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SOME NEW FAMILIES OF INTEGRAL GRAPHS

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Let $G_i, i = 1, 2, 3$ be finite simple graphs with vertex sets $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}, V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$. In this paper, for each ordered triple of graphs (G_1, G_2, G_3), we define a new composition $\psi(G_1, G_2, G_3)$: The vertex set of $\psi(G_1, G_2, G_3)$ is $(V(G_1) \biguplus V(G_2)) \times V(G_3)$ and adjacency between vertices of $\psi(G_1, G_2, G_3)$ is defined by:

1. (u_i, w_l) adj (v_k, w_l) for each $i = 1, 2, \ldots, n_1$ and $k = 1, 2, \ldots, n_2$ and $l = 1, 2, \ldots, n_3$.

2. (v_i, w_j) adj (v_k, w_l) if and only if

   i) $i = k$ and w_j adj w_l in G_3 or

   ii) $j = l$ and v_i adj v_k in G_2

3. (u_i, w_j) adj (u_k, w_j) whenever u_i adj u_k in G_1, for each $j = 1, 2, \ldots, n_3$.

We obtain the Adjacency spectrum, Laplacian spectrum and Q-spectrum of $\psi(G_1, G_2, G_3)$. As an application, many new infinite families of R-integral graphs where $R \in \{A, L, Q\}$ are constructed.

Key words : Adjacency spectrum; Laplacian spectrum; signless Laplacian spectrum; products of graphs; Integral graphs.

1. INTRODUCTION

Let G be a simple graph on n vertices. The spectrum of G is the set of n eigenvalues of the adjacency matrix $A(G)$ of G. It is denoted by $Spec(G)$. As $A(G)$ is real and symmetric, its eigenvalues are real and therefore can be ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For basic facts on the spectra of graphs, see [2, 4, 5].

A graph G is A-integral if the spectrum of $A(G)$ consists only of integers. In [3] constructions and properties of integral graphs are discussed in detail. The graphs K_p, C_4, C_6, the cocktail party graph $CP(n) = \overline{nK_2}$, $K_{n,n}$ (where bar stands for complement) are all examples of integral graphs. Some work on integral trees is given in [15]. Moreover, several graph operations such as Cartesian product, direct product and strong product on integral graphs can be used for constructing infinite families of integral graphs [3]. For other related works, see [7, 9, 11] and also the references cited therein.

Recall that the join $G \vee H$ of two graphs G and H is obtained by taking disjoint copies of G and H and making each vertex of G adjacent to every vertex of H. The Cartesian product $G = G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has $V(G) = V(G_1) \times V(G_2)$ as its vertex set and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$ [2]. Given two sets A_1 and A_2, the disjoint union of A_1 and A_2 is the set $A_1 \biguplus A_2 = \bigcup_{i=1}^{2}\{(x, i) : x \in A_i\}$.

Given the ordered triple of graphs (G_1, G_2, G_3), the graph $\psi(G_1, G_2, G_3)$ is defined as follows: Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$. The vertex set of $\psi(G_1, G_2, G_3) = (V(G_1) \biguplus V(G_2)) \times V(G_3)$ and adjacency is defined by:

1. (u_i, w_l) adj (v_k, w_l) for each $i = 1, 2, \ldots, n_1$ and $k = 1, 2, \ldots, n_2$ and $l = 1, 2, \ldots, n_3$.

2. (v_i, w_j) adj (v_k, w_l) if and only if

   i) $i = k$ and w_j adj w_l in G_3 or

   ii) $j = l$ and v_i adj v_k in G_2

3. (u_i, w_j) adj (u_k, w_j) whenever u_i adj u_k in G_1, for each $j = 1, 2, \ldots, n_3$.

In other words, $\psi(G_1, G_2, G_3)$ is simply the graph $\{|V(G_3)|(G_1 \vee G_2)\} \bigcup (G_2 \Box G_3)$, where $\bigcup$ denotes the union, not necessarily disjoint. Figure 1 displays $\psi(P_4, K_2, (K_4 - e))$. It is clear that the graph $\psi(G_1, G_2, G_3)$ is formed by taking n_3 copies of $G_1 \vee G_2$ and making the corresponding vertices in each of the n_3 copies of G_2 induce a copy of G_3. Note that $\psi(G_1, G_2, G_3)$ is of order $(n_1 + n_2)n_3$ and size $n_1n_2n_3 + n_2m_3$, where m_3 is the size of G_3.

We observe that the graph $K_{n,n+1} \equiv K_{n+1,n}$ studied by Wang in [15] is just the graph $\psi(\overline{K_n}, \overline{K_{n+1}}, K_2)$. It is shown in [15] that this graph is (adjacency) integral for all $n \geq 1$.

The main result of this paper is the determination of the spectrum of the graph $\psi(G_1, G_2, G_3)$.

