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ON A FUNCTIONAL CONNECTED TO THE LAPLACIAN IN A FAMILY OF PUNCTURED REGULAR POLYGONS IN $\mathbb{R}^2$

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Let $\mathfrak{p}_1$ and $\mathfrak{p}_0$ be closed, regular, convex, concentric polygons having n sides in $\mathbb{R}^2$ such that the circumradius of $\mathfrak{p}_0$ is strictly less than the inradius of $\mathfrak{p}_1$. We fix $\mathfrak{p}_1$ and vary $\mathfrak{p}_0$ by rotating it about its center. Let Ω be the interior of $\mathfrak{p}_1 \setminus \mathfrak{p}_0$. Let u be the solution of the stationary problem $-\Delta u = 1$ in Ω vanishing on the boundary. We show that the associated Dirichlet energy functional $J(\Omega)$ attains its extremum values when the axes of symmetry of $\mathfrak{p}_0$ coincide with those of $\mathfrak{p}_1$.

Key words : Laplacian operator; boundary value problems for second order elliptic equations; variational methods for second order elliptic equations; second order elliptic equations; variational methods.

1. INTRODUCTION

Let $R > 0$ be an arbitrary constant. Fix $n \in \mathbb{N}$ with $n \geq 3$. Let $\mathfrak{p}_1$ be a regular polygon of n sides in $\mathbb{R}^2$ circumscribed by the circle $C(0, R)$. Without loss of generality, let $(0, R) \in \mathbb{R}^2$ be one of the vertices of $\mathfrak{p}_1$. Let $r > 0$ be such that the circle $C(0, r)$ is contained in the interior of $\mathfrak{p}_1$. Throughout the paper the objects R, r and $\mathfrak{p}_1$ are fixed. Let $\mathfrak{p}_0$ denote any regular polygon of n sides circumscribed by the circle $C(0, r)$ and let $\mathcal{F}_n = \mathcal{F}_n(R, r)$ denote the family of domains $\Omega = (\mathfrak{p}_1 \setminus \mathfrak{p}_0)^\circ$, the interior of $\mathfrak{p}_1 \setminus \mathfrak{p}_0$ in $\mathbb{R}^2$.

We consider the following stationary problem on Ω:

$$\left.\begin{aligned} -\Delta u = 1 \text{ on } \Omega \text{ in the sense of distributions,} \\ u = 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{1.0.1}$$

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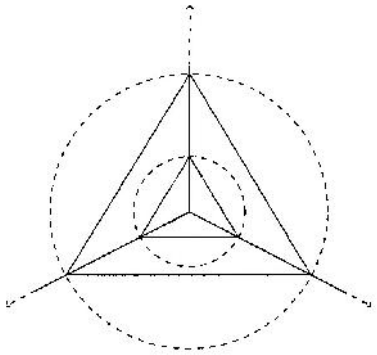
Let $u = y(\Omega)$ be the unique solution of problem (1.0.1). We study the extrema of the functional

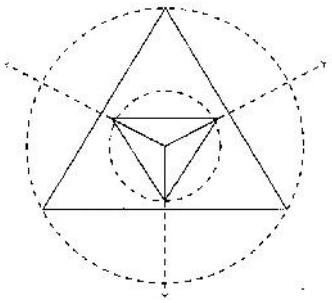
$$J(\Omega) = \int_{\Omega} \|\nabla y(\Omega)\|^2 \, dx$$

on $\mathcal{F}_n$ associated to (1.0.1).

Following [1], we introduce two special positions of $\mathfrak{p}_0$ in $\mathfrak{p}_1$: If a vertex of $\mathfrak{p}_0$ and and a vertex of $\mathfrak{p}_1$ lie on the same half-axis of symmetry emanating from the origin 0 then we say that $\mathfrak{p}_0$ occupies *on position* in $\mathfrak{p}_1$. If a vertex of $\mathfrak{p}_1$ lies on a half-axis of symmetry passing through the origin and the midpoint of one of the sides of $\mathfrak{p}_0$ then we say that $\mathfrak{p}_0$ occupies *off position* in $\mathfrak{p}_1$. In the general position of $\mathfrak{p}_0$ neither of the two conditions is satisfied. Rotation by an integral multiple of $2\pi/n$ is a symmetry of Ω at all positions of $\mathfrak{p}_1$. However the on and off positions possess additional reflectional symmetry along the axes mentioned above.



$\mathfrak{p}_0$ in 'on position' for n=3.



$\mathfrak{p}_0$ in 'off position' for n=3.

We state our main result.

***Theorem* 1.1** — *For $\Omega \in \mathcal{F}_n$, the functional $J(\Omega)$ is optimized exactly when the axes of symmetry of $\mathfrak{p}_0$ coincide with those of $\mathfrak{p}_1$. The minimizing configuration for $J(\Omega)$ corresponds to case when $\mathfrak{p}_0$ occupies 'on position' in $\mathfrak{p}_1$. The maximizing configuration for $J(\Omega)$ corresponds to the case when $\mathfrak{p}_0$ occupies 'off position' in $\mathfrak{p}_1$.*

The above result has been proved in [5] for a family of domains $\Omega = B_1 \setminus B_0$ where B_0 is an arbitrary closed ball contained in the interior B_1° of the closed ball B_1 in Euclidean space $\mathbb{R}^n$.

