Indian J. Pure Appl. Math., **46**(3): 287-307, June 2015

DOI: 10.1007/s13226-015-0123-x

ON A QUESTION OF URI SHAPIRA AND BARAK WEISS

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(*Received* 4 *April* 2014; *accepted* 29 *July* 2014)

Here it is proved that if $Q(x_1, \ldots, x_n)$ is a positive definite quadratic form which is reduced in the sense of Korkine and Zolotareff and has outer coefficients $B_1, \ldots, B_n$ satisfying $B_1 \geq 1,\ B_n \leq 1$ and $B_1 \cdots B_n = 1$, then its inhomogeneous minimum is at most $n/4$ for $n \leq 7$. This result implies a positive answer to a question of Shapira and Weiss for stable lattices and thereby provides another proof of Minkowski's Conjecture on the product of n real non-homogeneous linear forms in n variables for $n \leq 7$. Our result is an analogue of Woods' Conjecture which has been proved for $n \leq 9$. The analogous problem when $B_1 < 1$ is also investigated.

Key words : Lattice; covering; non-homogeneous; product of linear forms; critical determinant.

1. INTRODUCTION

Let

$$\begin{aligned} Q(x_1, \ldots, x_n) = \quad & B_1(x_1 + b_{21}x_2 + \cdots + b_{n1}x_n)^2 + B_2(x_2 + b_{32}x_3 + \cdots + b_{n2}x_n)^2 \\ & + \cdots + B_n x_n^2 \end{aligned} \tag{1.1}$$

be a positive definite quadratic form with real coefficients. The homogeneous minimum of Q is defined as

$$\lambda(Q) = \inf_{(u_1,\ldots,u_n)\in\mathbb{Z}^n\setminus\{0\}} Q(u_1, \ldots, u_n).$$

The inhomogeneous minimum of Q is defined as

$$\mu(Q) = \sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n} \ \inf_{(u_1,\ldots,u_n)\in\mathbb{Z}^n} Q(x_1 - u_1, \ldots, x_n - u_n).$$

[1]The author acknowledges the support of CSIR, India

[2]The author acknowledges the support of Indian National Science Academy during the period when the results of this paper were obtained

The form Q is said to be reduced in the sense of Korkine and Zolotaroff (or K-Z reduced) if for each $i,\ 1 \leq i \leq n$, B_i is the homogeneous minimum for the form

$$B_i(x_i + b_{i+1,i}x_{i+1} + \cdots + b_{ni}x_n)^2 + B_{i+1}(x_{i+1} + b_{i+2,i+1}x_{i+2} + \cdots)^2 + \cdots + B_n x_n^2.$$

Equivalently, a lattice $\mathbb{L}$ in $\mathbb{R}^n$ is called K-Z reduced if $\mathbb{L}$ has a basis of the form

$$(A_1, 0, 0, \ldots, 0), (a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}, A_n),$$

where $A_1, A_2, \ldots, A_n$ are all positive, and further for each $i = 1, 2, \ldots, n$ any two points of the lattice in $\mathbb{R}^{n-i+1}$ with basis

$$(A_i, 0, 0, \ldots, 0), (a_{i+1,i}, A_{i+1}, 0, \ldots, 0), \ldots, (a_{n,i}, a_{n,i+1}, \ldots, a_{n,n-1}, A_n)$$

are at a distance at least A_i apart.

(A positive definite quadratic form Q can be written as $Q(x_1, \ldots, x_n) = Q(X) = X'B'BX$, where B is a non singular matrix and the lattice corresponding to it is $\mathbb{L} = B\mathbb{Z}^n$.)

Conjecture (Woods) [13] — For a K-Z reduced form Q, if $B_1B_2 \cdots B_n = 1$ and $B_i \leqslant B_1$ for each i then $\mu(Q) \leq \frac{n}{4}$.

Equivalently, if a K-Z reduced lattice $\mathbb{L}$ has $A_1A_2 \cdots A_n = 1$ and $A_i \leqslant A_1$ for each i, then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\mathbb{L}$.

This conjecture has been proved for $n \leq 9$. (See Woods [12, 13, 14], Hans-Gill *et al*. [6,7] and Kathuria and Raka [8]). It is known that for any given n, a proof of well known conjecture on the product of n non-homogeneous linear forms in n variables, usually attributed to Minkowski follows from a proof of Woods' Conjecture for all $m \leq n$. The proof uses the results of Birch and Swinnerton-Dyer [1] and of McMullen [10]. For a history of Minkowski's conjecture see Gruber [5].

A lattice Λ (not necessarily K-Z reduced) is said to be a covering lattice for a set S if $\mathbb{R}^n \subseteq \bigcup_{A \in \Lambda} (S + A)$; equivalently if every translate of S contains a point of Λ. The covering radius of a lattice Λ is defined as the smallest real number λ such that Λ is a covering lattice for λS_n, where S_n is the closed unit sphere $\mid X \mid \leq 1$.

In geometric language one can state Minkowski's conjecture as

Conjecture (Minkowski) : Any lattice Λ of determinant 1 in $\mathbb{R}^n$ is a covering lattice for the set

$$S : |x_1x_2 \cdots x_n| \leq \frac{1}{2^n}.$$

In an effort to prove Minkowski's Conjecture, Shapira and Weiss [11] have proposed another approach by which it is enough to prove Minkowski's Conjecture for stable lattices. A lattice Λ of determinant 1 is called **stable** if any subgroup of Λ is of covolume at least 1.