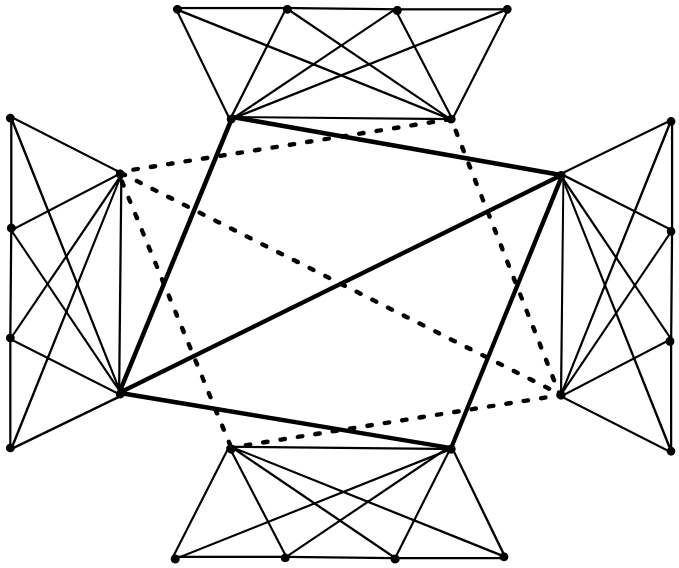


Figure 1: The graph $\psi(P_4, K_2, (K_4 - e))$

In this context it is appropriate to recall that one of the questions asked by Harary and Schwenk is the following [10]: *Which graphs have integral (adjacency) spectra?*. This question is known to be intractable. In fact, a result of Alon *et al.* [1] shows that only a fraction of $2^{-\Omega(n)}$ of the graphs on n vertices have an integral spectrum. Our results yield many new infinite families of graphs having integer spectra. There has been a revival of interest in integral graphs, in view of their applications in perfect state transfers in graphs [8, 14].

All graphs considered in this paper are simple and we follow [2, 5] for graph and spectral graph theoretic terminology.

2. The Adjacency Spectrum of the Graph $\psi(G_1, G_2, G_3)$

In this section we determine the adjacency spectrum of $\psi(G_1, G_2, G_3)$ of any three graphs G_1, G_2 and G_3 when G_1 and G_2 are regular and G_3 is an arbitrary graph. Throughout this paper, we set A_i to be the adjacency matrix of G_i, $i = 1, 2, 3$. We use $\mathbf{1}_k$ and $\mathbf{0}_k$ to denote the all-1 and all-0 vectors of length k respectively. Moreover $0_{j\times k}$ denotes the $j \times k$ zero matrix and $J_{j\times k}$, the $j \times k$ all-1 matrix.

Recall that the Kronecker product of two matrices $A = (a_{ij})$ of order m by n and $B = (b_{ij})$ of order p by q, is the mp by nq matrix got by replacing each entry a_{ij} of A by the double array $a_{ij}B$ got by multiplying each entry of B by a_{ij} [2].

***Theorem* 2.1** — *Let G_i be an r_i-regular graph, $1 \leq i \leq 2$. Let $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq$*

$\ldots \geq \lambda_{n_1}\}, Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \ldots \geq \mu_{n_2}\}$ *and* $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{n_3}\}$. *Then* $Spec(\psi(G_1, G_2, G_3))$ *consists of* $\lambda_i, i = 2, 3, \ldots, n_1$, *each of multiplicity* n_3, *all numbers of the form* $(\mu_j + \nu_l), j = 2, 3, \ldots, n_2; l = 1, 2, \ldots, n_3$ *together with* n_3 *pairs of numbers with each pair being the roots of the quadratic equation*

$$(x - r_1)(x - (r_2 + \nu_l)) - n_1 n_2 = 0,\ 1 \leq l \leq n_3. \tag{1}$$

PROOF : By definition, $\psi(G_1, G_2, G_3)$ contains n_3 copies of $G_1 \vee G_2$. In one such copy, label the vertices of G_1 followed by the vertices of G_2. Continue the labeling of vertices in the other copies of $G_1 \vee G_2$ in the same manner. Then the adjacency matrix $\mathcal{A}$ of G can be expressed as

$$\mathcal{A} = I_{n_3} \otimes \begin{bmatrix} A_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & A_2 \end{bmatrix} + A_3 \otimes \begin{bmatrix} 0_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & I_{n_2} \end{bmatrix}, \tag{2}$$

where $\otimes$ stands for the Kronecker product of two matrices. Note that in equation (2), the first term of the sum on the right gives the adjacency matrix of $G_1 \vee G_2$ while the second term corresponds to the adjacency matrix of $G_2 \Box G_3$. As regular graphs G_1 and G_2 have $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ as their respective eigenvectors corresponding to the eigenvalues r_1 and r_2 while all other eigenvectors are orthogonal respectively to $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$. (Note that G_i, $i = 1, 2$ need not be connected, and hence r_i need not be a simple eigenvalue of G_i.)