It is well known that for any bounded domain Ω having smooth boundary the solution $y(\Omega)$ of (1.0.1) belongs to $\mathcal{C}^\infty(\overline{\Omega}) \subset H^2(\Omega)$. However the domains we consider are non-convex polygons for which the solution of (1.0.1) is not a smooth function on $\overline{\Omega}$ and instead lies in the Sobolev space $H_0^{1+\delta}(\Omega)$, where $\delta \in \left]\frac{1}{2}, \frac{3}{5}\right[$ ([3]).

The Dirichlet energy functional $J(\Omega)$ associated to the stationary problem (1.0.1) is differentiable with respect to *variation of domain* when Ω has convex or C^2-boundary and expressions for the derivative are well-known ([8]) in such cases. Here we derive an expression for the derivative of $J(\Omega)$ when $\Omega \in \mathcal{F}_n$ in §3 and apply the standard technique of domain reflection pioneered by Serrin in [7] to prove our main result in §4.

2. PRELIMINARIES

Let Ω be a bounded polygonal domain in $\mathbb{R}^d$.

Definition 2.1 — For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d, |\alpha| = \alpha_1 + \ldots + \alpha_d$. A function $v \in L^2(\Omega)$ is said to be an *α-th weak partial derivative* of a function $u \in L^2(\Omega)$ if

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v\varphi \, dx, \; \forall \varphi \in \mathscr{D}(\Omega).$$

The α-th weak derivative of u is denoted by $\partial^\alpha u$, if it exists.

We define the Sobolev space $H^s(\Omega)$ for $s \geq 0$:

(i) For $s = m \in \mathbb{Z}^+$,

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \partial^\alpha u \in L^2(\Omega), \; \forall\, |\alpha| \leq m \right\}.$$

For $u, v \in H^m(\Omega)$, define

$$\langle u, v \rangle_{H^m\Omega} = \sum_{|\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

(ii) For $s = m + \sigma$, where $m \in \mathbb{Z}^+$ and $\sigma \in]0, 1[$,

$$H^s(\Omega) = \left\{ u \in H^m(\Omega) \;\middle|\; \sum_{|\alpha|=m} \int\int_{\Omega\times\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{\|x - y\|^{d+2\sigma}} dxdy < \infty \right\}.$$

For $u, v \in H^s(\Omega)$, define

$$\langle u, v \rangle_{H^s\Omega} = \langle u, v \rangle_{H^m(\Omega)} + \sum_{|\alpha|=m} \int\int_{\Omega\times\Omega} \frac{(\partial^\alpha u(x) - \partial^\alpha u(y)) \left(\overline{\partial^\alpha v(x) - \partial^\alpha v(y)} \right)}{\|x - y\|^{d+2\sigma}} dxdy.$$

$H^s(\Omega)$ is a Hilbert space with above inner product and the induced norm is called the Sobolev norm. The closure of $\mathscr{D}(\Omega)$ in $H^s(\Omega)$ is denoted by $H_0^s(\Omega)$.

Definition 2.2 — Let Ω be an open subset of $\mathbb{R}^d$. We say Ω has *Lipschitz (polygonal) boundary* if for every $p \in \partial\Omega$, there exists a neighborhood U of p in $\mathbb{R}^d$ such that $\partial\Omega \cap U$ is the graph of a Lipschitz (continuous piecewise linear) function.

Proposition 2.3 (cf. Theorem 1.2.7, p. 4 of [3]) — Let Ω be an open subset of $\mathbb{R}^d$ having Lipschitz boundary. Then $\mathscr{D}(\bar{\Omega})$ is dense in $H^s(\Omega)$ for all $s \geq 0$.

Definition 2.4 — Let $\mathscr{D}'(\Omega)$ denote the space of all *distributions* on Ω. For $T \in \mathscr{D}'(\Omega)$, we define $\Delta T \in \mathscr{D}'(\Omega)$ by

$$\langle \Delta T; \varphi \rangle := \langle T; \Delta\varphi \rangle , \quad \forall \varphi \in \mathscr{D}(\Omega).$$

Definition 2.5 — Let $u \in L^1_{loc}(\Omega)$. Define $T_u \in \mathscr{D}'(\Omega)$ by

$$\langle T_u; \varphi \rangle := \int_\Omega u\varphi \, dx, \quad \forall \varphi \in \mathscr{D}(\Omega).$$

For $u, v \in L^1_{loc}(\Omega)$, we say $\Delta u = v$ *in the sense of distributions* if $\Delta T_u = T_v$.

For $s \geq 0$, the following definition now makes sense:

Definition 2.6 — $E^s(\Delta, L^2(\Omega)) = \left\{ u \in H^s(\Omega) \;\middle|\; \Delta u \in L^2(\Omega) \right\}$.

Thus we have an operator $\Delta \colon E^s(\Delta, L^2(\Omega)) \to L^2(\Omega)$. For $u, v \in E^s(\Delta, L^2(\Omega))$, define

$$\langle u, v \rangle_{E^s(\Delta, L^2(\Omega))} = \langle u, v \rangle_{H^s(\Omega)} + \langle \Delta u, \Delta v \rangle_{L^2(\Omega)}.$$

$E^s(\Delta, L^2(\Omega))$ is a Hilbert space with this inner product. We denote by $E^s_0(\Delta, L^2(\Omega))$ ($s \geq 1$) the Hilbert space $E^s(\Delta, L^2(\Omega)) \cap H^1_0(\Omega)$ with the above inner product.

