomous with peu and très ramifié case respectively, très meaning 'not peu'.

One may look for lifts of $\bar{\rho}$ to characteristic zero that are weight 2 and flat at p, weight 2 and semistable at p or ordinary at p of any weight. In each case we form a deformation problem whose tangent space is given by a certain Selmer group. Flat corresponds to p not being in the level of the weight two form while semistable means p is in the level.

For the basics of deformation theory we refer the reader to [M2]. See [R2] and [T] for the particulars of Selmer groups used in the second half of this paper, though [T] uses the better language of dual Selmer groups.

In section 2 we recall the bare basics we need of Hida Theory and give the proof of Theorem 1. In section 3 we recall the deformation theory we will need, do a few Galois cohomology computations and prove Theorem 2. The main technical result of [KLR] plays a crucial role in this proof. We make no explicit use of '$R = T$' theorems in this paper. We do assume $\bar{\rho}$ is modular and the proof of Serre's conjecture of course involves '$R = T$' theorems.

The theory of $\mathcal{L}$-invariants has been generalized by various authors, e.g. Benois, Bertolini-Darmon-Iovita, Breuil, Coleman, Fontaine-Mazur, Stevens et al. to cases of higher weight using the eigencurve. The methods here may extend to these cases. Finally, my thanks go to Chandrashekhar Khare for catching an inaccuracy in a talk on this material that I gave at TIFR and to Benjamin Lundell and Eknath Ghate for helpful suggestions.

2. Hida Families

Given an eigenform f with multiplicative reduction at p, Hida proved there is a unique family (called a Hida family) containing f. There are various ways to consider the family. We will think of it as an integral domain $\mathbb{T}$ equipped with the weight space map $\Lambda = \mathbb{Z}_p[[T]] \to \mathbb{T}$ from the Iwasawa algebra. In fact, we need the normalization of $\mathbb{T}$ but abuse notation and refer to this normalization by $\mathbb{T}$ as well. There is also an attached Galois representation $\rho_{\mathbb{T}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T})$. Homomorphisms $\phi_k : \mathbb{T} \to \bar{\mathbb{Z}}_p$ for $k \in \mathbb{Z}_{\geq 2}$ satisfying $det \circ \phi_k \circ \rho_{\mathbb{T}} = \epsilon^{k-1}\omega^{2-k}$ correspond to Galois representations associated to cuspidal eigenforms of weight k in the family. We emphasize that $\Lambda = \mathbb{Z}_p[[T]] \to \mathbb{T}$ need not be an isomorphism. In general all we can say is that $\mathbb{T}$ is finite, flat and reduced over $\mathbb{Z}_p[[T]]$. Various quantities are locally p-adic analytic functions of the weight which in turn is, at classical points, locally analytic in T. We will take various derivatives at these classical points. A standard reference for Hida Theory is [H1]. See also [H2] for related work on $\mathcal{L}$-invariants.

Let γ be a topological generator of the Galois group over $\mathbb{Q}$ of its unique $\mathbb{Z}_p$ extension satisfying $\epsilon(\gamma) = 1 + p$. Proposition 3 is well-known.

Proposition 3. *Let $f \in S_2(\Gamma_0(Np))$ be an ordinary at p eigenform. Then f belongs to a Hida family $\mathbb{T}$ whose Galois representation satisfies $\rho_{\mathbb{T}}|_{G_p} = \begin{pmatrix} \beta\alpha\delta & * \\ 0 & \beta^{-1} \end{pmatrix}$ where α factors through the Galois group of the $\mathbb{Z}_p$ extension of $\mathbb{Q}_p$, β is unramified and δ factors through $Gal(\mathbb{Q}(\mu_p)/\mathbb{Q})$.*

Set $a = \alpha(\gamma)$ and $b = \beta(Frob_p)$. Note $a = 1 + T$ when we recall that $T \in \mathbb{T}$ via the map $\mathbb{Z}_p[[T]] \to \mathbb{T}$.

Proposition 4. *Let $\mathcal{P}$ be the set of kernels of the homomorphisms $\pi_2 : \mathbb{T} \to \bar{\mathbb{Z}}_p$ corresponding to weight two points. Then $(a - 1 - p) = \bigcap_{\wp \in \mathcal{P}} \wp$.*

Proof. By Proposition 3 and its discussion the composite $\rho_{\mathbb{T}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}) \to GL_2(\mathbb{T}/(a-1-p))$ has determinant the cyclotomic character. Also $\mathbb{T}/(a-1-p)$ is finite flat and reduced over $\mathbb{Z}_p$. As $a-1-p \in \wp$ for all $\wp \in \mathcal{P}$, we see $(a-1-p) \subseteq \bigcap_{\wp \in \mathcal{P}} \wp$. Since weight two Hecke algebras and spaces of weight two cusp forms are dual, the surjection $\mathbb{T}/(a-1-p) \twoheadrightarrow \mathbb{T}/\left(\cap_{\wp \in \mathcal{P}} \wp\right)$ is between free $\mathbb{Z}_p$-modules of equal rank and hence an isomorphism so $(a-1-p) = \bigcap_{\wp \in \mathcal{P}} \wp$. □

