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COMPOSITION OPERATORS ON HARDY-SOBOLEV SPACES

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In this paper, we obtain some growth estimates for Hardy-Sobolev functions of the unit ball. We also give the representation of the spectra of composition operators C_φ on $H^2_\beta(\mathbb{B}_n)$ and describe the Fredholmness of C_φ equivalently.

Key words : Hardy-Sobolev space; composition operator; spectrum; Fredholm operator.

1. Introduction

For any points $z = (z_1, z_2, ..., z_n)$ and $w = (w_1, w_2, ..., w_n)$ in $\mathbb{C}^n$, we denote

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + ... + z_n\overline{w_n}$$

and

$$|z|^2 = \langle z, z\rangle = |z_1|^2 + |z_2|^2 + ... + |z_n|^2.$$

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, and $\mathbb{S}_n = \partial\mathbb{B}_n$ its boundary. The normalized Lebesgue measure on $\mathbb{B}_n$ and $\mathbb{S}_n$ will be denoted as dv and $d\sigma$ respectively. Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions in $\mathbb{B}_n$.

For any $\beta \in \mathbf{R}$, we consider the special fractional radial differential operator $\mathcal{R}^\beta$ defined on $H(\mathbb{B}_n)$ as

$$\mathcal{R}^\beta f(z) = \sum_{k=0}^{\infty}(1 + k)^\beta f_k(z),$$

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where $f(z)=\sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f.

Given $0<r<1$ and $0<p<+\infty$, we define

$$H_\beta^p(\mathbb{B}_n)=\{f\in H(\mathbb{B}_n)\mid \mathcal{R}^\beta f\in H^p\}$$

and the norm of f in $H_\beta^p(\mathbb{B}_n)$ is

$$\|f\|_{H_\beta^p}=\|\mathcal{R}^\beta f\|_{H^p}=\sup_{0<r<1}\left\{\int_{\mathbb{S}_n}|\mathcal{R}^\beta f(r\zeta)|^p d\sigma(\zeta)\right\}^{\frac{1}{p}},$$

then $(H_\beta^p(\mathbb{B}_n),\|\cdot\|_{H_\beta^p})$ is a Banach space called as Hardy-Sobolev space.

For $p=\infty$, we define the Hardy-Sobolev space H_β^∞ as

$$H_\beta^\infty=\{f\mid \mathcal{R}^\beta f\in H^\infty\}.$$

Let φ be a self-map of $\mathbb{B}_n$, the composition operator C_φ for $f\in H_\beta^p(\mathbb{B}_n)$ is defined by $C_\varphi f=f\circ\varphi$.

For any point $z\in\mathbb{B}_n$, $\alpha\in\mathbf{N}^n$, by direct calculation, we have

$$\begin{aligned}
\|z^\alpha\|_{H_\beta^2} &= \sup_{0<r<1}\left\{\int_{\mathbb{S}_n}|\mathcal{R}^\beta(z^\alpha)(r\zeta)|^2 d\sigma(\zeta)\right\}^{\frac{1}{2}}\\
&= \sup_{0<r<1}\left\{\int_{\mathbb{S}_n}|(1+|\alpha|)^\beta(r\zeta)^\alpha|^2 d\sigma(\zeta)\right\}^{\frac{1}{2}}\\
&= (1+|\alpha|)^\beta\left\{\int_{\mathbb{S}_n}|\zeta^\alpha|^2 d\sigma(\zeta)\right\}^{\frac{1}{2}}\\
&= (1+|\alpha|)^\beta M_\alpha^{\frac{1}{2}}
\end{aligned}$$

where

$$M_\alpha=\int_{\mathbb{S}_n}|\zeta^\alpha|^2 d\sigma(\zeta)=\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

Set $e_\alpha(z)=\dfrac{z^\alpha}{\|z^\alpha\|_{H_\beta^2}}$, then $\{e_\alpha\}_{\alpha\in\mathbf{N}^n}$ is the orthogonal base of $H_\beta^2(\mathbb{B}_n)$. For any points $z\in\mathbb{B}_n$ and $w\in\mathbb{B}_n$, it is easy for us to get the reproducing kernel of $H_\beta^2(\mathbb{B}_n)$ is

$$K_z(w)=\sum_{\alpha\in\mathbf{N}^n} e_\alpha(z)\overline{e_\alpha(w)}=\sum_{\alpha\in\mathbf{N}^n}\frac{z^\alpha\overline{w}^\alpha}{(1+|\alpha|)^{2\beta}M_\alpha},$$

and

$$\mathcal{R}^\beta K_z(w)=\mathcal{R}^\beta\Big[\sum_{\alpha\in\mathbf{N}^n}\frac{z^\alpha\overline{w}^\alpha}{(1+|\alpha|)^{2\beta}M_\alpha}\Big]=\sum_{\alpha\in\mathbf{N}^n}\frac{z^\alpha\overline{w}^\alpha}{(1+|\alpha|)^{\beta}M_\alpha}.$$