Let X and Y be eigenvectors of G_1 and G_2 orthogonal to $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ respectively so that $\mathbf{1}_{n_1}^T X = 0$ and $\mathbf{1}_{n_2}^T Y = 0$. Let λ, μ be eigenvalues of G_1 and G_2 corresponding to $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ respectively. Let ν be any eigenvalue of G_3 and Z, an eigenvector corresponding to ν.

The following relation for matrices is well known [12]. For matrices A, B, C and D

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$$

whenever the products AC and BD are defined.

Now we show that the eigenvalue λ of G_1 is an eigenvalue of $\mathcal{A}$ repeated n_3 times. To see this, consider

$$\mathcal{A} \cdot \left(Z \otimes \begin{bmatrix} X \\ 0 \end{bmatrix} \right) = \left(I_{n_3} \otimes \begin{bmatrix} A_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & A_2 \end{bmatrix} + A_3 \otimes \begin{bmatrix} 0_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & I_{n_2} \end{bmatrix} \right)$$

$$\cdot \left(Z \otimes \begin{bmatrix} X \\ 0 \end{bmatrix} \right) = \left(Z \otimes \begin{bmatrix} A_1 X \\ 0 \end{bmatrix} \right) + \left(\nu Z \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$= Z \otimes \begin{bmatrix} \lambda X \\ 0 \end{bmatrix}$$

$$= \lambda Z \otimes \begin{bmatrix} X \\ 0 \end{bmatrix}.$$

Similarly $Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\mu + \nu$ of $\mathcal{A}$. This is because

$$\mathcal{A} \cdot \left(Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix} \right) = \left(I_{n_3} \otimes \begin{bmatrix} A_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & A_2 \end{bmatrix} + A_3 \otimes \begin{bmatrix} 0_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & I_{n_2} \end{bmatrix} \right)$$

$$\cdot \left(Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix} \right) = \left(Z \otimes \begin{bmatrix} 0 \\ A_2 Y \end{bmatrix} \right) + \left(A_3 Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix} \right)$$

$$= \left(Z \otimes \begin{bmatrix} 0 \\ \mu Y \end{bmatrix} \right) + \left(\nu Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix} \right)$$

$$= (\mu + \nu) Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix}.$$

Thus for each eigenvalue $\lambda \neq r_1$ of G_1, there are n_3 eigenvalues λ with corresponding eigenvector $Z \otimes \begin{bmatrix} X \\ 0 \end{bmatrix}$ for $\psi(G_1, G_2, G_3)$. Similarly for each eigenvalue $\mu \neq r_2$ of G_2, there are n_3 eigenvalues $\mu + \nu$ with corresponding eigenvector $Z \otimes \begin{bmatrix} 0 \\ Y \end{bmatrix}$ for $\psi(G_1, G_2, G_3)$. In this way we obtain a total of $(n_1 + n_2)n_3 - 2n_3$ mutually orthogonal eigenvectors of $\mathcal{A}$. All these eigenvectors are orthogonal to the vectors $Z \otimes \begin{bmatrix} J_{n_1 \times 1} \\ 0 \end{bmatrix}$ and $Z \otimes \begin{bmatrix} 0 \\ J_{n_2 \times 1} \end{bmatrix}$, which means that they span the space (over $\mathbb{R}$) spanned by the remaining $2n_3$ eigenvectors of $\mathcal{A}$. This implies that the remaining $2n_3$ eigenvectors of $\mathcal{A}$ have the form $Z \otimes \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{n_2 \times 1} \end{bmatrix}$ for some $(\alpha, \beta) \neq (0, 0)$. Now using the fact that $A_i \cdot \mathbf{1} = r_i \mathbf{1}$,

$i = 1, 2$ we get the system of equations

$$\begin{aligned} \alpha r_1 + \beta n_2 &= x\alpha, \\ \alpha n_1 + \beta\,(r_2 + \nu) &= x\beta. \end{aligned}$$

Substituting $\beta = \dfrac{\alpha(x - r_1)}{n_2}$ in the latter equation and eliminating α we get the quadratic equation in x

$$(x - r_1)(x - (r_2 + \nu)) - n_1 n_2 = 0 \tag{3}$$

whose roots are indeed the remaining two eigenvalues of $\mathcal{A}$ each of multiplicity n_3. □