Proposition 2.7 (cf. Lemma 3.1, p. 261 of [2]) — Let Ω be a bounded open subset of $\mathbb{R}^d$ having Lipschitz boundary. Then $\mathscr{D}(\bar{\Omega})$ is dense in $E^s(\Delta, L^2(\Omega))$ for any $0 \leq s < 2$ with $s \neq \dfrac{3}{2}, \dfrac{1}{2}$.

For the remainder of this section Ω denotes a bounded open set in $\mathbb{R}^2$ which has polygonal boundary and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ be open line segments such that $\partial\Omega = \bigcup_{j=1}^{N} \overline{\Gamma}_j$. Let n_j denote the outward unit normal to Ω on the side $\Gamma_j (1 \leq j \leq N)$.

Proposition 2.8 (Trace operator: cf. Theorem 1.4.2, p. 12 of [3]) — Fix $s > \dfrac{1}{2}$. The map $u \longmapsto u_{|_{\Gamma_j}}$ from $\mathscr{D}(\bar{\Omega})$ extends as a bounded linear operator from $H^s(\Omega)$ into $H^{s-\frac{1}{2}}(\Gamma_j)$.

Remark 2.9 : By Proposition 2.8, $\forall u \in H^1_0(\Omega), u_{|_{\Gamma_j}} = 0$.

We need the Green's Identitities:

Proposition 2.10 (cf. Proposition 3.4, p. 264 of [2])

Let $s > \frac{1}{2}$. Let $v \in E^{1+s}(\Delta, L^2(\Omega))$ and $u \in H^1(\Omega)$ be arbitrary. Then

$$\int_\Omega u\Delta v = -\int_\Omega \langle \nabla u, \nabla v\rangle \, dx + \sum_{j=1}^{N} \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j} ds.$$

Proposition 2.11 (cf. Proposition 3.5, p. 264 of [2]) — Let $s > \frac{1}{2}$. Let $u \in E_0^{1+s}(\Delta, L^2(\Omega))$ and $v \in E^s(\Delta, L^2(\Omega))$ be arbitrary. Then

$$\int_\Omega u\Delta v - v\Delta u \, dx = -\sum_{j=1}^{N} \int_{\Gamma_j} v_{|\Gamma_j} \frac{\partial u}{\partial n_j} \, ds \, .$$

Proposition 2.12 (cf. Proposition 3.6, p. 265 of [2]) — For every $f \in L^2(\Omega)$ there exists a unique solution $u \in H_0^1(\Omega)$ of the Dirichlet boundary value problem:

$$\left.\begin{aligned} \Delta u = f \text{ on } \Omega \text{ in the sense of distributions,} \\ u = 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{2.12.1}$$

Let $\Lambda = \dfrac{\pi}{\max_j \omega_j}$, where ω_j denotes the interior angle at the j-th vertex of Ω. Then $\Lambda > \frac{1}{2}$. It may be noted that for a non-convex polygonal domain, $\Lambda < 1$. We state the following regularity result.

Proposition 2.13 (cf. Corollary 2.4.4 and Remark 2.4.6, pp. 58-59 [3]) — Let $s_0 = \min\{1, \Lambda\} > \frac{1}{2}$. If $u \in H_0^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$, then $u \in H^{1+s}(\Omega), \forall s < s_0$.

By the above result $E_0^{1+s}(\Delta, L^2(\Omega)) = E_0^{1+t}(\Delta, L^2(\Omega)), \forall\ 0 \le s, t \le s_0$. We fix an arbitrary constant $\delta \in \left]\frac{1}{2}, \frac{3}{5}\right[$ for the remainder of this paper.

Proposition 2.14 (cf. Proposition 4.1, p. 267 of [2]) — $\Delta\colon E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is an invertible bounded operator.

Proposition 2.15 (The Maximum Principle, cf. Theorem 5, p. 61 of [6]) — Let $u(x_1, x_2, \ldots, x_d)$ satisfy the differential inequality

$$L[u] \equiv \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} \ge 0$$

in a domain $D \subset \mathbb{R}^d$ where L is uniformly elliptic. Suppose the coefficients a_{ij} and b_i are uniformly bounded. If u attains a maximum M at a point of D, then $u \equiv M$ in D.

Proposition 2.16 (The Hopf Maximum Principle, cf. Theorem 7, p. 65 [6]) — Let u satisfy the inequality

$$L[u] \equiv \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} \geq 0$$

in a domain $D \subset \mathbb{R}^d$ where L is uniformly elliptic. Suppose that $u \leq M$ in D and that $u = M$ at a boundary point P. Assume that P lies on the boundary of a ball K_1 in D. If u is continuous in $D \cup P$ and an outward directional derivative $\partial u/\partial \nu$ exists at P, then $\partial u/\partial \nu > 0$ at P unless $u \equiv M$.