Proof of Theorem 1. Suppose all weight two points of $\mathbb{T}$ are of level Np. Then $U_p^2 = 1$ on these forms and after a possible quadratic twist we may assume $U_p = 1$ at all weight two forms. Thus $b = 1$ at all these points. Namely we have the implications $\pi_2(a) = 1 + p \implies \pi_2(b) = 1$ for all $\pi_2 : \mathbb{T} \to \bar{\mathbb{Z}}_p$ of weight two. We rewrite this as $a - 1 - p \in \wp \implies b - 1 \in \wp$ for all $\wp \in \mathcal{P}$. Taking intersections over all $\wp \in \mathcal{P}$ and using Proposition 4, our assumption that all weight two points of $\mathbb{T}$ have level Np gives $b - 1 \in \bigcap_{\wp \in \mathcal{P}} \wp = (a-1-p)$. We write $b - 1 = (a-1-p)x$ for some $x \in \mathbb{T}$. Differentiating this equation with respect to T and evaluating at f, we have, as $a|_f = 1 + p$ and $a = 1 + T$,

$$\left.\frac{db}{dT}\right|_f = \left.\left((a-1-p)\,\frac{dx}{dT} + \frac{da}{dT}x\right)\right|_f = \left.\left(\frac{da}{dT}x\right)\right|_f = x|_f.$$

So

$$v_p\left(\left.\frac{db}{dT}\right|_f\right) = v_p\left(x|_f\right) \geq 0 \tag{2.1}$$

as $x \in \mathbb{T}$ is integral. Note $\dfrac{db}{dk} = \dfrac{db/dT}{dk/dT}$ and, as a function of k, $a(k) = (1+p)^{k-1}$. Thus $(k-1)\log_p(1+p) = \log_p a$ so $\left.\dfrac{dk}{dT}\right|_f = \dfrac{1}{(1+p)\log_p(1+p)}$. Recall from [GS] that $\mathcal{L}_f = -2\left.\dfrac{db}{dk}\right|_f$. Thus

$$\mathcal{L}_f = -2\left.\frac{db}{dk}\right|_f = -2\left.\frac{db/dT}{dk/dT}\right|_f = -2(1+p)\log_p(1+p)\left.\frac{db}{dT}\right|_f = \left(-2(1+p)x\log_p(1+p)\right)|_f. \tag{2.2}$$

As $v_p(\log_p(1+p)) = 1$, we see from (2.1) that $v_p(\mathcal{L}_f) \geq 1$, a contradiction. Thus if $v_p(\mathcal{L}_f) < 1$ there is a weight two point of level N on $\mathbb{T}$. □

Corollary 5. *(Greenberg-Stevens) Suppose now that f corresponds to an elliptic curve E with multiplicative reduction at p. Recall that, up to a quadratic twist,* $\bar{\rho}_E|_{G_p} =$

$\begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$. *If $*$ is très ramifié then $v_p(\mathcal{L}_E) \geq 1$. If $v_p(\mathcal{L}_E) < 1$ then the $*$ is is peu ramifié.*

Proof. As with any element of $p\mathbb{Z}_p$ we may write $q_E = p^{p^s t}\zeta_{p-1}(1+p^u x)$ where $s,t \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{Z}_{\geq 1}$, $x \in \mathbb{Z}_p$, $p \nmid t, x$ and ζ_{p-1} is a p-1st root of unity. Then by (1.1) $\mathcal{L}_E = \dfrac{\log_p(1+p^u x)}{p^s t}$ so $v_p(\mathcal{L}_E) = u - s$. In the très ramifié case $s = 0$ so $v_p(\mathcal{L}_E) \geq 1$.

We already know by Theorem 1 that if $v_p(\mathcal{L}_E) < 1$ then there is a weight two point on $\mathbb{T}$ of level prime to p. Thus $\bar{\rho}|_{G_p}$ comes from a finite flat group scheme over $\mathbb{Z}_p$ and we must be in the peu ramifié case, namely $s \geq 1$. □

Remark 1. It is easy to find elliptic curves over $\mathbb{Q}$ to which Corollary 5 applies. For instance choose a semistable at p curve E with q_E be p-adically close to $p^p(1+p)$. Theorem 1 applies when $s > u \geq 2$, even though in this case $\bar{\rho}_E|_{G_p}$ is decomposable, hence peu ramifié. When $u \geq s+1 \geq 2$ we have that $v_p(\mathcal{L}_E) \geq 1$ but $\bar{\rho}|_{G_p}$ is again decomposable. Mazur has proved that p can be removed from the level of $\bar{\rho}$ in this case but we do not know whether the weight 2 level N point lies on the same component as f_E. Finally, it is conjectured that $\mathcal{L}_E$ is never 0.

3. Deformation Theory

We informally summmarize the technical details of this section. Throughout this section $p \geq 5$, $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ is an odd absolutely irreducible representation, and, up to an unramified quadratic twist, $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ and is flat (peu ramifié) and indecomposable. The example of a semistable at p elliptic curve with $v_p(q_E) \equiv 0$ mod p but q_E is not a pth power in $\mathbb{Z}_p$ ($u = 1$ in Remark 1) remains our prototype. Recall S is the union of $\{p, \infty\}$ and the ramified primes of $\bar{\rho}$. In the various deformation problems we will impose local conditions at all places v. For $v \in S \backslash \{p\}$, $v \neq p$ our local condition is that of being *minimally ramified* as in [D] and appropriate quotients of the local deformation rings are smooth of adequate dimension for a global characteristic zero deformation to exist, so we simply refer the reader to [R2] (Proposition 1) and [T]. The point of the plethora of definitions below is that the various deformation problems have mod p Selmer and dual Selmer groups attached to them. These are described in terms of global Galois cohomomology and are relevant to Theorem 2.