Therefore,

$$
\begin{aligned}
\|K_z\|_{H^2_\beta} &= \sup_{0<r<1}\left\{\int_{\mathbb{S}_n} |\mathcal{R}^\beta K_z(rw)|^2 d\sigma(w)\right\}^{\frac{1}{2}} \\
&= \sup_{0<r<1}\left\{\int_{\mathbb{S}_n} |\sum_{\alpha\in\mathbf{N}^n} \frac{z^\alpha \overline{rw}^\alpha}{(1+|\alpha|)^\beta M_\alpha}|^2 d\sigma(w)\right\}^{\frac{1}{2}} \\
&= \left\{\int_{\mathbb{S}_n} |\sum_{\alpha\in\mathbf{N}^n} \frac{z^\alpha \overline{w}^\alpha}{(1+|\alpha|)^\beta M_\alpha}|^2 d\sigma(w)\right\}^{\frac{1}{2}} \\
&= \left\{\int_{\mathbb{S}_n} \sum_{\alpha\in\mathbf{N}^n} \frac{|z^\alpha|^2 |w^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha} d\sigma(w)\right\}^{\frac{1}{2}} \\
&= \left\{\sum_{\alpha\in\mathbf{N}^n} \frac{|z^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha^2} \int_{\mathbb{S}_n} |w^\alpha|^2 d\sigma(w)\right\}^{\frac{1}{2}} \\
&= \left\{\sum_{\alpha\in\mathbf{N}^n} \frac{|z^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

For any point $a \in \mathbb{B}_n$, we define the Möbius transform on $\mathbb{B}_n$ as

$$
\varphi_a(z) = \begin{cases} -z, & a = 0; \\ \\ \dfrac{a - P_a(z) - S_a Q_a(z)}{1 - \langle z, a\rangle}, & a \neq 0, \end{cases}
$$

where $S_a = \sqrt{1-|a|^2}$, $P_a(z) = \dfrac{\langle z,a\rangle}{|a|^2} a$ and $Q_a = I - P_a$. As we known, φ_a is an automorphism of $\mathbb{B}_n$ and has the following properties:

$$
\varphi_a(0) = a\ ,\ \varphi_a(a) = 0\ ,
$$

$$
\varphi_a \circ \varphi_a(z) = z\ ,\ 1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}\ .
$$

As general analytic function spaces, the Hardy-Sobolev spaces have very complex constructions and the classical methods on classical spaces such as Hardy spaces or Bergman spaces can not be applied to them directly. Naturally, to characterize the operators on these spaces are even more difficult. In recent years, there are a series of papers investigate these spaces. For instance, Tchoundja [1] gives a characterization of Carleson measures for some Hardy-Sobolev spaces by means of Green's formula. Cho and Zhu [2] show the relation and difference between Hardy-Sobolev spaces and holomorphic mean Lipschitz spaces in terms of fractional radial differential operators and integral operators. Ahern and Cohn [3] equivalently characterize the exceptional sets for Hardy-Sobolev spaces by the sets of Hausdorff capacity zero and the sets of Bessel capacity zero. The purpose of this paper is to discuss more unknown properties about Hardy-Sobolev spaces and the composition operators on them. Definitely speaking, we mainly answer the following three questions:

1. How fast does a H^p_β-function grow near the boundary?

2. Is there any way to calculate the spectra of composition operators on H^2_β?

3. When does C_φ is Fredbolm?

In section 2, we solve the question 1 and obtain some growth estimates for $H^p_\beta(\mathbb{B}_n)$-functions. In section 3, we give a definite answer for the question 2, that is, we will show the representation of the spectra of composition operators C_φ on $H^2_\beta(\mathbb{B}_n)$. In section 4, we describe the Fredholmness of C_φ equivalently.

2. Some Estimates About H^p_β-Functions, $(0 < p < \infty)$

To obtain a growth estimate for H^p_β-functions, we need the following two lemmas.

Lemma 2.1 — ([1]) Suppose β and t be two positive real numbers such that $t - \beta > 0$, let $f(z) = \dfrac{1}{(1 - \langle z, w\rangle)^t}$. Then the following conclusion holds:

(1) There exists a positive constant C such that

$$|\mathcal{R}^\beta f(z)| \le \frac{C}{|1 - \langle z, w\rangle|^{t+\beta}}$$

and

$$|\mathcal{R}^{-\beta} f(z)| \le \frac{C}{|1 - \langle z, w\rangle|^{t-\beta}}$$

for all $w \in \mathbb{B}_n$.

Lemma 2.2 — ([4]) Suppose $0 < p < \infty$, $f \in H^p(\mathbb{B}_n)$, then

$$|f(z)| \le 2^{n/p}\|f\|_p(1 - |z|)^{-n/p}$$

for any $z \in \mathbb{B}_n$.