3. Material Derivative and Hadamard Formula

Recall that $\mathcal{F}_n$ is the family of punctured sets $\Omega = (\mathfrak{p}_1 \backslash \mathfrak{p}_0)^\circ$ where $\mathfrak{p}_1$ is a fixed closed, convex, regular polygon having n sides circumscribed on the circle $C(0,R)$ and $\mathfrak{p}_0$ a similar polygon circumscribed on $C(0,r)$ $(R > r > 0)$. Let $B(0,s)$ denote the open disk of radius $s > 0$ centered at the origin 0 in $\mathbb{R}^2$.

The *in-radius* of $\mathfrak{p}_1$ is defined as $\sup\{s > 0 \;\big|\; C(0,s) \subset \mathfrak{p}_1\}$. We denote the in-radius of $\mathfrak{p}_1$ by $inrad(\mathfrak{p}_1)$. Let $r_1 = \dfrac{r + inrad(\mathfrak{p}_1)}{2}$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a radial, non-negative $\mathcal{C}^\infty$-function having compact support such that $\varphi(x) = 1, \forall x \in B(0,r_1)$ and $\varphi(x) = 0, \forall x \in \mathbb{R}^2 \setminus \overline{B}(0, inrad(\mathfrak{p}_1))$.

The sides are numbered such that $\Gamma_1, \ldots, \Gamma_n$ are contained in the inner polygonal boundary and $\Gamma_{n+1}, \ldots, \Gamma_{2n}$ are contained in the outer polygonal boundary. Following the same convention, let $Q_1, \ldots, Q_n$ denote the vertices on the inner boundary and let $Q_{n+1}, \ldots, Q_{2n}$ denote the vertices on the outer boundary. Let ω_j denote the interior angle at vertex Q_j for $j = 1, \ldots, 2n$.

We denote by R_θ a counter-clockwise rotation of $\mathbb{R}^2$ by an angle of θ. Consider the vector field V on $\mathbb{R}^2$ defined by

$$V(x) = \varphi(x) R_{\pi/2}(x), \;\left(x \in \mathbb{R}^2\right).$$

Throughout this section we fix $\Omega \in \mathcal{F}_n$ arbitrarily and y denotes the unique solution $y(\Omega)$ of (1.0.1).

Let $\{\psi_t\}_{t \in \mathrm{R}}$ be the 1-parameter group of diffeomorphisms of $\mathbb{R}^2$ associated with V. Thus ψ_t is given explicitly by the formula

$$\psi_t(x) = R_{t\varphi(x)}(x) \;\forall\; x \in \mathbb{R}^2.$$

Let Ω_t denote $\psi_t(\Omega)$. Observe that $\mathcal{F}_n = \{\Omega_t \;\big|\; t \in \mathbb{R}\}$. Let $y_t \in H_0^1(\Omega_t)$ be the unique solution of (1.0.1) with $\Omega = \Omega_t$ and let $y^t (t \in \mathbb{R})$ denote the composite $y_t \circ \psi_{t|_\Omega}$. It follows by Proposition

2.13 that $y^t \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ and by the Elliptic Regularity Theorem $y^t \in \mathcal{C}^\infty(\Omega)$. Following [4] we prove the following result.

Proposition 3.1 — The map $t \longmapsto y^t$ is a $\mathcal{C}^1$-curve in $E_0^{1+\delta}(\Delta, L^2(\Omega))$ from a neighborhood of zero in $\mathbb{R}$.

PROOF : For each $t \in \mathbb{R}$ let $\gamma_t(x) = det((d\psi_t)_x)$ where $(d\psi_t)_x$ denotes the total derivative of ψ_t at $x \in \mathbb{R}^2$.

Let $\varphi \in \mathscr{D}(\Omega)$ be arbitrary. Since ψ_t^{-1} is a diffeomorphism, $\varphi \circ \psi_t^{-1} \in \mathscr{D}(\Omega_t)$. Therefore

$$\begin{aligned} \int_{\Omega_t} \varphi \circ \psi_t^{-1}\, dx &= \int_{\Omega_t} (-\Delta y_t)(\varphi \circ \psi_t^{-1})\, dx \qquad \text{(by (1.0.1))} \\ &= \int_{\Omega_t} \left\langle \nabla y_t, \nabla(\varphi \circ \psi_t^{-1}) \right\rangle dx \qquad \text{(by Proposition 2.10).} \end{aligned} \tag{3.1.1}$$

By the change of variable $\psi_t \colon \Omega \to \Omega_t$ we obtain

$$\begin{aligned} \int_{\Omega_t} \left\langle \nabla y_t, \nabla(\varphi \circ \psi_t^{-1}) \right\rangle dx &= \int_\Omega \left\langle (\nabla y_t) \circ \psi_t, \left(\nabla(\varphi \circ \psi_t^{-1})\right) \circ \psi_t \right\rangle \gamma_t\, dx \\ &= \int_\Omega \left\langle (d\psi_t^{-1})_x^* \nabla(y_t \circ \psi_t), (d\psi_t^{-1})_x^* \nabla(\varphi \circ \psi_t^{-1} \circ \psi_t) \right\rangle \gamma_t\, dx \\ &= \int_\Omega \left\langle A_t \nabla y^t, \nabla\varphi \right\rangle dx\,. \end{aligned} \tag{3.1.2}$$

where $A_t(x) = \gamma_t(x)\, ((d\psi_t)_x^* (d\psi_t)_x)^{-1}$ (the superscript $*$ indicates the adjoint of the linear map).