One can allow ramification at nice primes (see Definition 7) and insist on deformations that

- have determinant ϵ (fixing the weight to be two),
- are 'new' at all nice primes (see Definition 7),
- minimally ramified at $v \in S \backslash \{p\}$
- and either flat or semistable at p.

These deformation functors are representable. The local (versal) deformation rings are all smooth. See [R2] and [T]. For the semistable case when $v = p$, see the discussion of the **not flat** case in Propostion 3 and Table 3 of [R2]. The proofs there apply to the semistable deformation ring for $\bar{\rho}|_{G_p}$ flat as well. If one chooses the nice primes to annihilate the dual Selmer group (which is possible) the functor is then represented by $\mathbb{Z}_p$. This means there is exactly one characteristic zero deformation of $\bar{\rho}$ with local properties described above. This applies in weight two and is done in [R2] and [T]. It turns out an arbitrary weight ordinary theory often exists. Here we sketch this theory for our $\bar{\rho}|_{G_p}$ indecomposable and peu ramifié. Lundell has worked out most other cases in his thesis. See [L]. The nice primes that we need satisfy nonempty Chebotarev conditions and hence exist in abundance.

For our flat (peu ramifié) $\bar{\rho}$ we can annihilate the dual Selmer group of the flat or semistable deformation problem (both weight two) by allowing ramification at carefully chosen nice primes as in [T] and [R2]. One can also annihilate the dual Selmer group for the ordinary weight two deformation problem, but the local at p ordinary deformation ring is **not** smooth so Galois cohomological techniques do not (as far as I know) yield a characteristic zero lift. If, however, one tries this for the ordinary *arbitrary weight* deformation problem, the dual Selmer group will be annihilated but the Selmer group will be one dimensional which implies the relevant deformation ring, $R_T^{ord,T\backslash S-new}$, is a quotient of $\mathbb{Z}_p[[U]]$. By work of Diamond and Taylor [DT] this ring will have a characteristic zero point corresponding to an ordinary modular form of weight two. This point lies on a unique Hida family whose infinitely many classical characteristic zero points are parametrized by $R_T^{ord,T\backslash S-new}$ and thus $R_T^{ord,T\backslash S-new} \simeq \mathbb{Z}_p[[U]]$ surjects onto this Hida family. This surjection is then necessarily an isomorphism, so $R_T^{ord,T\backslash S-new} \simeq \mathbb{Z}_p[[U]]$ is a Hida family of suitable tame level that includes the nice primes in $T\backslash S$. For the case considered here, the local at p ordinary arbitrary weight deformation ring can be shown to be smooth, but the above argument does not use this smoothness.

Using results from [KLR] we will construct examples where $\mathbb{Z}_p[[U]]$ has at least two flat points and two semistable points. All flat points might be conjugate. All semistable points may be conjugate as well. Thus the map $\mathbb{Z}_p[[T]] \to \mathbb{Z}_p[[U]]$ has degree at least 4, proving Theorem 2.

Definition 6. *By $Ad^0\bar{\rho}$ we mean the $G_{\mathbb{Q}}$-module of 2×2 trace zero matrices over $\mathbb{F}_p$ with action $g.X = \bar{\rho}(g)X\bar{\rho}(g)^{-1}$. The symbol $Ad\bar{\rho}$ denotes the same action on the set of all 2×2 matrices. Let $U^0 = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ and $U^1 = \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}$ be G_p-stable subspaces of $Ad^0\bar{\rho}$. Let $\tilde{U} \subset Ad\bar{\rho}$ be the G_p-stable subspace of $Ad\bar{\rho}$ consisting of matrices of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$.*

Definition 7. *Let q be a prime for which $\bar{\rho}$ is unramified, $q \not\equiv \pm 1 \mod p$, and $\bar{\rho}(Frob_q)$ has eigenvalues with ratio q. We call such primes* **nice**. *The subspace $\mathcal{N}_q \subset H^1(G_q, Ad^0\bar{\rho})$ and smooth quotient of the local deformation ring are as in p. 543 of [R1]. Note $\mathcal{N}_q$ is spanned by s_q in the notation of [R1]. Let τ_q be a generator of tame inertia of G_q. In the weight two problems our local deformation condition for nice primes are given by $Frob_q \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau_q \mapsto \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. For arbitrary weight problems we allow twists by unramified (at G_q of course) characters.*

Definition 8. *Let $T \supseteq S$ be a finite set of places, with those places $q \in T\backslash S$ being as in Definition 7 and having local condition described there.*
a) Let $R_{T,2}^{fl,T\backslash S-new}$ represent the functor consisting of global deformations of $\bar{\rho}$ unramified outside T, minimally ramified at $v \in S\backslash\{p\}$, new at $v \in T\backslash S$, with determinant the cyclotomic character ϵ, and flat at p. This is the weight two case where p is not in the level.
b) Let $R_{T,2}^{st,T\backslash S-new}$ represent the functor consisting of global deformations of $\bar{\rho}$ unramified outside T, minimally ramified at $v \in S\backslash\{p\}$, new at $v \in T\backslash S$, with determinant the cyclotomic character ϵ,and semistable at p, that is all deformations when restricted to G_p are of the form $\begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$. This is the weight two case where p is in the level.
c) Let $R_T^{ord,T\backslash S-new}$ represent the functor consisting of deformations that are, up to twist by a power of ϵ, minimally ramified at $v \in S\backslash\{p\}$ and new at $v \in T\backslash S$. At

$v = p$ we require the representation is ordinary. Finally, the determinant of the global representation must factor through $Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. We impose no weight condition.