The quadratic equation (1) can be expressed as $x^2 - (r_1 + r_2 + \nu)x + r_1(r_2 - \nu) - n_1 n_2 = 0$. Hence the roots are given by

$$\begin{aligned} x &= \frac{r_1 + r_2 + \nu \pm \sqrt{(r_1 + r_2 + \nu)^2 - 4(r_1(r_2 - \nu) - n_1 n_2)}}{2} \\ &= \frac{r_1 + r_2 + \nu \pm \sqrt{r_1^2 + r_2^2 + \nu^2 + 2r_1 r_2 + 2r_1\nu + 2r_2\nu - 4r_1 r_2 + 4r_1\nu + 4n_1 n_2}}{2} \\ &= \frac{r_1 + r_2 + \nu \pm \sqrt{(r_1 - r_2 + \nu)^2 + 4n_1 n_2}}{2}, \text{ where } \nu \in Spec(G_3). \end{aligned}$$

Thus we have the following Corollary.

Corollary 2.2 — Let G_i, $i = 1, 2, 3$ be three integral graphs with G_i being r_i-regular, $i = 1, 2$. Then $\psi(G_1, G_2, G_3)$ is integral if and only if the expression $(r_1 - r_2 + \nu)^2 + 4n_1 n_2$, where $\nu \in Spec(G_3)$ is a perfect square. □

Corollary 2.3 — Let G_i, $i = 1, 2, 3$ be three integral graphs with G_i being r-regular, $i = 1, 2$. Then $\psi(G_1, G_2, G_3)$ is integral if and only if $4n_1 n_2 + \nu^2$ is a perfect square for every $\nu \in Spec(G_3)$. □

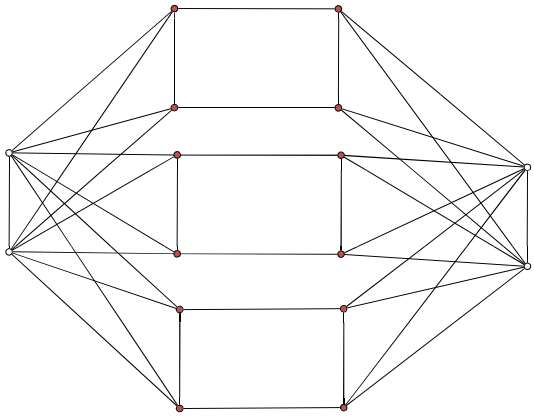
We now exhibit some new families of integral graphs based on the results proved above. In the sequel, we have used the software newGRAPH 1.1.3 developed by Dragan Stevanović, Vladimir Brankov, Dragoš Cvetković and Slobodan Simić [13] to draw the graphs.

Proposition 2.4 — Let $a \in \mathbb{N}$, $i = 1, 2, 3$ and $b = \dfrac{iap^2 \pm p}{i} \in \mathbb{N}, p = 1, 2, \ldots$ Then the infinite families $\psi(aK_i, bK_i, K_2)$ and $\psi(bK_i, aK_i, K_2)$ are integral.

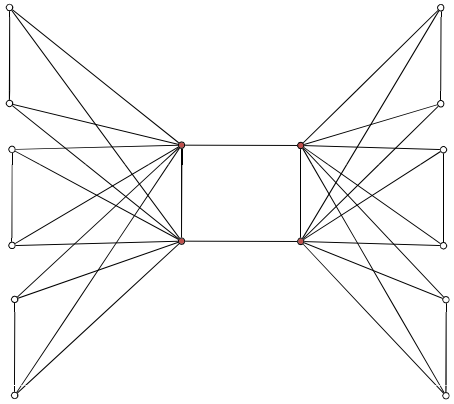
Table 1 gives some possible pairs of values for a and b for $i = 2, 3$ while Figure 2 displays the first two members of this family for $i = 2, 3$.

a	1	1	1	1	1	2	2	2	2	2	...
b when i = 2	3	5	14	18	33	7	9	30	34	69	...
b when i = 3	8	10	34	38	78	17	19	70	74	159	...

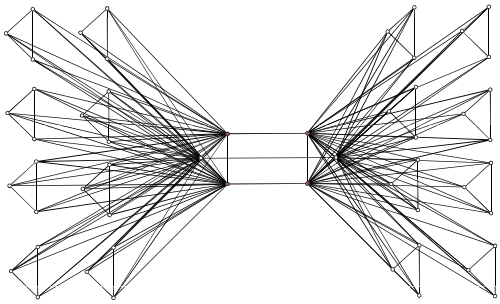
Table 1: Integral graphs from the family $\psi(aK_i, bK_i, K_2)$ for $i = 2, 3$.