From (3.1.1) and (3.1.2) we obtain

$$\begin{aligned} \int_\Omega \left\langle A_t \nabla y^t, \nabla\varphi \right\rangle dx &= \int_{\Omega_t} \varphi \circ \psi_t^{-1}\, dx \\ &= \int_\Omega \varphi \gamma_t\, dx\,. \end{aligned} \tag{3.1.3}$$

Let Ω' be any relatively compact open subset of Ω having $\mathcal{C}^\infty-$ boundary and containing $supp(\varphi)$. By the Divergence Theorem of Gauss applied to the smooth vector field $X = \varphi(A_t \nabla y^t)$ defined on Ω' we obtain

$$\int_{\Omega'} \left\langle A_t \nabla y^t, \nabla\varphi \right\rangle dx + \int_{\Omega'} (div(A_t \nabla y^t))\varphi\, dx = 0.$$

Hence

$$\int_\Omega \left\langle A_t \nabla y^t, \nabla\varphi \right\rangle dx = -\int_\Omega (div(A_t \nabla y^t))\varphi\, dx\,. \tag{3.1.4}$$

From (3.1.3) and (3.1.4) we get

$$\int_\Omega \gamma_t \varphi \, dx = - \int_\Omega (div(A_t \nabla y^t)) \varphi \, dx, \forall \varphi \in \mathscr{D}(\Omega). \tag{3.1.5}$$

It follows that

$$- div(A_t \nabla y^t) = \gamma_t \text{ on } \Omega. \tag{3.1.6}$$

Define $\delta_{A_t} \colon E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ by

$$\delta_{A_t} u(x) = - div(A_t \nabla u)(x).$$

Note that $\delta_{A_t} u(x) = -\Delta u(x)$ for $x \in \Omega \cap B(0, r_1)$ and $x \in \Omega \setminus \overline{B}(0, inrad(\mathfrak{p}_1))$. Thus each $\delta_{A_t} \colon E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is a bounded operator.

Next define $F \colon \mathbb{R} \times E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ by

$$F(t, u) = -\delta_{A_t}(u) + \gamma_t.$$

Then F is a $\mathcal{C}^1$-map and by (3.1.6) $F(t, y^t) = 0$. Also $(D_2 F)_{(0,y)}(0, u) = \Delta u$ and hence by Proposition 2.14 $(D_2 F)_{(0,y)} \colon E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is an isomorphism. By the Implicit Function Theorem $t \longmapsto y^t$ is a $\mathcal{C}^1$-map in a neighborhood of zero in $\mathbb{R}$. □

Let Ω' be any relatively compact open subset of Ω with $\mathcal{C}^\infty$-boundary.

Lemma 3.2 (cf. [8]) — Let $t \longmapsto f(t)$ be a differentiable curve in $L^2(\Omega)$, with $f(0) \in H^1(\Omega)$. Let $\{\psi_t\}_{t \in \mathrm{R}}$ be the one parameter group of diffeomorphisms of $\mathbb{R}^2$ associated with a compactly supported smooth vector field V defined on $\mathbb{R}^2$. Then for t sufficiently close to zero $t \longmapsto f(t) \circ \psi_{t|_{\Omega'}}$ is a differentiable curve in $L^2(\Omega')$ and at $t = 0$

$$\left.\frac{d(f(t) \circ \psi_t)}{dt}\right|_{t=0} = \left.\frac{df}{dt}\right|_{t=0} + \langle \nabla(f(0)), V \rangle \in L^2(\Omega').$$

As a consequence of Lemma 3.2 we have the following two corollaries.

Corollary 3.3 — The map $t \longmapsto y_{t|_{\Omega'}}$ is a $\mathcal{C}^1$-curve in $L^2(\Omega')$ from a neighborhood of zero in $\mathbb{R}$ and

$$\left.\frac{d}{dt}\right|_{t=0} \left(y_{t|_{\Omega'}}\right) = \left(\left.\frac{dy^t}{dt}\right|_{t=0} - \langle \nabla y, V \rangle\right)_{|_{\Omega'}} \in L^2(\Omega').$$

PROOF : Observe that $y_t = y^t \circ \psi_t^{-1} = y^t \circ \psi_{-t}$. By Proposition 3.1 the map $t \longmapsto y^t$ is a $\mathcal{C}^1$ curve in $H^1(\Omega)$, so $t \longmapsto y_t$ satisfies the conditions of Lemma 3.2. The result now follows from the

observation that $\{\psi_{-t}\}_{t\in\mathbb{R}}$ is the one parameter group of diffeomorphisms of $\mathbb{R}^2$ associated with the vector field $-V$. □

Definition 3.4 — The *material derivative* of $y = y(\Omega)$ in the direction of V is defined by

$$\dot{y}(\Omega, V) := \left.\frac{dy^t}{dt}\right|_{t=0} \in E_0^{1+\delta}(\Delta, L^2(\Omega)).$$

Definition 3.5 — The *shape derivative* y' in the direction of V is defined by

$$y' = \dot{y} - \langle \nabla y, V \rangle .$$

Corollary 3.6 —

$$y'_{|\Omega'} = \left.\frac{d}{dt}\right|_{t=0} \left(y_{t|\Omega'}\right). \tag{3.6.1}$$

Consider the domain functional $J(\Omega_t)$ ($t \in \mathbb{R}$) defined in §1. By Proposition 2.10 we have

$$J(\Omega_t) = \int_{\Omega_t} y_t \, dx .$$

Definition 3.7 — The *Eulerian derivative* $dJ(\Omega, V)$ of $J(\Omega_t)$ at $t = 0$ is defined as

$$dJ(\Omega, V) := \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

For the sake of completeness we include a proof of the following Proposition which can be found in [8].