For convenience, we simply denote $f \lesssim g$ if there exists a positive constant C which doesn't depend on the essential parameters such that $f \le Cg$. The following theorem tells us how fast an arbitrary function from H^p_β grows near the boundary.

***Theorem* 2.3** — *For $0 < p < \infty$ and $0 < \beta < n/p$, then for any $f \in H^p_\beta$, we have*

$$|\mathcal{R}^\beta f(z)| \lesssim \frac{\|f\|_{H^p_\beta}}{(1 - |z|)^{n/p}}$$

and

$$|f(z)| \lesssim \frac{\|f\|_{H^p_\beta}}{(1-|z|)^{n/p-\beta}}.$$

PROOF : For any $f \in H^p_\beta$, we know that $\mathcal{R}^\beta f \in H^p$ and $\|\mathcal{R}^\beta f\|_p = \|f\|_{H^p_\beta}$, then

$$|\mathcal{R}^\beta f(z)| \lesssim \frac{\|\mathcal{R}^\beta f\|_p}{(1-|z|)^{n/p}} = \frac{\|f\|_{H^p_\beta}}{(1-|z|)^{n/p}}$$

because of the conclusion in Lemma 2.2. Consequently, it follows

$$|f(z)| = |\mathcal{R}^{-\beta}\mathcal{R}^\beta f(z)| \lesssim \|f\|_{H^p_\beta} |\mathcal{R}^{-\beta}\frac{1}{(1-|z|)^{n/p}}| \lesssim \frac{\|f\|_{H^p_\beta}}{(1-|z|)^{n/p-\beta}}$$

from the fact given in Lemma 2.1 as $0 < \beta < n/p$. The proof is completed. □

3. SPECTRA OF COMPOSITION OPERATORS ON $H^2_\beta(\mathbb{B}_n)$

From the definition introduced in section 1, we know that

$$H^2_\beta(\mathbb{B}_n) = \{f \in H(\mathbb{B}_n) \mid \mathcal{R}^\beta f \in H^2\}.$$

Obviously, $H^2_\beta(\mathbb{B}_n)$ is a reflexive Hilbert space and the inner product in it is induced by

$$\langle f, g\rangle_{H^2_\beta} = \int_{\mathbb{S}_n} \mathcal{R}^\beta f \overline{\mathcal{R}^\beta g} d\sigma, \ \ \forall f \in H^2_\beta(\mathbb{B}_n), \ g \in H^2_\beta(\mathbb{B}_n).$$

From now on, the inner product in $H^2_\beta(\mathbb{B}_n)$ is simply denoted as $\langle \cdot, \cdot \rangle$.

In this section, we are to compute the spectra of composition operators on $H^2_\beta(\mathbb{B}_n)$. At the beginning, we give an equivalent characterization of the composition operator on $H^2_\beta(\mathbb{B}_n)$.

***Theorem* 3.1** — *Suppose T is a bounded linear operator on $H^2_\beta(\mathbb{B}_n)$. Then T is a composition operator on $H^2_\beta(\mathbb{B}_n)$ if and only if T^*K_z is a reproducing kernel for any $z \in \mathbb{B}_n$, where K_z is the reproducing kernel of $H^2_\beta(\mathbb{B}_n)$.*

PROOF : If T is a composition operator on $H^2_\beta(\mathbb{B}_n)$, then there exists a holomorphic map φ mapping $\mathbb{B}_n$ into itself such that $T = C_\varphi$. For any $f \in H^2_\beta(\mathbb{B}_n)$, $z \in \mathbb{B}_n$, we have

$$\langle f, C^*_\varphi K_z\rangle = \langle C_\varphi f, K_z\rangle = C_\varphi f(z) = f \circ \varphi(z) = \langle f, K_{\varphi(z)}\rangle,$$

which implies that $C^*_\varphi K_z = K_{\varphi(z)}$.

Conversely, if T^*K_z is a reproducing kernel, then for any $f \in H^2_\beta(\mathbb{B}_n)$, we have

$$Tf(z) = \langle Tf, K_z\rangle = \langle f, T^*K_z\rangle.$$

Since T^*K_z is a reproducing kernel we know that there is a map ϕ mapping $\mathbb{B}_n$ into itself such that $T^*K_z = K_{\phi(z)}$, then

$$Tf(z) = \langle f, K_{\phi(z)}\rangle = f(\phi(z)).$$

Let $f_j(z) = z_j$, then

$$Tf_j(z) = \langle f_j, T^*K_{\phi(z)}\rangle = f_j(\phi(z)) = \phi_j(z),$$

therefore, ϕ_j is a holomorphic function and $T = C_\phi$. This completes the proof. □