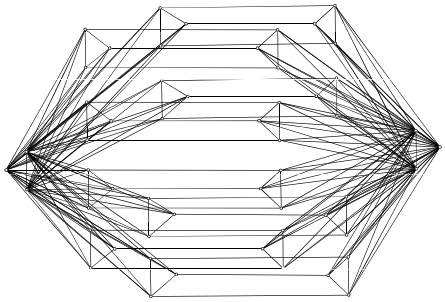
(a) $\psi(K_2, 3K_2, K_2)$ with spectrum $\{5^1, 4^1, 2^2, 0^5, -1^2, -2^4, -3^1\}$



(b) $\psi(3K_2, K_2, K_2)$ with spectrum $\{5^1, 4^1, 1^4, 0, -1^6, -2^2, -3^1\}$



(c) $\psi(K_3, 8K_3, K_2)$ with spectrum $\{11^1, 10^1, 2^{14}, 0^2, -1^{32}, -2^2, -6^1, -7^1\}$



(d) $\psi(8K_3, K_3, K_2)$ with spectrum $\{11^1, 10^1, 3^7, 1^7, 0^{16}, -1^4, -2^{16}, -6^1, -7^1\}$

Figure 2: Integral graphs from the family $\psi(aK_i, bK_i, K_2)$ for $i = 2, 3$.

Proposition 2.5 — For $a \in \mathbb{N}$, $i, j \in \{3, 4, 6\}$ and $b = \dfrac{iap^2 \pm p}{j} \in \mathbb{N}, p = 1, 2, \ldots,$ the infinite families $\psi(aC_i, bC_j, K_2)$ and $\psi(bC_i, aC_j, K_2)$ are integral.

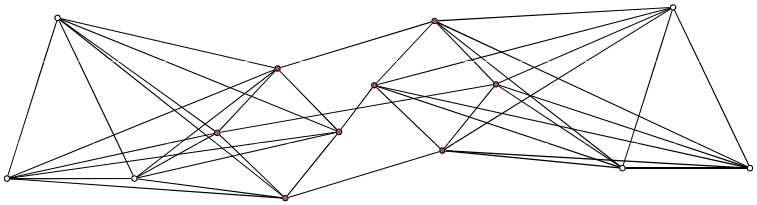
Table 2 gives some possible pairs of values for a and b that satisfy Proposition 2.5. Figure 3 displays some members of this family.

Proposition 2.6 — For $a \in \mathbb{N}$ and $b = \dfrac{6ap^2 \pm p}{6} \in \mathbb{N}, p = 1, 2, \ldots,$ the infinite families $\psi(aG, bH, K_2)$ where $G \neq H \in \{K_{3,3}, C_3 \square K_2\}$ are integral.

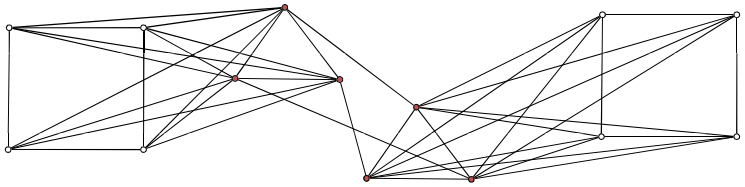
Proposition 2.7 — For $a \in \mathbb{N}$ the infinite family $\psi(K_{2a-1}, CP(a), K_2)$ and $\psi(CP(a), K_{2a-1}, K_2)$ are integral.

a	1	1	1	1	1	2	2	2	2	2	...
b when i=3, j=4	1	6	11	13	20	23	25	94	98	213	...
b when i=3, j=6	4	5	17	19	39	35	37	142	146	321	...

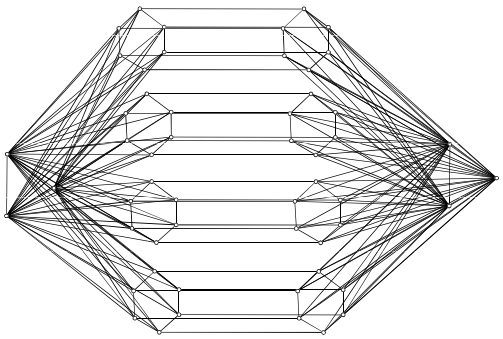
Table 2: Integral graphs from the family $\psi(aC_i, bC_j, K_2)$ for $i = 3, j = 4, 6$.



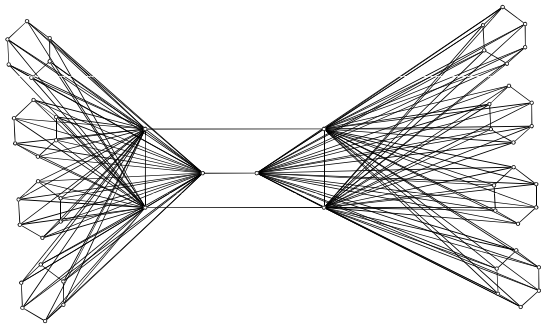
(a) $\psi(C_3.C_4, K_2)$ with spectrum $\{6^1, 5^1, 1^2, -1^8, -2^1, -3^1\}$