Proposition 3.8 — The functional $J(\Omega_t)$ is differentiable at $t = 0$ and

$$dJ(\Omega, V) = \int_\Omega y' \, dx .$$

PROOF : Let ω denote the volume form on $\mathbb{R}^2$. Let $L_V\omega$ denote the Lie derivative of ω with respect to V and $i_V\omega$ denote the interior multiplication of ω with respect to V. Then

$$\left.\frac{d}{dt}\left(\psi_t^*\omega\right)\right|_{t=0} =: L_V\omega = (di_V + i_V d)\omega = d(i_V\omega) = div(V)\omega$$

and

$$\begin{aligned} dJ(\Omega, V) &= \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t} \\ &= \lim_{t\to 0} \int_\Omega \left(\frac{y^t \gamma_t \omega - y\omega}{t} \right) \qquad (\text{ where } \gamma_t = det(d\psi_t)) \\ &= \int_\Omega \left(\frac{d}{dt}(y^t \gamma_t \omega) \right) \Big|_{t=0} \qquad (\text{by Proposition } 3.1) \\ &= \int_\Omega (\dot{y} + y\, div(V))\, \omega \\ &= \int_\Omega \left(y' + \langle \nabla y, V \rangle + y\, div(V) \right) \omega \\ &= \int_\Omega y'\omega + \int_\Omega d(y\; i_V \omega) \\ &= \int_\Omega y'\omega. \end{aligned}$$

As in §2 let $\Gamma_1, \ldots, \Gamma_{2n}$ be the open segments of the boundary $\partial\Omega = \bigcup_{j=1}^{2n} \overline{\Gamma}_j$ and let n_j denote the outward unit normal to Ω on the side Γ_j. The sides Γ_j are numbered such that $\Gamma_1, \ldots, \Gamma_n$ are contained in the inner polygonal boundary and $\Gamma_{n+1}, \ldots, \Gamma_{2n}$ are contained in the outer polygonal boundary.

Proposition 3.9 — The shape derivative $y' \in H^\delta(\Omega)$ is a weak solution of the following Dirichlet Boundary Value Problem:

$$\begin{aligned} &\Delta v = 0 \text{ on } \Omega \\ &v_{|\Gamma_j} = -\frac{\partial v}{\partial n_j} \langle V, n_j \rangle \in H^{\delta - 1/2}(\Gamma_j), \forall j = 1. \ldots, 2n. \end{aligned} \tag{3.9.1}$$

Thus $y' \in E^\delta(\Delta, L^2(\Omega))$.

PROOF : Fix $\varphi \in \mathscr{D}(\Omega)$ arbitrarily. Choose a relatively compact open subset $\Omega' \subset \Omega$ having $\mathcal{C}^\infty$ boundary such that $supp(\varphi) \subset \Omega'$. Choose $a > 0$ such that $\overline{\Omega'} \subset \Omega_t,\ \forall\, |t| < a$. Then for each $|t| < a, \varphi \in \mathfrak{D}(\Omega_t)$ and we have

$$\begin{aligned} \int_{\Omega_t} \varphi\, dx &= -\int_{\Omega_t} (\Delta y_t) \varphi\, dx \qquad (\text{by } (1.0.1)) \\ &= -\int_{\Omega_t} y_t (\Delta \varphi)\, dx \qquad (\text{by Definition } 2.5). \end{aligned}$$

Hence

$$\int_{\Omega'} \varphi\, dx = \int_{\Omega'} y_t (\Delta \varphi)\, dx\,. \tag{3.9.2}$$

By Corollary 3.6 we can differentiate both sides of (3.9.2) with respect to t (at $t = 0$) to obtain

$$\int_{\Omega'} y'(\Delta\varphi)\, dx = 0.$$

Since $\varphi \in \mathscr{D}(\Omega)$ is arbitrary, $\Delta y' = 0$ in the sense of distributions.

Since $y' \in E^{\delta}(\Delta, L^2(\Omega))$, ${y'}_{|\Gamma_j} \in H^{\delta-1/2}, \forall j = 1, \ldots, 2n$ by Proposition 2.8. Further

$$\begin{aligned} {y'}_{|\Gamma_j} &= \dot{y}_{|\Gamma_j} - \langle \nabla y, V\rangle_{|\Gamma_j} \\ &= -\langle \nabla y, V\rangle_{|\Gamma_j} \qquad (\because \dot{y} \in E_0^{1+\delta}(\Delta, L^2(\Omega)) \subset H_0^1(\Omega) \text{ and by Remark 2.9}) \\ &= -\frac{\partial y}{\partial n_j}\langle V, n_j\rangle \qquad (\because y \in H_0^1(\Omega)). \end{aligned}$$ □

Now we derive what is commonly known as the 'Hadamard formula'.