Now, we turn to calculate the spectra of C_φ. Firstly, we decompose the $H^2_\beta(\mathbb{B}_n)$. Suppose $e_\alpha = \frac{z^\alpha}{\|z^\alpha\|_{H^2_\beta}}$, then $\{e_\alpha\}_{\alpha\in\mathbf{N}^n}$ is the orthogonal base of $H^2_\beta(\mathbb{B}_n)$. We arrange the $\{e_\alpha\}_{|\alpha|=k}$ in dictionary order. Set

$$H_k = \bigvee_{\alpha\in\mathbf{N}^n, |\alpha|=k} \{z^\alpha\}, \quad k = 0, 1, 2, ...,$$

then $\{e_\alpha\}_{|\alpha|=k}$ is the orthogonal base of H_k whose dimension is

$$\dim H_k = C^k_{n+k-1} = \frac{(n+k-1)(n+k-2)\cdot ...\cdot n}{k!},$$

and $H^2_\beta(\mathbb{B}_n) = \bigoplus_{k=0}^\infty H_k$.

Take T is a bounded linear operator on $H^2_\beta(\mathbb{B}_n)$ and let

$$S_k(T) = (\langle Te_\alpha, e_\gamma\rangle)_{|\alpha|=|\gamma|=k}.$$

If $T = C_\varphi$, then we denote $S_k(\varphi) = S_k(C_\varphi)$. Secondly, we still need the following proposition.

Proposition 3.2 — Suppose φ is a holomorphic map mapping from $\mathbb{B}_n$ into itself which satisfies $\varphi(0) = 0$. Then according to the above decomposition of $H^2_\beta(\mathbb{B}_n) = \bigoplus_{k=0}^\infty H_k$, C_φ has the following form

$$C_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & ... \\ * & S_1(\varphi) & 0 & 0 & ... \\ * & * & S_2(\varphi) & 0 & ... \\ * & * & * & S_3(\varphi) & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

relative to the base $\{e_\alpha\}_{\alpha=(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbf{N}^n}$, where $S_k(\varphi)$ is a $C^k_{n+k-1}\times C^k_{n+k-1}$-matrix whose entries satisfying

$$\langle C_\varphi e_\alpha, e_\gamma\rangle = \sum_{\alpha=(\alpha_i^j),\sum_{j=1}^n|\alpha_i^j|=\gamma_i,\sum_{i=1}^n|\alpha_i^j|=\alpha_j}\prod_{i,j=1}^n(\frac{\partial\varphi_j}{\partial z_i})^{\alpha_i^j}.$$

PROOF : Set $\varphi=\varphi_1\varphi_2...\varphi_n$. Since $\varphi(0)=0$, then φ_j can be written as

$$\varphi_j(z)=\sum_{i=1}^n\frac{\partial\varphi_j}{\partial z_i}(0)z_i+\psi_j(z)\quad(j=1,2,...,n),$$

where each term of the Fourier series of $\psi_j(z)$ has at least degree two. Therefore, it is not difficult to find that

$$\langle C_\varphi z^\alpha, z^\gamma\rangle=\langle\varphi^\alpha,z^\gamma\rangle=0$$

for any $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbf{N}^n$ and $\gamma=(\gamma_1,\gamma_2,...,\gamma_n)\in\mathbf{N}^n$ with $|\alpha|>|\gamma|$. If $|\alpha|=|\gamma|$, then we have

$$\begin{aligned}
&\langle C_\varphi z^\alpha, z^\gamma\rangle=\langle\varphi^\alpha,z^\gamma\rangle=\langle\varphi_1\varphi_2...\varphi_n,z^\gamma\rangle\\
=\;&\Big\langle[\sum_{i=1}^n\frac{\partial\varphi_1}{\partial z_i}(0)z_i+\psi_1(z)]^{\alpha_1}...[\sum_{i=1}^n\frac{\partial\varphi_n}{\partial z_i}(0)z_i+\psi_n(z)]^{\alpha_n},z^\gamma\Big\rangle\\
=\;&\langle\prod_{j=1}^n[\sum_{i=1}^n\frac{\partial\varphi_j}{\partial z_i}(0)z_i+\psi_j(z)]^{\alpha_j},z^\gamma\rangle\\
=\;&\langle\prod_{j=1}^n\sum_{k=0}^{\alpha_j}C^k_{\alpha_j}[\sum_{i=1}^n\frac{\partial\varphi_j}{\partial z_i}(0)z_i]^{\alpha_j-k}[\psi_j(z)]^k,z^\gamma\rangle\\
=\;&\langle\prod_{j=1}^n[\sum_{i=1}^n\frac{\partial\varphi_j}{\partial z_i}(0)z_i]^{\alpha_j},z^\gamma\rangle\\
=\;&\langle\prod_{j=1}^n\{\sum_{\sum_{i=1}^n|\alpha_i^j|=\alpha_j}[\frac{\partial\varphi_j}{\partial z_i}(0)z_i]^{\alpha_i^j}\},z^\gamma\rangle\\
=\;&\langle\sum_{\alpha=(\alpha_i^j)_{i,j=1}^n,\sum_{j=1}^n|\alpha_i^j|=\gamma_i,\sum_{i=1}^n|\alpha_i^j|=\alpha_j}\{\prod_{i,j=1}^n[\frac{\partial\varphi_j}{\partial z_i}(0)]^{\alpha_i^j}\}z^\gamma,z^\gamma\rangle\\
=\;&\sum_{\alpha=(\alpha_i^j)_{i,j=1}^n,\sum_{j=1}^n|\alpha_i^j|=\gamma_i,\sum_{i=1}^n|\alpha_i^j|=\alpha_j}\prod_{i,j=1}^n[\frac{\partial\varphi_j}{\partial z_i}(0)]^{\alpha_i^j}\|z^\gamma\|^2_{H^2_\beta}.
\end{aligned}$$