In each case the determinant condition excludes nebentype of order p at primes other than p. If the reader prefers, he can just assume that S contains no primes that are $1 \mod p$. Nice primes are not $\pm 1 \mod p$ so this assumption suffices. Similarly, if she assumes all $v \in S$ satisfy $v \not\equiv \pm 1 \mod p$, then $H^2(G_v, Ad^0\bar{\rho}) = 0$ (see [R1]) and the local at v deformation problems are unobstructed and any local at v deformation is allowed in our global deformation problem.

Associated to these deformation rings are their Selmer groups. Recall I_p is the inertia group at p.

Definition 9. *a) Define $\mathcal{N}^{fl}_{p,2} \subset H^1(G_p, Ad^0\bar{\rho})$ as in p. 127 of [R2]. The flat (weight 2) mod p Selmer group is $Sel^{fl}_{T,2} = Ker\left(H^1(G_T, Ad^0\bar{\rho}) \to \left(\oplus_{v\in T\setminus\{p\}} \frac{H^1(G_v,Ad^0\bar{\rho})}{\mathcal{N}_v}\right) \oplus \frac{H^1(G_p,Ad^0\bar{\rho})}{\mathcal{N}^{fl}_{p,2}}\right)$.*
b) Define $\mathcal{N}^{st}_{p,2} = Ker\left(H^1(G_p, Ad^0\bar{\rho}) \to H^1(G_p, Ad^0\bar{\rho}/U^0)\right)$. The semistable weight 2 mod p Selmer group is
$Sel^{st}_{T,2} = Ker\left(H^1(G_T, Ad^0\bar{\rho}) \to \left(\oplus_{v\in T\setminus\{p\}} \frac{H^1(G_v,Ad^0\bar{\rho})}{\mathcal{N}_v}\right) \oplus \frac{H^1(G_p,Ad^0\bar{\rho})}{\mathcal{N}^{st}_{p,2}}\right)$.
c) Set $\tilde{\mathcal{N}}^{ord}_p = Ker\left(H^1(G_p, Ad\bar{\rho}) \to H^1(I_p, Ad\bar{\rho}/\tilde{U})\right)$. For $v \neq p$ we define $\tilde{\mathcal{N}}_v$ as follows: Let λI be the $\mathbb{F}_p[G_\mathbb{Q}]$-submodule of scalar matrices of $Ad\bar{\rho}$. Note $H^1(G_v, \lambda I) = Hom(G_v, \mathbb{F}_p)$. Then $\tilde{\mathcal{N}}_v$ is the sum of $\mathcal{N}_v \subseteq H^1(G_v, Ad^0\bar{\rho}) \subset H^1(G_v, Ad\bar{\rho})$ and those elements of $Hom(G_v, \mathbb{F}_p)$ that are trivial on I_v. The ordinary **arbitrary weight** *mod p Selmer group is*
$Sel^{ord}_T = Ker\left(H^1(G_T, Ad\bar{\rho}) \to \left(\oplus_{v\in T\setminus\{p\}} \frac{H^1(G_v,Ad\bar{\rho})}{\tilde{\mathcal{N}}_v}\right) \oplus \frac{H^1(G_p,Ad\bar{\rho})}{\tilde{\mathcal{N}}^{ord}_p}\right)$.
c') Set $\mathcal{N}^{ord}_{p,2} = Ker\left(H^1(G_p, Ad^0\bar{\rho}) \to H^1(I_p, Ad^0\bar{\rho}/U^0)\right)$. The ordinary **weight** *2 mod p Selmer group is*
$Sel^{ord}_{T,2} = Ker\left(H^1(G_T, Ad^0\bar{\rho}) \to \left(\oplus_{v\in T\setminus\{p\}} \frac{H^1(G_v,Ad^0\bar{\rho})}{\mathcal{N}_v}\right) \oplus \frac{H^1(G_p,Ad^0\bar{\rho})}{\mathcal{N}^{ord}_{p,2}}\right)$. *The associated global deformation ring is denoted R^{ord}_2.*

Note the numerical subscript '2' indicates a weight two deformation problem with fixed determinant. Each deformation ring is a quotient of $\mathbb{Z}_p[[T_1, \ldots, T_d]]$ where d is the dimension of the relevant Selmer group. The elements of the Selmer group correspond to infinitesimal deformations of $\bar{\rho}$, that is deformations to the dual numbers $\mathbb{F}_p[x]/(x^2)$ satisfying all local conditions. In case c) we have no weight restriction. For $v \neq p$ the condition $\tilde{\mathcal{N}}_v$ just allows for arbitrary unramified twists in our local deformations.