***Theorem* 3.10 —** *Define $j(t) = J(\Omega_t)$. Then $j(t)$ is differentiable at $t = 0$ and its derivative is given by*

$$j'(0) = \sum_{j=1}^{n} \int_{\Gamma_j} \left(\frac{\partial y}{\partial n_j}\right)^2 \langle V, n\rangle\, ds\,.$$

PROOF : By Proposition 3.8, $j'(0) = \int_\Omega y'\, dx$. Since $-\Delta y = 1$ on Ω by (1.0.1) and $\Delta y' = 0$ on Ω by (3.9.1) we have

$$\begin{aligned} \int_\Omega y'\, dx &= \int_\Omega \big((\Delta y')y - (\Delta y)y'\big)\, dx \\ &= -\sum_{j=1}^{2n} \int_{\Gamma_j} {y'}_{|\Gamma_j} \frac{\partial y}{\partial n_j}\, ds \qquad (\text{ by Proposition } 2.11) \\ &= \sum_{j=1}^{n} \int_{\Gamma_j} \left(\frac{\partial y}{\partial n_j}\right)^2 \langle V, n_j\rangle\, ds \qquad (\text{ by Proposition } 3.9) \end{aligned}$$ □

4. PROOF OF THE MAIN THEOREM

Recall $\mathcal{F}_n, \mathfrak{p}_1, R$ and r from §1. Let $\mathfrak{p}_2$ be a regular polygon having n sides which has $(0, r) \in \mathbb{R}^2$ as one of its vertices, circumscribed on the circle $C(0, r)$. Let $\Omega_0 = (\mathfrak{p}_1 \setminus \mathfrak{p}_2)^\circ$ be fixed throughout this section.

Let V be the vector field defined in §3 and $\{\psi_t\}$ the 1-parameter group of diffeomorphisms of $\mathbb{R}^2$ associated with V. Let $\Omega_t = \psi_t(\Omega_0)$. Then $\mathcal{F}_n = \{\Omega_t \;\big|\; t \in \mathbb{R}\}$. Since $R_{2\pi/n}(\mathfrak{p}_2) = \mathfrak{p}_2$, $\psi_{2\pi/n}(\Omega) = \Omega, \forall\ \Omega \in \mathcal{F}_n$. Therefore $\mathcal{F}_n = \left\{\Omega_t \;\middle|\; t \in \left]-\frac{\pi}{n}, \frac{\pi}{n}\right]\right\}$.

Recall $J(\Omega_t) = \int_{\Omega_t} y_t \, dx$ from §3 where y_t is the unique solution of (1.0.1) with $\Omega = \Omega_t$. Define $j(t) = J(\Omega_t)$, for $t \in \mathbb{R}$.

We use the following notations:

1. For $p, q \in \mathbb{R}^2$, $[p, q] = \{p + t(q - p) \mid 0 \leq t \leq 1\}$ denotes the closed line segment joining p & q. Half-open or open segments are represented similarly.

2. $\Gamma_{1,t}, \ldots, \Gamma_{n,t}$ denote the open segments of the inner polygonal boundary of Ω_t as in §3 numbered counter-clockwise and $n_{j,t}$ denotes the outward unit normal to side $\Gamma_{j,t}$. The side $\Gamma_{1,0}$ of Ω_0 meets the side $\Gamma_{n,0}$ at vertex $(0, r) \in \mathbb{R}^2$ and $\Gamma_{j,t} = R_t(\Gamma_{j,0})$ for $j = 1, \ldots, n$. $\Gamma_{n+1}, \ldots, \Gamma_{2n}$ denote the open segments of the outer polygonal boundary of Ω which is the same for every $\Omega \in \mathcal{F}_n$.

Since $\psi_{s+t} = \psi_s \circ \psi_t$, $j(t)$ is differentiable as a function of t by Theorem 3.10 applied to $\Omega = \Omega_t$ and

$$j'(t) = \sum_{j=1}^{n} \int_{\Gamma_{j,t}} \left(\frac{\partial y_t}{\partial n_{j,t}} \right)^2 \langle V, n_{j,t} \rangle \, ds \, .$$

Since $R_{2\pi/n}$ is a symmetry of Ω_t

$$j'(t) = n \int_{\Gamma_{1,t}} \left(\frac{\partial y_t}{\partial n_{1,t}} \right)^2 \langle V, n_{1,t} \rangle \, ds \, . \tag{4.0.1}$$

Let m_t be the midpoint of $\Gamma_{1,t} = [b_t, c_t]$, where b_t and c_t denote the left and right endpoints of $\Gamma_{1,t}$ respectively. See Figure 1. Let σ_t denote the reflection map of $\mathbb{R}^2$ about the line passing through m_t and the origin 0.

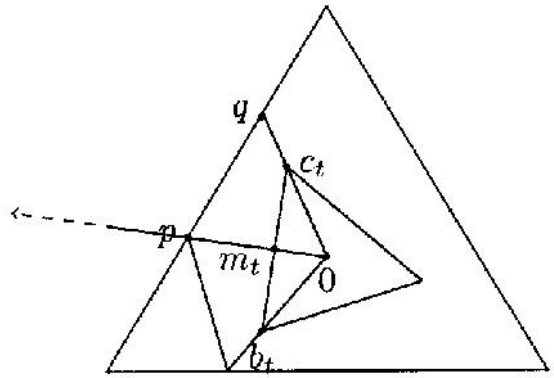


Figure 1

Since $n_{1,t}$ is constant along $\Gamma_{1,t}$ and $n_{1,t} = \dfrac{m_t}{\|m_t\|}$, σ_t fixes $n_{1,t}$. Therefore $x + \sigma_t(x)$ is a scalar multiple of $n_{1,t}$ for each $x \in \mathbb{R}^2$.