Thus

$$\langle C_\varphi e_\alpha, e_\gamma\rangle = \sum_{\alpha=(\alpha_i^j),\sum_{j=1}^n|\alpha_i^j|=\gamma_i,\sum_{i=1}^n|\alpha_i^j|=\alpha_j}\prod_{i,j=1}^n(\frac{\partial\varphi_j}{\partial z_i})^{\alpha_i^j},$$

which implies that

$$C_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ * & S_1(\varphi) & 0 & 0 & \dots \\ * & * & S_2(\varphi) & 0 & \dots \\ * & * & * & S_3(\varphi) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

under the base $\{e_\alpha\}_{\alpha\in N^n}$ and this finishes the proof. $\square$

Now, we give the main results of this section.

***Theorem* 3.3** — *Suppose φ is a holomorphic map mapping from $\mathbb{B}_n$ into itself which satisfies $\varphi(0) = 0$. Then*

$$\sigma_p(C_\varphi) = \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi)).$$

PROOF : It is easy to see that $\bigcup_{k=0}^{\infty} \sigma(S_k(\varphi)) \subseteq \sigma_p(C_\varphi)$ from the Proposition 3.2. To complete the proof, we just need to show that $\sigma_p(C_\varphi) \subseteq \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi))$. Suppose $\lambda \in \sigma_p(C_\varphi)$, then there is a non-zero function $f \in H^2_\beta(\mathbb{B}_n)$ such that $(C_\varphi - \lambda)f = 0$. According to the decomposition of $H^2_\beta(\mathbb{B}_n)$, we can find $f_j \in H_j (j = 0, 1, ...)$ such that $f = \bigoplus_{j=0}^{\infty} f_j$. Let $l = \min\{j | f_j \neq 0\}$, then $f = \bigoplus_{j=l}^{\infty} f_j$ satisfying $f_l \neq 0$. Consequently, by Proposition 3.2, we have

$$(C_\varphi - \lambda)f = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & \dots \\ * & S_1(\varphi)-\lambda & 0 & 0 & \dots \\ * & * & S_2(\varphi)-\lambda & 0 & \dots \\ * & * & * & S_3(\varphi)-\lambda & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} 0 \\ \dots \\ 0 \\ f_{l_k} \\ \dots \end{pmatrix} = 0,$$

and $(S_{l_k}(\varphi) - \lambda) f_{l_k} = 0$, i.e. $\lambda \in \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi))$. Therefore, $\sigma_p(C_\varphi) \subseteq \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi))$ and the proof of this theorem is completed. $\square$

***Theorem* 3.4** — *Suppose φ is a holomorphic map mapping from $\mathbb{B}_n$ into itself which satisfies $\varphi(0) = 0$. If C_φ is a compact operator on $H^2_\beta(\mathbb{B}_n)$, then*

$$\sigma(C_\varphi) = \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi)) \bigcup \{0, 1\}.$$

PROOF : It is easy to get that $\bigcup_{k=0}^{\infty} \sigma(S_k(\varphi)) \bigcup \{0, 1\} \subseteq \sigma(C_\varphi)$ from the conclusion in Theorem 3.3. On the other hand, if $\lambda \in \sigma(C_\varphi)$, without loss of generality, we assume that $\lambda \neq 0, 1$. Since C_φ

is compact, then $\lambda \in \sigma_p(C_\varphi)$. By using the Theorem 3.3 again, we obtain that $\lambda \in \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi))$. The proof is completed. □

Corollary 3.5 — Suppose φ is a holomorphic map mapping from $\mathbb{B}_n$ into itself and $z_0 \in \mathbb{B}_n$ is a fixed point of φ. If C_φ is a compact operator on $H^2_\beta(\mathbb{B}_n)$, then

$$\sigma(C_\varphi) = \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi_{z_0} \circ \varphi \circ \varphi_{z_0})) \bigcup \{0, 1\},$$

where φ_{z_0} is the M$\ddot{o}$bius transform of $\mathbb{B}_n$ defined in section 1.