(b)$\psi(C_4, C_3, K_2)$ with spectrum $\{6^1, 5^1, 0^6, -1^1, -2^5\}$



(c) $\psi(C_3, 4C_6, K_2)$ with spectrum $\{11^1, 10^1, 3^3, 2^8, 1^3, 0^{16}, -1^8, -2^8, -3^4, -6^1, -7^1\}$



(d) $\psi(4C_6, C_3, K_2)$ with spectrum $\{11^1, 10^1, 2^6, 1^{16}, 0^2, -1^{16}, -2^{10}, -6^1, -7^1\}$

Figure 3: Integral graphs from the family $\psi(aC_i, bC_j, K_2)$ from Proposition 2.5

Proposition 2.8 — For $a \in \mathbb{N}$ the infinite families

(1) $\psi(G, G, K_{3a,3a})$ where the graph G is either K_{2a} or its complement,

(2) $\psi(G, G, K_{8a,8a})$ where the graph G is either K_{3a} or its complement,

(3) $\psi(iK_a, iK_a, K_{8a,8a})$ for $i = 2, 3$,

(4) $\psi(K_{a,a}, K_{a,a}, K_{3a,3a})$, and

(5) $\psi(G, H, K_{3a,3a})$ where both G and H are in $\{K_a, \overline{K_a}, K_{2b}, \overline{K_{2b}}\}$ and $G \neq H$ are all integral.

Proposition 2.9 — For $a = 4s^2 - t^2$, $b = 4st$, $c = 4s^2 + t^2 \in \mathbb{N}$, (a, b, c) forms a pythagorean triple so that $a^2 + b^2 = c^2$. Then the infinite families

(1) $\psi(\overline{K_u}, \overline{K_v}, K_{a,a})$,

(2) $\psi(\overline{K_v}, \overline{K_u}, K_{a,a})$, where $uv = 4(st)^2$, and

(3) $\psi(K_d, K_d, K_{a,a})$, where $d = 2st$

are all integral.

Proposition 2.10 — For $a = q(s^2 - t^2)$, $b = 2qst$, $c = q(s^2 + t^2) \in \mathbb{N}$, (a, b, c) forms a pythagorean triple so that $a^2 + b^2 = c^2$. Then the infinite families

(1) $\psi(\overline{K_u}, \overline{K_v}, K_{a,a})$,

(2) $\psi(\overline{K_v}, \overline{K_u}, K_{a,a})$, where $uv = (qst)^2$, and

(3) $\psi(K_d, K_d, K_{a,a})$, where $d = qst$

are all integral.

3. The Laplacian Spectrum of the Graph $\psi(G_1, G_2, G_3)$

For any graph G of order n with adjacency matrix A and degree sequence $\{d_1, d_2, \ldots, d_n\}$, the Laplacian matrix L of G is defined by: $L = \Delta - A$, where Δ = diagonal matrix$(d_1, d_2, \ldots, d_n)$. The spectrum of L is called the Laplacian spectrum of G. In this section we obtain the Laplacian spectrum of $\psi(G_1, G_2, G_3)$ when G_i are r_i-regular, $i = 1, 2, 3$. The degrees of vertices of $\psi(G_1, G_2, G_3)$ are then given by:

(1) $\deg(u_i, w_j) = r_1 + n_2$

(2) $\deg(v_i, w_j) = r_2 + n_1 + r_3$.

Therefore, labeling the vertices of $\psi(G_1, G_2, G_3)$ as in Section 2, its Laplacian matrix $\mathcal{L}$ of $\psi(G_1, G_2, G_3)$ can be expressed in the form

$$\begin{aligned}
\mathcal{L} &= I_{n_3} \otimes \begin{bmatrix} (r_1 + n_2) I_{n_1} - A_1 & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & (r_2 + n_1) I_{n_2} - A_2 \end{bmatrix} + (r_3 I_{n_3} - A_3) \otimes \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \\
&= I_{n_3} \otimes \begin{bmatrix} n_2 I_{n_1} + L(G_1) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & n_1 I_{n_2} + L(G_2) \end{bmatrix} + L(G_3) \otimes \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}
\end{aligned}$$

We then have the following theorem whose proof is analogous to that of Theorem 2.1 and hence is omitted.