Proposition 10. *$\dim H^1(G_p, Ad^0\bar{\rho}) = 3$, $\dim H^1(G_p, Ad\bar{\rho}) = 5$, $\dim H^1(G_p, U^1) = 2$ and $\dim H^1(G_p, Ad^0\bar{\rho}/U^1) = 1$.*

Proof. These are exercises in local Galois cohomology. Use of local duality, the Euler-Poincaré characteristic and some diagram chasing are required. □

Proposition 11. *$\dim \mathcal{N}^{fl}_{p,2} = \dim \mathcal{N}^{st}_{p,2} = 1$.*

Proof. See [R2]. The 'not flat' proof there works in our 'st' case and the local at p semistable deformation ring is smooth in one variable. For the flat case, that $\bar{\rho}|_{G_p}$ is indecomposable is used. □

Proposition 12. *For our $\bar{\rho}|_{G_p}$ flat and indecomposable, $\mathcal{N}^{ord}_{p,2} = H^1(G_p, U^1)$.*

Proof. First, we prove that $\dim \mathcal{N}^{ord}_{p,2} \leq 2$. Consider the exact sequence of G_p-modules

$$0 \to U^1 \to Ad^0\bar{\rho} \to Ad^0\bar{\rho}/U^1 \to 0.$$

As $Ad^0\bar{\rho}/U^1 \simeq \mathbb{F}_p(\bar{\epsilon}^{-1})$, we easily see $H^0(G_p, Ad^0\bar{\rho}/U^1) = 0$ so the map $H^1(G_p, U^1) \to H^1(G_p, Ad^0\bar{\rho})$ is an injection and we may think of $H^1(G_p, U^1)$ as contained in $H^1(G_p, Ad^0\bar{\rho})$. By Proposition 10 we know $\dim H^1(G_p, U^1) = 2$, $\dim H^1(G_p, Ad^0\bar{\rho}) = 3$ and $\dim H^1(G_p, Ad^0\bar{\rho}/U^1) = 1$. Thus it suffices to show all classes of $H^1(G_p, Ad^0\bar{\rho}/U^1)$ are ramified as their inverse images in $H^1(G_p, Ad^0\bar{\rho})$ are then not ordinary. For any $\mathbb{F}_p[G_p]$-module M, it is standard that $\dim H^0(G_p, M) = \dim H^1_{nr}(G_p, M)$ where this *unramified cohomology* is the image of the inflation map $H^1(G_p/I_p, M^{I_p}) \to H^1(G_p, M)$. So $H^1_{nr}(G_p, Ad^0\bar{\rho}/U^1) = 0$ and all nonzero elements of $H^1(G_p, Ad^0\bar{\rho}/U^1)$ are ramified. Thus $\dim \mathcal{N}^{ord}_{p,2} \leq \dim H^1(G_p, U^1) = 2$.

We now show $\mathcal{N}^{ord}_{p,2}$ contains both one dimensional spaces $\mathcal{N}^{st}_{p,2}$ and $\mathcal{N}^{fl}_{p,2}$ and that they are independent and both lie in $H^1(G_p, U^1)$. This will complete the proof.

Clearly $\mathcal{N}^{st}_{p,2} \subseteq \mathcal{N}^{ord}_{p,2}$. To a nonzero element of $\mathcal{N}^{fl}_{p,2}$, the corresponding deformation to the dual numbers with basis $\{e_1, e_2, xe_1, xe_2\}$ is $\begin{pmatrix} \bar{\epsilon} & * & - & - \\ 0 & 1 & - & - \\ 0 & 0 & \bar{\epsilon} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Recall, for instance from [Ray], that if Galois module comes from a finite flat group scheme over $\mathbb{Z}_p$, so do all it subquotients.

The above Galois module contains the subquotient $\begin{pmatrix} 1 & - \\ 0 & \bar{\epsilon} \end{pmatrix}$. The connected part of a finite flat group scheme over $\mathbb{Z}_p$ is always a subobject. Here the étale part is a subobject as well, so the $-$ is trivial. Our lift to the dual numbers is then $\begin{pmatrix} \bar{\epsilon} & * & - & - \\ 0 & 1 & 0 & - \\ 0 & 0 & \bar{\epsilon} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and reducible, that is it comes from $H^1(G_p, U^1)$. It can be written $\begin{pmatrix} \bar{\epsilon}(1+x\phi) & *+x* \\ 0 & 1-x\phi \end{pmatrix} \subset GL_2(\mathbb{F}_p[x]/(x^2))$.

To show this deformation to the dual numbers is ordinary, it suffices to show ϕ is unramified. Suppose ϕ is ramified. The rank one $\mathbb{F}_p[x]/(x^2)$-module that is a subobject with Galois action given by $\bar{\epsilon}(1+x\phi)$ corresponds to a finite flat group scheme killed by p. We use Fontaine's discriminant bounds. By the Corollaire to Théorème A of [F] we know that the normalized valuation of the different of the corresponding extension of $\mathbb{Q}_p$ is less than $\dfrac{p}{p-1}$. But the extension is abelian and is, up to an unramified twist (which does not change differents), $\mathbb{Q}_p(\zeta_{p^2})$ (we are assuming ϕ is ramified). The different of this field is $f^{'}(\zeta_{p^2})$ where $f(x) = \dfrac{x^{p^2}-1}{x^p-1}$ is the (essentially Eisenstein) minimal polynomial of ζ_{p^2}. Observe

$$v_p\left(f^{'}(\zeta_{p^2})\right) = v_p\left(\frac{p^2\zeta_{p^2}^{p^2-1}}{(\zeta_{p^2}^p-1)}\right) = 2 - \frac{1}{p-1} = \frac{2p-3}{p-1} > \frac{p}{p-1}.$$

Thus $\bar{\epsilon}(1+x\phi)$ being flat implies ϕ is unramified.