PROOF : Since z_0 is a fixed point of φ, we have $\varphi(z_0) = z_0$, then

$$\varphi_{z_0} \circ \varphi \circ \varphi_{z_0}(0) = \varphi_{z_0} \circ \varphi(z_0) = \varphi_{z_0}(z_0) = 0.$$

By Theorem 3.4, we know that

$$\sigma(C_{\varphi_{z_0} \circ \varphi \circ \varphi_{z_0}}) = \bigcup_{k=0}^{\infty} \sigma(S_k(\varphi_{z_0} \circ \varphi \circ \varphi_{z_0})) \bigcup \{0, 1\}.$$

Further, φ_{z_0} is the M$\ddot{o}$bius transform of $\mathbb{B}_n$, then $\sigma(C_\varphi) = \sigma(C_{\varphi_{z_0} \circ \varphi \circ \varphi_{z_0}})$ and this proves the corollary. □

Example : If C_φ is a compact operator on $H^2_\beta(\mathbb{B}_n)$ satisfying $\varphi(0) = 0$ and $\frac{\partial \varphi_j}{\partial z_i}(0) = 0 \ (i \neq j)$, then according to the Proposition 3.2, we know that $S_k(\varphi)$ is a diagonal matrix which has diagonal entries

$$\{\prod_{i,j=1}^{n} (\frac{\partial \varphi_j}{\partial z_i}(0))^{\alpha_j}\}_{\sum \alpha_j = k}.$$

In this case, by Theorem 3.4, it is not difficult for us to obtain that

$$\sigma(C_\varphi) = \{(\frac{\partial \varphi_1}{\partial z_1}(0))^{\alpha_1} (\frac{\partial \varphi_2}{\partial z_2}(0))^{\alpha_2} ... (\frac{\partial \varphi_n}{\partial z_n}(0))^{\alpha_n} | \alpha_j \in \mathbf{Z}^+, j = 1, 2, ..., n\} \bigcup \{0, 1\}.$$

In fact, if the Jacobi matrix of φ at 0 $J_C\varphi(0)$ is lower triangular, we can see easily that all $S_k(\varphi)$ are lower triangular and C_φ has a lower triangular form, moreover, the diagonal entries of $S_k(\varphi)$ consist of all the possible k-times products of the diagonal entries of $J_C\varphi(0)$ in this case. Thus

$$\sigma(S_k(\varphi)) = \{\lambda^\alpha | \lambda = (\lambda_1 \lambda_2 ... \lambda_n), \alpha \in \mathbf{N}^n, \sum \alpha_j = k\},$$

where λ_i is an eigenvalue of $J_C\varphi(0)$ for $i = 1, 2, ..., n$. This implies that $\sigma_p(C_\varphi)$ consist of all the possible finite products of the eigenvalues of $J_C\varphi(0)$. For general cases, if $J_C\varphi(0)$ is not lower triangular, then we can transform it to be lower triangular by some unitary transformation, that is, we can find a unitary matrix U such that $UJ_C\varphi(0)U^{-1}$ is a lower triangular matrix and set $\psi = U \circ \varphi \circ U^{-1}$, we can obtain a similar conclusion holds in this case. Further, if there exists a point $z_0 \in \mathbb{B}_n$ such that z_0 is a fixed point of φ, then we can see that $\psi = \varphi_{z_0} \circ \varphi \circ \varphi_{z_0}$ and $\sigma(C_\psi) = \sigma(C_\varphi)$. Thus we just need to discuss the spectrum of C_ψ which satisfies $\psi(0) = 0$. Obviously, a similar conclusion holds, too.

At last, we will show an important property of the compact composition operators on $H^2_\beta(\mathbb{B}_n)$, which plays a significant role in calculating the spectra of compact composition operators.

***Theorem* 3.6** — *Suppose φ is a holomorphic map mapping from $\mathbb{B}_n$ into itself. If C_φ is a compact operator on $H^2_\beta(\mathbb{B}_n)$, then φ has a fixed point in $\mathbb{B}_n$.*