***Theorem* 3.1** — *Let $G_i, i = 1, 2, 3$ be r_i-regular graphs on n_i vertices with Laplacian spectra $\{0 = \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{n_1}\}$, $\{0 = \eta_1 \leq \eta_2 \leq \ldots \leq \eta_{n_2}\}$ and $\{\tau_1 \leq \tau_2 \leq \ldots \leq \tau_{n_3}\}$ respectively. Then the Laplacian spectrum of $\psi(G_1, G_2, G_3)$ consists of $(n_2 + \theta_i), i = 2, 3, \ldots, n_1$, each with multiplicity n_3, $(n_1 + \eta_j + 1), j = 2, 3, \ldots, n_2$, each with multiplicity n_3 together with the $2n_3$ numbers which are roots of the quadratic equation*

$$x^2 - (n_1 + n_2 + \tau_i)x + n_2\tau_i = 0, i = 1, 2, \ldots, n_3. \quad (4)\ \square$$

Using the above theorem, we now exhibit some new infinite families of Laplacian integral graphs.

Proposition 3.2 — For $a, b \in \mathbb{N}$ the infinite families

(1) $\psi(G, H, K_{2b})$ where $G \in \{K_a, \overline{K_a}\}$ and $H \in \{K_{a+b}, \overline{K_{a+b}}\}$

(2) $\psi(G, H, K_{a+b})$ where $G \in \{K_a, \overline{K_a}\}$ and $H \in \{K_{2b}, \overline{K_{2b}}\}$

(3) $\psi(G, H, K_{a^2-1})$ where $G, H \in \{K_a, \overline{K_a}\}$

(4) $\psi(G, H, K_a)$ where $G \in \{K_a, \overline{K_a}\}$ and $H \in \{K_{a^2-1}, \overline{K_{a^2-1}}\}$

(5) $\psi(G, H, K_{a^2})$ where $G \in \{K_{a-1}, \overline{K_{a-1}}\}$ and $H \in \{K_{a+1}, \overline{K_{a+1}}\}$

(6) $\psi(G, H, K_{a+1})$ where $G \in \{K_{a-1}, \overline{K_{a-1}}\}$ and $H \in \{K_{a^2}, \overline{K_{a^2}}\}$ are all Laplacian integral.

Proposition 3.3 — For $a \in \mathbb{N}$ and $i \neq j = 2, 3$ the infinite families $\psi(G, K_{ia^2+1}, K_{ja^2+1})$ and $\psi(G, \overline{K_{ia^2+1}}, K_{ja^2+1})$, where the graph G is either K_{a^2} or its complement are Laplacian integral if and only if a is a positive integer satisfying the Pell's equation $b^2 - 3a^4 = 1$.

Proposition 3.4 — For $a, b \in \mathbb{N}$, and $i \neq j \in \{a - 1, a + 1\}$, the infinite families $\psi(G, K_i, K_j)$ and $\psi(G, \overline{K_i}, K_j)$, where the graph G is either K_{2ab} or its complement are Laplacian integral if and only if a is a positive integer satisfying the Pell's equation $c^2 - b(b + 2)a^2 = 1$.

Proposition 3.5 — For $a, b \in \mathbb{N}$, and $i \neq j \in \{a - 1, a + 1\}$, the infinite families $\psi(G, K_i, K_j)$ and $\psi(G, \overline{K_i}, K_j)$, where the graph G is either $K_{a(2b+1)}$ or its complement are Laplacian integral if and only if a is a positive integer satisfying the Pell's equation $c^2 - (2b + 1)(2b + 5)a^2 = 4$.

Proposition 3.6 — For $a, b \in \mathbb{N}$, and $i \neq j \in \{a - b, a + b\}$, the infinite families $\psi(G, K_i, K_j)$ and $\psi(G, \overline{K_i}, K_j)$, where the graph G is either K_a or its complement are Laplacian integral if and only if a is a positive integer satisfying the Pell's equation $c^2 - 5a^2 = 4b^2$.

Proposition 3.7 — For $a, b \in \mathbb{N}$, and $i \neq j \in \{3a, a + b\}$, the infinite families $\psi(G, K_i, K_j)$ and $\psi(G, \overline{K_i}, K_j)$, where the graph G is either K_{2a} or its complement are Laplacian integral if and only

if a is a positive integer satisfying the Pell's equation $c^2 - 24a^2 = b^2$.

The above proposition holds good for the pair of integers (i, j) (where $i \neq j$) $\in \{a - b, 3a\}$ whenever $b = 1, 2, \ldots, a - 2$.

Proposition 3.8 — For $a \in \mathbb{N}$, and $i \neq j \in \{a^2 - 1, a^2\}$, the infinite families of graphs $\psi(G, K_i, K_j)$ and $\psi(G, \overline{K_i}, K_j)$ where the graph $G \in \{K_{a^2+1}, \overline{K_{a+1}}\}$ are Laplacian integral if and only if a is a positive integer satisfying the Pell's equation $c^2 - 5a^2 = 4b^2$.