It could be that $\mathcal{N}_{p,2}^{fl} = \mathcal{N}_{p,2}^{st}$. This happens only if ϕ is trivial. It is routine to see that cohomology classes in $\mathcal{N}_{p,2}^{st}$ cut out a $\mathbb{Z}/p\mathbb{Z}$ extension $K/\mathbb{Q}_p(\zeta_p)$ and $Gal(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ acts on $Gal(K/\mathbb{Q}_p(\zeta_p))$ via the cyclotomic character. Indeed, our $\bar{\rho}$ cuts out such an extension, but the cohomology class cuts out an independent extension. A local class field theory computation shows there are two such independent extensions of $\mathbb{Q}_p(\zeta_p)$, so a lift to the dual numbers of $\bar{\rho}$ coming from an element of $\mathcal{N}_{p,2}^{st}$ cuts out all of them and thus necessarily contains the splitting field of the Eisenstein polynomial $g(x) = x^p - p$, which also cuts out such an extension. Since g is Eisenstein, the different of this splitting field is just $g^{'}(\delta)$ where δ is a root of $g(x)$. But

$$v_p\left(g^{'}(\delta)\right) = 1 + \frac{p-1}{p} = \frac{2p-1}{p} > \frac{p}{p-1},$$

Fontaine's bound. Thus $\mathcal{N}_p^{fl} \neq \mathcal{N}_p^{st}$ and these one dimensional subspaces of $H^1(G_p, Ad^0\bar{\rho})$ span the two dimensional space $H^1(G_p, U^1)$. So $\mathcal{N}_{p,2}^{ord}$ has dimension at least 2 and equals $H^1(G_p, U^1)$.

□

Remark 2. For $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ and très ramifié we **do not** have that $H^1(G_p, U^1) = \mathcal{N}_p^{ord}$. Indeed, $H^1(G_p, U^1)$ contains classes ramified at both diagonal elements. In this case $\mathcal{N}_p^{ord} = \mathcal{N}_p^{st}$ and their common dimension is 1. This is also a moderately complicated exercise in Galois cohomology.

Proposition 13. $\dim \tilde{\mathcal{N}}_p^{ord} = 3$. *This space is spanned by* $H^1(G_p, U^1)$ *and the unramified twist (see below).*

Proof. We are working with the full adjoint and the only new (local) cohomology classes that one has with the full adjoint are the ramified and unramified homomorphisms to $\mathbb{F}_p$, the two twists when we view elements of $H^1(G_p, Ad\bar{\rho})$ as deformations to the dual numbers, $\mathbb{F}_p[x]/(x^2)$. Clearly the unramified twist is ordinary. Any ordinary linear combination of the ramified twist and an element of $H^1(G_p, Ad^0\bar{\rho})$ cannot come from $H^1(G_p, Ad^0\bar{\rho}) \backslash H^1(G_p, U^1)$ by the last part of the proof of Proposition 12. But any linear combination of the ramified twist with an element of $H^1(G_p, U^1) = \tilde{\mathcal{N}}_p^{ord}$ will, by Proposition 12, be ramified on both diagonal elements and hence not ordinary. Thus $\tilde{\mathcal{N}}_p^{ord}$ is spanned by $H^1(G_p, U^1)$ and the unramified twist. □

Remark 3. For $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ and très ramifié we **do** have that $\dim \mathcal{N}_p^{ord} = 3$. In addition to the one dimensional space $\mathcal{N}_{p,2}^{st}$, there is of course the unramified twist. The third dimension is obtained as follows. Given a nonordinary element of $H^1(G_p, U^1)$ (which exists for $\bar{\rho}$ très ramifié), adding a suitable multiple of the ramified twist can be used to remove ramification at the lower right entry but keep ramification in the upper left entry.

Proposition 14. *Let* $T \supseteq S$ *be a finite set of primes such that* $T \backslash S$ *consists of nice primes. Then* $j \in Sel_T^{ord}$ *actually lies in* $Sel_{T,2}^{ord}$, *that is* $j \in H^1(G_T, Ad^0\bar{\rho})$ *as opposed to* $H^1(G_T, Ad\bar{\rho})$.

Proof. Let $j \in Sel_T^{ord}$. As $Ad\bar{\rho}$ is the direct sum of $Ad^0\bar{\rho}$ and the scalars, $j = g + h$ where $g \in H^1(G_T, Ad^0\bar{\rho})$ and $h \in H^1(G_T, \lambda I)$. As we insist that all global determinants factor through the cyclotomic extension of $\mathbb{Q}_p$ (no p-power nebentype is allowed away from p), h factors through the $\mathbb{Z}/p$ extension of $\mathbb{Q}$ ramified only at p. As j is ordinary $h = 0$ by the proof of Proposition 13. □

For M an $\mathbb{F}_p[G_T]$-module denote by M^* its $\mathbb{G}_m$-dual and for $\mathcal{L}_v \subset H^1(G_v, M)$ denote by $\mathcal{L}_v^{\perp} \subset H^1(G_v, M^*)$ the annihilator of $\mathcal{L}_v$ under the local duality pairing. Then we define the Selmer group

$$Sel_T^{\mathcal{L}} = Ker\left(H^1(G_T, M) \to \oplus_{v \in T} \frac{H^1(G_v, M)}{\mathcal{L}_v}\right)$$

and the dual Selmer

$$DualSel_T^{\mathcal{L}_v^{\perp}} = Ker\left(H^1(G_T, M) \to \oplus_{v \in T} \frac{H^1(G_v, M)}{\mathcal{L}_v^{\perp}}\right).$$

The following is Proposition 1.6 of [W].