PROOF : Suppose $\{\lambda_k\}$ is an increasing positive sequence and $\lambda_k \to 1$ as $k \to \infty$. For any $w \in \mathbb{B}_n$, write $\varphi_k(w) = \varphi(\lambda_k w)$, then $\{\varphi_k\}$ be elements of the ball algebra on $\mathbb{B}_n$. By the Brower fixed point theorem, φ_k has fixed points in $\overline{\mathbb{B}_n}$. We may assume that w_k is a relative fixed point of $\varphi_k (k = 1, 2, ...)$, that is $\varphi_k(w_k) = w_k \in \overline{\mathbb{B}_n}$. Without loss of generality, we suppose that $w_k \to a$ as $k \to \infty$. Therefore, $\varphi(a) = a$. To complete the proof, we only need to show that $|a| < 1$. Otherwise, if $|a| = 1$, then $\lambda_k w_k \to a \in \partial\mathbb{B}_n$. Since

$$\frac{K_{\lambda_k w_k}}{\|K_{\lambda_k w_k}\|_{H^2_\beta}} \xrightarrow{w} 0$$

in $H^2_\beta(\mathbb{B}_n)$ and C^*_φ is compact on $H^2_\beta(\mathbb{B}_n)$, we have

$$\frac{\|C^*_\varphi K_{\lambda_k w_k}\|_{H^2_\beta}}{\|K_{\lambda_k w_k}\|_{H^2_\beta}} \to 0.$$

Note

$$\|C^*_\varphi K_{\lambda_k w_k}\|_{H^2_\beta} = \|K_{\varphi(\lambda_k w_k)}\|_{H^2_\beta}$$

and

$$\frac{\|K_{\varphi(\lambda_k w_k)}\|_{H^2_\beta}}{\|K_{\lambda_k w_k}\|_{H^2_\beta}} = \frac{\{\sum_{\alpha \in \mathbf{N}^n} \frac{|[\varphi(\lambda_k w_k)]^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}\}^{\frac{1}{2}}}{\{\sum_{\alpha \in \mathbf{N}^n} \frac{|(\lambda_k w_k)^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}\}^{\frac{1}{2}}} = \{\frac{\sum_{\alpha \in \mathbf{N}^n} \frac{|w_k^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}}{\sum_{\alpha \in \mathbf{N}^n} \frac{|(\lambda_k w_k)^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}}\}^{\frac{1}{2}},$$

we get

$$\frac{\sum_{\alpha \in \mathbf{N}^n} \frac{|w_k^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}}{\sum_{\alpha \in \mathbf{N}^n} \frac{|(\lambda_k w_k)^\alpha|^2}{(1+|\alpha|)^{2\beta} M_\alpha}} \to 0$$

as $k \to \infty$, which implies that $|w_k| < |\lambda_k w_k|$ as $k \to \infty$. However, $0 < \lambda_k < 1$, which is a contradiction. Therefore, a is the fixed point of φ in $\mathbb{B}_n$. $\square$

Combining this conclusion with corollary 3.5, we can compute the spectrum of any compact composition operator on $H^2_\beta(\mathbb{B}_n)$.

4. Fredholmness of composition operators on $H^2_\beta(\mathbb{D})$

In this section, we are going to investigate some other properties of composition operators on $H^2_\beta(\mathbb{B}_n)$ as $n = 1$. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, and $\mathbb{T} = \partial\mathbb{D}$ its boundary. From section 1, we know that the reproducing kernel on $H^2_\beta(\mathbb{D})$ is

$$K_z(w) = \sum_{m=0}^{\infty} \frac{z^m \overline{w}^m}{(1+m)^{2\beta}}$$

and

$$\|K_z\|_{H^2_\beta} = \{\sum_{m=0}^{\infty} \frac{|z|^{2m}}{(1+m)^{2\beta}}\}^{\frac{1}{2}}$$

which has the following properties:

Proposition 4.1 — For any $z \in \mathbb{D}$, the following proposition are true:

1. $\|K_z\|_{H^2_\beta} \geq 1$;

2. $\|K_z\|_{H^2_\beta}$ is an increasing function for $|z|$;

3. If $\beta \leq \frac{1}{2}$, then for any point $z \in \mathbb{D}$, we have $\|K_z\|_{H^2_\beta}$ is uniformly convergent on any compact subset of $\mathbb{D}$ and $\lim\limits_{|z|\to 1} \|K_z\|_{H^2_\beta} = \infty$. If $\beta > \frac{1}{2}$, then for any point $z \in \overline{\mathbb{D}}$, we have $\|K_z\|_{H^2_\beta} < \infty$.

The following conclusion gives some equivalent descriptions about the Fredholmness of composition operators on $H^2_\beta(\mathbb{D})$.

***Theorem* 4.2** — *Suppose $\beta \leq \frac{1}{2}$ and φ is a holomorphic map mapping from $\mathbb{D}$ into itself. If C_φ is the composition operator on $H^2_\beta(\mathbb{D})$. Then the following are equivalent:*

(1) φ is a disk automorphism;

(2) C_φ is invertible;

(3) C_φ is a Fredholm operator.

PROOF : Clearly, $(1) \Rightarrow (2) \Rightarrow (3)$, we just need to prove that $(3) \Rightarrow (1)$. Suppose that (3) holds, we notice that φ is not a constant function. Otherwise, the co-dimension of C_φ must be infinite from the fact that the closure of the range of C_φ consist of just the constant functions, which is contradicted with the Fredholmness of C_φ. Now, we claim that φ is a bijection on $\mathbb{D}$.