Let $\Gamma = [m_t, c_t[$. Then $\Gamma_{1,t} = \Gamma \cup \sigma_t(\Gamma)$. For any $x \in \Gamma$,

$$\begin{aligned}\langle V(x) + V(\sigma_t(x)), n_{1,t}\rangle &= \left\langle R_{\pi/2}(x) + R_{\pi/2}(\sigma_t(x)), n_{1,t}\right\rangle \\ &= \left\langle R_{\pi/2}(x + \sigma_t(x)), n_{1,t}\right\rangle \\ &= 0 \quad (\because x + \sigma_t(x) \in \mathbb{R}n_{1,t}).\end{aligned}$$

Hence

$$\langle V(x), n_{1,t}\rangle = -\langle V(\sigma_t(x)), n_{1,t}\rangle, \forall x \in \Gamma. \tag{4.0.2}$$

Therefore by (4.0.2)

$$j'(t) = n\left(\int_\Gamma \left(\frac{\partial y_t}{\partial n_{1,t}}\right)^2 \langle V, n_{1,t}\rangle\, ds - \int_\Gamma \left(\frac{\partial (y_t \circ \sigma_t)}{\partial n_{1,t}}\right)^2 \langle V, n_{1,t}\rangle\, ds\right). \tag{4.0.3}$$

Note that σ_t is a symmetry of Ω_t for $t = 0, \pm\dfrac{\pi}{n}$ and hence $y_t \circ \sigma_t = y_t, \forall t = 0, \pm\dfrac{\pi}{n}$. Therefore by (4.0.1) and (4.0.3), $j'(t)$ vanishes at $t = 0, \pm\dfrac{\pi}{n}$. Since $\sigma_0(\Omega_t) = \Omega_t$, $j(t)$ is an even function of t. Hence the result follows if we show that $j'(t) > 0, \forall t \in \left]0, \dfrac{\pi}{n}\right[$.

So let $t \in \left]0, \dfrac{\pi}{n}\right[$. We identify $z = (a, b) \in \mathbb{R}^2$ with $a + ib \in \mathbb{C}$. So $n_{1,t} = \dfrac{m_t}{\|m_t\|} = -e^{i(t+\pi/2+\pi/n)}$. For $z \in \Gamma$, $V(z) = iz$ and $Arg(e^{-it}z) \in \left]\frac{\pi}{2}, \frac{\pi}{2} + \dfrac{\pi}{n}\right[$. It follows that

$$\begin{aligned}Re\left(V(z)\overline{n_{1,t}}\right) &= -Re\left(ize^{-i(t+\pi/2+\pi/n)}\right) = -Re\left((e^{-i\pi/n})(e^{-it}z)\right) \\ &= -\cos\left(Arg(e^{-it}z) - \frac{\pi}{n}\right) < 0.\end{aligned}$$

Hence

$$\langle V(x), n_{1,t}\rangle < 0, \forall x \in \Gamma. \tag{4.0.4}$$

Let x^* denote $\sigma_t(x)$ for $x \in \mathbb{R}^2$. Using (4.0.4) to prove that $j'(t) > 0$ it remains to show that $\left|\dfrac{\partial y_t}{\partial n_{1,t}}(x^*)\right| > \left|\dfrac{\partial y_t}{\partial n_{1,t}}(x)\right|, \forall x \in \Gamma$.

By (1.0.1) , $\Delta(-y_t) = 1 > 0$ on Ω_t and $-y_t(x) = 0, \forall x \in \partial\Omega_t$. Hence by the Maximum Principle (Proposition 2.15), $y_t(x) > 0, \forall x \in \Omega_t$. Then Proposition 2.16 applies and we have

$$\frac{\partial y_t}{\partial n_{1,t}}(x) < 0, \ \forall x \in \Gamma. \tag{4.0.5}$$

Let $p, q \in \Gamma_{n+1}$ be the points where the lines extending $[0, m_t]$ and $[0, c_t]$ meet Γ_{n+1}. Consider the quadrilateral G having the vertices c_t, m_t, p, q. Then $\sigma_t(G) \subset \Omega_t$. We define $w(x) = y_t(x) - y_t(x^*)\ \forall\ x \in G$. Then

$$\begin{aligned} \Delta w &= 0, && \text{on } G, \\ w &< 0, && \text{on }]p, q[, \\ w &= 0, && \text{on } \Gamma. \end{aligned}$$

Hence by the Hopf Maximum Principle $\dfrac{\partial w}{\partial n_{1,t}}(x) > 0, \forall x \in \Gamma$. By (4.0.5), $\left|\dfrac{\partial y_t}{\partial n_{1,t}}(x^*)\right| > \left|\dfrac{\partial y_t}{\partial n_{1,t}}(x)\right|, \forall x \in \Gamma$. □

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