Proposition 15.

$$\begin{aligned} \dim Sel_T^{\mathcal{L}} - \dim DualSel_T^{\mathcal{L}_v^{\perp}} &= \dim H^0(G_T, M) - \dim H^0(G_T, M^*) \\ &\quad - \sum_{v \in T} \left(\dim \mathcal{L}_v - \dim H^0(G_v, M)\right). \end{aligned}$$

For $M = Ad^0\bar{\rho}$ we have for $v \in T$, $v \neq p, \infty$ that $\dim \mathcal{N}_v = \dim H^0(G_v, M)$ and that the relevant local deformation problem is smooth of the right dimension. By this we mean it is possible to annihilate the dual Selmer group as in [R2] and [T] and obtain a unique deformation to $\mathbb{Z}_p$. If we study $v \neq p$ for $M = Ad\bar{\rho}$ we simply include unramified twists into our local condition $\tilde{\mathcal{N}_v}$. For $v \neq p$, $v \not\equiv 1 \mod p$, this balances the increase in dimension of H^0 by 1 when we switch from $Ad^0\bar{\rho}$ to $Ad\bar{\rho}$. When $v \equiv 1 \mod p$, observe $\dim H^1(G_v, Ad\bar{\rho}) - \dim H^1(G_v, Ad^0\bar{\rho}) = 2$, not 1. This causes no problems as for such primes there is a *global* ramified at v cohomology class with values in $\lambda I \subset Ad\bar{\rho}$ which 'counters' the extra ramified at v local dimension in the Selmer group map.

When studying the local problems at p with $Ad^0\bar{\rho}$, we need to specify whether we are working with $\mathcal{N}_{fl}$ or $\mathcal{N}_{st}$. By Proposition 15 one has for $* \in \{fl, st\}$

(3.1) $\dim Sel_{T,2}^{*} - \dim DualSel_{T,2}^{*} = \dim H^0(G_T, Ad^0\bar{\rho}) - \dim H^0(G_T, Ad^0\bar{\rho}^*)$
$+\dim \mathcal{N}_p^* - \dim H^0(G_p, Ad^0\bar{\rho}) + \dim \mathcal{N}_\infty - \dim H^0(G_\infty, Ad^0\bar{\rho}) = 0-0+1-0+0-1 = 0.$

With $Ad^0\bar{\rho}$ and the ordinary condition, we have

(3.2) $\dim Sel_{T,2}^{ord} - \dim DualSel_{T,2}^{ord} = \dim H^0(G_T, Ad^0\bar{\rho}) - \dim H^0(G_T, Ad^0\bar{\rho}^*)$
$+\dim \mathcal{N}_p^{ord} - \dim H^0(G_p, Ad^0\bar{\rho}) + \dim \mathcal{N}_\infty - \dim H^0(G_\infty, Ad^0\bar{\rho}) = 0-0+2-0+0-1 = 1.$

Finally, with the full adjoint and the ordinary condition

$$\dim Sel_T^{ord} - \dim DualSel_T^{ord} = \dim H^0(G_T, Ad\bar{\rho}) - \dim H^0(G_T, Ad\bar{\rho}^*) \tag{3.3}$$
$$+\dim \mathcal{N}_p^{ord} - \dim H^0(G_p, Ad\bar{\rho}) + \dim \mathcal{N}_\infty - \dim H^0(G_\infty, Ad\bar{\rho}) = 1-0+3-1+0-2 = 1.$$

Corollary 16. *Let f be any classical eigenform with residual representation $\bar{\rho}$ satisfying our running hypotheses and let $\mathbb{T}$ be its Hida family. The weight space map $\mathbb{Z}_p[[T]] \to \mathbb{T}$ is* **not** *an isomorphism.*

Proof. If the map were degree one, it would be an isomorphism and there would be exactly one ordinary $\mathbb{Z}_p$-valued point of weight two. Then $\dim Sel_{T,2}^{ord} = 0$, which contradicts (3.2). □

Remark 4. Thus for $\bar{\rho}|_{G_p}$ peu ramifié indecomposable, the map to weight space along *every* member of the Hida community of any tame level N is of degree greater than 1. This partially generalizes the Greenberg-Stevens result, Theorem 1. Whether one can remove p from the level at a weight two point along each component is open.

The main Theorem of [R2] (reproved in [T]) is:

Theorem. Let $\bar{\rho}$ be odd, absolutely irreducible and peu ramifié indecomposable. Let $*$ be either fl or st. There exist sets of primes $Q = \{q_1, ..., q_r\}$, depending on $\bar{\rho}$ and $*$, such that on setting $T = S \cup Q$, $\dim Sel_{T,2}^{*,Q-new} = 0$ and $R_{T,2}^{*,Q-new} \simeq \mathbb{Z}_p$.

We also have the Theorem below. It is essentially joint with Khare and Larsen from [KLR], but the precise statement we need follows more directly from [R3].

Theorem. Let $d \in \mathbb{N}$ be given. Let $*$ be either fl or st. There exist sets of $W = \{w_1, ..., w_s\}$, depending on $\bar{\rho}$, $*$ and d such that, on setting $T = S \cup W$, $\dim Sel_{T,2}^{*,W-new} = d$.