Firstly, φ is one to one. Assume that φ is not one to one on $\mathbb{D}$, then there exist points ζ and ζ' in $\mathbb{D}$ with $\zeta \neq \zeta'$ such that $\varphi(\zeta) = \varphi(\zeta')$. Let V and V' be open neighborhoods of ζ and ζ' in $\mathbb{D}$ satisfies that $V \bigcap V' = \emptyset$, then the set $W = \varphi(V) \bigcap \varphi(V')$ is a non-empty open set since φ is analytic and not constant. And we can find a sequence $\{z_k\}$ in W with $z_k \neq z_l$ if $k \neq l$. Suppose $\{\zeta_k\}$ is the sequence in V and $\{\zeta'_k\}$ is the sequence in V' such that $\zeta_k = \varphi^{-1}(z_k) \bigcap V$ and $\zeta'_k = \varphi^{-1}(z_k) \bigcap V'$ respectively. Let K_{ζ_k} and K'_{ζ_k} be the reproducing kernel at ζ_k and ζ'_k. Then for any $f \in H^2_\beta(\mathbb{D})$, we have

$$\langle K_{\zeta_k} - K_{\zeta'_k}, C_\varphi f \rangle = \langle C^*_\varphi (K_{\zeta_k} - K_{\zeta'_k}), f \rangle = \langle K_{\varphi(\zeta_k)} - K_{\varphi(\zeta'_k)}, f \rangle = 0,$$

this shows that $K_{\zeta_k} - K_{\zeta'_k} \in Ran(C_\varphi)$. But since $\{K_{\zeta_k} - K_{\zeta'_k}\}$ is independent, we have that $\dim[Ker(C_\varphi)^*] = \dim[Ran(C_\varphi)]^\perp = \infty$, which contradicts that C^*_φ is Fredholm. Thus, φ is one to one.

Secondly, φ is onto as $\beta \leq \frac{1}{2}$. In fact, if $\varphi(\mathbb{D}) \neq \mathbb{D}$, then $\mathbb{D} \bigcap \partial[\varphi(\mathbb{D})]$ should not be empty. Suppose $a \in \mathbb{D} \bigcap \partial[\varphi(\mathbb{D})]$, that is we can find a sequence $\{\zeta_k\} \subseteq \mathbb{D}$ such that $|\zeta_k| \to 1$ and $\varphi(\zeta_k) \to a$ as $k \to \infty$. Since $a \in \mathbb{D}$, we know that

$$\|C^*_\varphi K_{\zeta_k} - K_a\|_{H^2_\beta} = \|K_{\varphi(\zeta_k)} - K_a\|_{H^2_\beta} \to 0$$

as $k \to \infty$. Set $g_k = \frac{K_{\zeta_k}}{\|K_{\zeta_k}\|_{H^2_\beta}}$, then $\|g_k\|_{H^2_\beta} = 1$, $\|C^*_\varphi K_{\zeta_k}\|_{H^2_\beta}$ is bounded and $\frac{1}{\|K_{\zeta_k}\|_{H^2_\beta}} \to 0$ as $k \to \infty$. Therefore,

$$\|C^*_\varphi g_k\|_{H^2_\beta} = \frac{\|C^*_\varphi K_{\zeta_k}\|_{H^2_\beta}}{\|K_{\zeta_k}\|_{H^2_\beta}} \to 0$$

as $k \to \infty$. Since $(H^2_\beta)^* = H^2_\beta$ and C_φ is a Fredholm composition operator, there is a bounded operator T and a compact operator K on $H^2_\beta(\mathbb{D})$ such that $TC^*_\varphi - K = I$. Consequently, $\|TC^*_\varphi(g_k)\|_{H^2_\beta} \to 0$ from the fact that $\|C^*_\varphi g_k\|_{H^2_\beta} \to 0$, which induces that

$$\|Kg_k + g_k\|_{H^2_\beta} \to 0 \qquad (4.1).$$

Since K is compact and $\{g_k\}$ is a bounded sequence on H^2_β, we know that $\{Kg_k\}$ has a convergent subsequence, without loss of generality, we may assume that $\{Kg_k\}$ converges to g in H^2_β, then $\{g_k\}$ converges to $-g$ by (4.1). However, it is not difficult to check that $\{g_k\} \xrightarrow{w} 0$ in H^2_β, we get $g = 0$ and $\|g_k\|_{H^2_\beta} \to \|g\|_{H^2_\beta} = 0$ as $k \to \infty$, which is contradicted with the fact that $\|g_k\|_{H^2_\beta} = 1$ for any $k \in \mathbf{Z}^+$. Hence, φ is onto when $\beta \leq \frac{1}{2}$. The proof is completed. □

Question : How about the case as $\beta > \frac{1}{2}$?

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