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SYMMETRY IDENTITIES OF *q*-BERNOULLI POLYNOMIALS OF THE SECOND KIND

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In this paper, we give identities of symmetry for the q-Bernoulli polynomials which are derived from the symmetric properties of the p-adic invariant integrals on $\mathbb{Z}_p$.

Key words : q-Bernoulli polynomial of the second kind; identity of symmetry.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to1}[x]_q = x$. As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1}$$

(see [1-14]).

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

By (1), we easily get

$$(B+1)^n - B_n = \delta_{1,n}, \quad B_0 = 1, \tag{2}$$

with the usual convention about replacing B^n by B_n (see [1-14]).

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differential functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{3}$$

For $f_1(x) = f(x+1)$, we have

$$I_0(f_1) = I_0(f) + f'(0), \tag{4}$$

where $f'(0) = \left.\dfrac{df(x)}{dx}\right|_{x=0}$ (see [7, 9]).

Let us take $f(x) = e^{tx}$. Then, by (4), we get

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_0(x) = \frac{t}{e^t+1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{5}$$

By (5), we get

$$\int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_0(y) = \frac{t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{6}$$

Thus, from (5) and (6), we have

$$\int_{\mathbb{Z}_p} (x+y)^n\, d\mu_0(y) = B_n(x), \quad \int_{\mathbb{Z}_p} x^n d\mu_0(x) = B_n, \tag{7}$$

where $n \geq 0$ (see [1-16]).

Let $f_n(x) = f(x+n)$, for $n \in \mathbb{N}$. Then we have

$$\int_{\mathbb{Z}_p} f_n(x)\, d\mu_0(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) + \sum_{l=0}^{n-1} f'(l). \tag{8}$$

The q-Bernoulli numbers of the second kind are defined by Kim as follows :

$$\int_{\mathbb{Z}_p} [x]_q^n\, d\mu_0(x) = \beta_{n,q} \quad \text{for } n \geq 0. \tag{9}$$

The q-Bernoulli polynomials of the second kind are also defined by Kim as follows :

$$\int_{\mathbb{Z}_p} [x+y]_q^n\, d\mu_0(y) = \beta_{n,q}(x), \quad n \geq 0, \tag{10}$$

(see [7]).

Thus, we note that

$$\beta_{0,q}=1,\quad (q\beta_q+1)^n-\beta_{n,q}=\frac{\log q}{q-1}\delta_{1,n},$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [7, 8, 9, 11]).

From (10), we note that

$$\begin{aligned}\beta_{n,q}(x) &= \int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_0(x) \qquad (11)\\ &= \sum_{l=0}^{n}\binom{n}{l}q^{lx}\int_{\mathbb{Z}_p}[y]_q^l\,d\mu_0(y)\,[x]_q^{n-l}\\ &= \sum_{l=0}^{n}\binom{n}{l}q^{lx}\,[x]_q^{n-l}\,\beta_{l,q}.\end{aligned}$$

In [4], several identities of symmetry for the Bernoulli polynomials are given by (5) and (6).

In this paper, we investigate several further interesting properties of symmetry for the p-adic invariant integral on $\mathbb{Z}_p$ and give identities of symmetry for the q-Bernoulli polynomials of the second kind.

2. Identities of Symmetry for q-Bernoulli Polynomials

By (10), we easily get

$$\beta_{n,q}(x)=\frac{\log q}{q-1}\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{l}{[l]_q}q^{lx}, \qquad (12)$$

where $n\geq 0$ (see [7-9]).

Let $w_1,\ w_2$ be natural numbers. Then we see that

$$\begin{aligned}&\frac{1}{w_1}\int_{\mathbb{Z}_p}e^{[w_1w_2x+w_2j+w_1y]_qt}d\mu_0(y) \qquad (13)\\ &=\lim_{N\to\infty}\frac{1}{w_1w_2p^N}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1(i+w_2y)]_qt}.\end{aligned}$$

From (13), we have

$$\begin{aligned}&\frac{1}{w_1}\sum_{j=0}^{w_1-1}\int_{\mathbb{Z}_p}e^{[w_1w_2x+w_2j+w_1y]_qt}d\mu_0(y) \qquad (14)\\ &=\lim_{N\to\infty}\frac{1}{w_1w_2p^N}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1(i+w_2y)]_qt}.\end{aligned}$$

By the same method as (14), we get

$$\frac{1}{w_2}\sum_{j=0}^{w_2-1}\int_{\mathbb{Z}_p} e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_0(y) \tag{15}$$
$$=\lim_{N\to\infty}\frac{1}{w_1w_2p^N}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1} e^{[w_1w_2x+w_1j+w_2(i+w_1y)]_q t}.$$

Therefore, by (14) and (15), we obtain the following theorem.

***Theorem* 1** — *Let* $w_1,\ w_2\in\mathbb{N}$. *Then we have*

$$\frac{1}{w_1}\sum_{j=0}^{w_1-1}\int_{\mathbb{Z}_p} e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_0(y)$$
$$=\frac{1}{w_2}\sum_{j=0}^{w_2-1}\int_{\mathbb{Z}_p} e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_0(y).$$

Corollary 2 — For $n\geq 0$, we have

$$\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_0(y)$$
$$=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\int_{\mathbb{Z}_p}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_0(y).$$

Therefore, by (10) and Corollary 2, we obtain the following theorem.

***Theorem* 3** — *For* $n\geq 0$, *we have*

$$\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\beta_{n,q^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)$$
$$=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\beta_{n,q^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).$$

From (10), we can derive the following equation :

$$\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_0(y) \tag{16}$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i\, q^{w_2(n-i)j}\int_{\mathbb{Z}_p}[w_2x+y]_{q^{w_1}}^{n-i}d\mu_0(y)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i\, q^{w_2(n-i)j}\beta_{n-i,\,q^{w_1}}(w_2x).$$

Thus, by (16), we get

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_0(y) \\ =&\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^{n-i}}{w_1}[w_2]_q^i\sum_{j=0}^{w_1-1}[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,q^{w_1}}(w_2x) \\ =&\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^{n-i}}{w_1}[w_2]_q^i T_{n,i}(w_1|q^{w_2})\beta_{n-i,q^{w_1}}(w_2x),\end{aligned} \tag{17}$$

where

$$T_{n,i}(w|q)=\sum_{j=0}^{w-1}q^{(n-i)j}[j]_q^i$$

(see [11]).

By the same method as (17), we get

$$\begin{aligned}&\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\int_{\mathbb{Z}_p}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_0(y) \\ =&\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^{n-i}}{w_2}[w_1]_q^i T_{n,i}(w_2|q^{w_1})\beta_{n-i,q^{w_2}}(w_1x).\end{aligned} \tag{18}$$

Therefore, by (17) and (18), we obtain the following theorem.

***Theorem* 4** — *For $n \geq 0$, we have*

$$\begin{aligned}&\frac{1}{w_1}\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{n-i}[w_2]_q^i T_{n,i}(w_1|q^{w_2})\beta_{n-i,q^{w_1}}(w_2x) \\ =&\frac{1}{w_2}\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^{n-i}[w_1]_q^i T_{n,i}(w_2|q^{w_1})\beta_{n-i,q^{w_2}}(w_1x).\end{aligned}$$

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