Remark 5. This involves a fairly complicated argument where either one or two nice primes are successively added to the level to make the Selmer group one dimension larger at each stage, but we do not know whether it is one or two. I do not know how to make the Selmer group larger by adding just one nice prime, though I expect this happens for a positive proportion of nice primes. After much futile searching, I suspect this set is *not* determined by Chebotarev conditions. In [R3] at nice primes q the set $\mathcal{N}_q$ is as in this paper, but for $v \in S$, $\mathcal{N}_v = 0$ so the Selmer group there (which is made arbitrarily large) is *contained* in the Selmer group of this paper. In fact, the Selmer group in [R3] is not directly computed. Instead the universal ring corresponding to the Selmer group with trivial local condition at $v \in S$ (with a varying set of nice primes at which ramification is allowed, depending on d) is shown to be a quotient of $R = \mathbb{Z}_p[[T_1, ..., T_d]]$ and the deformation to $R/(p, m_R^2)$ is onto.

Proof of Theorem 2. The choice of $* \in \{fl, st\}$ is irrelevant, so make an arbitrary choice. We only need $d = 4$ in the Theorem of [KLR] and proceed on this assumption. We want to study the ordinary arbitrary weight deformation problem for $\bar{\rho}$. Since $Sel_T^{ord,W-new} \supseteq Sel_{T,2}^{*,W-new}$ for $* \in \{fl, st\}$, $\dim Sel_T^{ord,W-new} \geq 4$. By Proposition 14, all elements of $Sel_T^{ord,W-new}$ have values in $Ad^0\bar{\rho}$. Proposition 13 implies the map $Sel_{T,2}^{ord,W-new} \to \mathcal{N}_p^{ord}$ has nontrivial kernel. Let g be in this kernel and augment it to a basis of $Sel_{T,2}^{ord,W-new}$. By (3.3) $\dim DualSel_T^{ord} = 3$. Following [R2] and [T], add in more nice primes to simultaneously annihilate the basis elements of Selmer other than g and the elements of a basis of $DualSel_T^{ord}$. Let X be the union of W and the other new primes just added in. Relabel T to be $S \cup X$. Then $\dim Sel_T^{ord,T\setminus S-new} = 1$ and $\dim DualSel_T^{ord,T\setminus S-new} = 0$.

As $g|_{G_p} = 0$, it spans $Sel_T^{ord,T\setminus S-new} = Sel_{T,2}^{st,T\setminus S-new} = Sel_{T,2}^{fl,T\setminus S-new}$. Now $R_T^{ord,T\setminus S-new}$ is a quotient of $\mathbb{Z}_p[[U]]$ and comes with an attached representation of $G_{\mathbb{Q}}$. By [DT] it has a characteristic zero point corresponding to a classical eigenform. This eigenform belongs to a Hida family the rest of whose (infinitely many) classical points factor through this attached representation. Thus $R_T^{ord,T\setminus S-new} = \mathbb{Z}_p[[U]]$ *is* our Hida family. It has as quotients $R_{T,2}^{st,T\setminus S-new}$ and $R_{T,2}^{fl,T\setminus S-new}$. As each of these rings has nontrivial Selmer group (spanned by the cohomology class g), each ring has $\mathbb{Z}_p$-rank at least two. This proves that the degree of the map to weight space $\mathbb{Z}_p[[T]] \to R_T^{ord,T\setminus S-new}$ is at least 4. □

Corollary 17. *There are infinitely many classical points on the Hida family constructed in the proof of Theorem 2 whose corresponding modular forms have field of Fourier coefficients over $\mathbb{Q}_p$ totally ramified of degree at least* 4*. There are also infinitely many classical points where a prime of the field of Fourier coefficients lies above p with residue degree* 1*.*

Proof. Recall the weight space map $\mathbb{Z}_p[[T]] \to \mathbb{Z}_p[[U]]$ and $a = 1 + T \in \mathbb{Z}_p[[U]]$. When viewed as a power series in the variable U, we see that $a = 1 + p$ has at least four solutions, at least two that are flat and at least two that are semistable. Each set could be conjugate.

We study $a-1-p \in \mathbb{Z}_p[[U]]$. By the Weierstrass Preparation Theorem, $a-1-p = p^n j(U)y(U)$ where $j(U)$ is a distinguished polynomial of degree $d \geq 4$ and $y(U)$ is a unit. Since our Hida family has weight three points, the equation $a = (1+p)^{3-1}$ has solutions in $\bar{\mathbb{Z}}_p$. Finding weight three points is then solving $(1+p)^2 - 1 - p = p^n j(U)y(U)$ which becomes $p(1+p) = p^n j(U)y(U)$. If $n \geq 1$, then after dividing by p we would have a unit equal to a multiple of a nontrivial distinguished polynomial, a contradiction. Thus $n = 0$.

For suitable (in fact most) classical weights k, the constant term of $a-(1+p)^{k-1}$ will have valuation exactly 1. Thus we have $a-(1+p)^{k-1} = j_k(U)y_k(U)$ where $j_k(U)$ is a distinguished *Eisenstein* polynomial and $y_k(U)$ is a unit. The roots of $j_k(U)$ correspond to the weight k points of our Hecke algebra and each generates a totally ramified extension of $\mathbb{Q}_p$ of degree $d \geq 4$.

For the second part, for suitable k in a small open neighborhood of $\mathbb{Z}_p$ the valuation of the constant term of $a-(1+p)^{k-1}$ can be made arbitrarily large. For such a classical k, the Newton polygon of $a-(1+p)^{k-1} \in \mathbb{Z}_p[[U]]$ has its initial slope as it largest and is of length one, that is $a-(1+p)^{k-1}$ has a root in $\mathbb{Z}_p$, the desired result. $\square$

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