4. The Signless Laplacian Spectrum of $\psi(G_1, G_2, G_3)$

For any graph G of order n with adjacency matrix A and degree sequence $\{d_1, d_2, \ldots, d_n\}$, the signless Laplacian matrix Q of G is defined by: $Q = \Delta + A$, where Δ = diagonal matrix$(d_1, d_2, \ldots, d_n)$. The spectrum of Q is called the signless Laplacian spectrum of G. Labeling the vertices of $\psi(G_1, G_2, G_3)$ as in Section 2, the signless Laplacian matrix $\mathcal{Q}$ of $\psi(G_1, G_2, G_3)$ can be expressed in the form

$$\mathcal{Q} = I_{n_3} \otimes \begin{bmatrix} (r_1 + n_2) I_{n_1} + A_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & (r_2 + n_1) I_{n_2} + A_2 \end{bmatrix} + (r_3 I_{n_3} + A_3) \otimes \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$$

$$= I_{n_3} \otimes \begin{bmatrix} n_2 I_{n_1} + Q(G_1) & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & n_1 I_{n_2} + Q(G_2) \end{bmatrix} + Q(G_3) \otimes \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$

where $Q(G_i)$ is the signless Laplacian matrix of the graph G_i for $i = 1, 2, 3$.

Imitating the proof of Theorem 2.1, we get the following characterization for the graph $\psi(G_1, G_2, G_3)$ to be signless Laplacian integral.

***Theorem* 4.1** — *Let $G_i, i = 1, 2$ be two integral r_i-regular graphs each of order n_i. Then the graph $\psi(G_1, G_2, G_3)$ is signless Laplacian integral if and only if for each l, $1 \leq l \leq n_3$, $(2(r_1 - r_2) + (n_2 - n_1) - \nu_l)^2 + 4n_1n_2$ is a perfect square, for each $\nu_l \in Q$-spectrum of the integral graph G_3.* □

All regular integral graphs are signless Laplacian integral [6] but only a few non-regular signless Laplacian integral graphs are known. We now give some infinite families of **non-regular** signless Laplacian integral graphs.

Corollary 4.2 — Let G_i, $i = 1, 2, 3$ be r_i-regular integral graphs on n_i vertices. For $p \in \mathbb{N}$, $r_1 = r_2 + p$, $n_1 = n_2 + p$ and $G_3 \simeq K_{p+1}$, the graph $\psi(G_1, G_2, G_3)$ is signless Laplacian integral if and only if $4n_2(n_2 + p) + 1$ is a perfect square.

Using Corollary 4.2, we exhibit the following infinite families of non-regular signless Laplacian integral graphs.

Proposition 4.3 — For $p = 1$, the infinite family of non-regular graphs $\psi(K_{n+1}, K_n, K_2)$ is signless Laplacian integral.

Proposition 4.4 — When $4n_2(n_2 + p) = (a + 1)(a - 1)$ for some $a = 4t \pm 1$ and $n_2 = t$, the infinite families of non-regular graphs $\psi(K_{4t\pm2}, K_t, K_{(3t\pm2)+1})$ are signless Laplacian integral.

Proposition 4.5 — When $n_2(n_2+p) = n_2h(n_2h\pm1)$ for some $h = n_2+1, n_2+2, \ldots$, the infinite families of non-regular graphs $\psi(K_{n_2h^2\pm h}, K_{n_2}, K_{n_2(h^2-1)\pm h+1})$ are signless Laplacian integral.

Proposition 4.6 — When $4n_2 = a + 1$ and $n_2 + p = a - 1$ for some $a = 4t - 1$ and $n_2 = 2m$, for some $m \in \mathbb{N}$, the infinite families of non-regular graphs $\psi(CP(4m - 1), CP(m), K_{6m-1})$ are signless Laplacian integral.

Proposition 4.7 — For $n_2 = p = 2m$, the infinite families $\psi(K_{m,m,m,m}, K_{m,m}, K_{2m+1})$ and for $n_2 = p = 2m + 1$, the infinite families of non-regular graphs $\psi(K_{2(2m+1)}, K_{2m+1}, K_{2m+2})$ where $m \in \mathbb{N}$ satisfies the Pell's equation $c^2 - 8p^2 = 1$ are signless Laplacian integral.

Proposition 4.8 — For $n_2 = 2p = 2m$, the infinite families of non-regular graphs $\psi(K_{2m,2m,2m}, K_{2m,2m}, K_{2m+1})$ and for $n_2 = 2p = 2m + 1$, the infinite families of non-regular graphs $\psi(K_{2m+1,2m+1,2m+1}, K_{2m+1,2m+1}, K_{2m+2})$ where $m \in \mathbb{N}$ satisfies the Pell's equation $c^2 - 24p^2 = 1$ are signless Laplacian integral.

5. Conclusion

In this paper we have defined a new composition on an ordered set of three graphs and have discovered several new classes of integral graphs and thus adding them to the existing classes of integral graphs. Several other new operations on graphs can be expected to produce such nice families of integral graphs.

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