Shapira and Weiss [11] showed that if all stable lattices in $\mathbb{R}^n$ have covering radius at most $\sqrt{n}/2$, then Minkowski's Conjecture is true in dimension n (see Corollary 5.1 of [11]).

Shapira and Weiss [11] further proved that for $n \leq 7$ covering radius of any stable lattice in $\mathbb{R}^n$ is at most $\frac{\sqrt{n}}{2}$, using the results of locally extremal lattices due to Dutour-Sikiric [3] and Dutour-Sikiric *et al*. [4]. It is clear that a stable K-Z reduced lattice satisfies $A_1 \geq 1$, $A_1A_2 \geq 1, \ldots$, $A_1A_2 \cdots A_{n-1} \geq 1$. In a lecture delivered in our department, Barak Weiss asked the following question:

If a K-Z reduced lattice $\mathbb{L}$ has $A_1A_2 \cdots A_i \geq 1$ for $i = 1, 2, \ldots, n$ and $A_1A_2 \cdots A_n = 1$, then does any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contain a point of $\mathbb{L}$?

In Section 3, we show that for $n \leq 7$, this question has a positive answer and thereby provide another proof of Minkowski's Conjecture for $n \leq 7$. In fact we prove the result under a weaker hypothesis.

***Theorem* 1** — *For $n \leq 7$, if a K-Z reduced lattice $\mathbb{L}$ has $A_1 \geq 1$, $A_n \leq 1$ and $A_1A_2 \cdots A_n = 1$, then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\mathbb{L}$.*

In Section 4, we investigate the upper bounds on covering radii of K-Z reduced lattices under the condition $A_1 < 1$. We prove

***Theorem* 2** — *For each $n \geq 2$, there exist K-Z reduced lattices $\mathbb{L}$ of determinant 1 having $A_1 < 1$ and $A_n > 1$ whose covering radius is $> \sqrt{n}/2$.*

***Theorem* 3** — *For $n = 3, 4$ the covering radius of $\mathbb{L}$ is $\leq \sqrt{n}/2$, where $\mathbb{L}$ is any K-Z reduced lattice of determinant 1 with $A_1 < 1$ and $A_n \leq 1$. For $n \geq 8$, there exist K-Z reduced lattices $\mathbb{L}$ of determinant 1 having $A_1 < 1$ and $A_n \leq 1$ whose covering radius is $> \sqrt{n}/2$.*

Under the hypothesis of Theorem 3 one can easily get some partial results for $n = 5, 6, 7$. For example, one can show that for $n = 5$ if $A_1 < 1$ and $A_5 \leq 1$ and if any one of A_2, A_3, A_4 is ≤ 1, then the covering radius of the lattice is $\leq \sqrt{5}/2$.

2. Preliminary Lemmas

For a unit sphere S_n with center O in $\mathbb{R}^n$, let $\Delta(S_n)$ be the critical determinant of S_n, defined as

$$\Delta(S_n) = \inf\{d(\Lambda) : \Lambda \text{ has no point other than } O \text{ in the interior of } S_n\}.$$

Let γ_n be the Hermite's constant i.e. γ_n is the smallest real number such that for any positive definite quadratic form Q in n variables of determinant D, there exist integers $u_1, u_2, \ldots, u_n$ not all zero satisfying

$$Q(u_1, u_2, \ldots, u_n) \leq \gamma_n D^{1/n}.$$

It is well known that $\Delta^2(S_n) = \gamma_n^{-n}$.

Let $\mathbb{L}$ be a lattice in $\mathbb{R}^n$ reduced in the sense of Korkine and Zolotareff. Let $A_1, A_2, \ldots, A_n$ be defined as in Section 1. We state below some preliminary lemmas. Lemmas 1 and 2 are due to Woods [12] and Lemma 3 is due to Korkine and Zolotareff [9]. In Lemma 4, the case $n = 3$ is a classical result of Gauss; $n = 4$ and 5 are due to Korkine and Zolotareff [9] while $n = 6$ and 7 are due to Blichfeldt [2].

Lemma 1 — If $2\Delta(S_{n+1})A_1^n \geq d(\mathbb{L})$, then any closed sphere of radius

$$R = A_1\{1 - (A_1^n \Delta(S_{n+1})/d(\mathbb{L}))^2\}^{1/2}$$

in $\mathbb{R}^n$ contains a point of $\mathbb{L}$.

Lemma 2 — For a fixed integer i with $1 \leqslant i \leqslant n-1$, denote by $\mathbb{L}_1$ the lattice in $\mathbb{R}^i$ with reduced basis

$$(A_1, 0, 0, \ldots, 0), (a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{i,1}, a_{i,2}, \ldots, a_{i,i-1}, A_i)$$

and denote by $\mathbb{L}_2$ the lattice in $\mathbb{R}^{n-i}$ with the reduced basis

$$(A_{i+1}, 0, 0, \ldots, 0), (a_{i+2,i+1}, A_{i+2}, 0, \ldots, 0), \ldots, (a_{n,i+1}, a_{n,i+2}, \ldots, a_{n,n-1}, A_n)$$

If any sphere in $\mathbb{R}^i$ of radius r_1 contains a point of $\mathbb{L}_1$ and if any sphere in $\mathbb{R}^{n-i}$ of radius r_2 contains a point of $\mathbb{L}_2$ then any sphere in $\mathbb{R}^n$ of radius $(r_1^2 + r_2^2)^{1/2}$ contains a point of $\mathbb{L}$.

Lemma 3 — For all relevant i, $A_{i+1}^2 \geq \frac{3}{4}A_i^2$ and $A_{i+2}^2 \geq \frac{2}{3}A_i^2$.

Lemma 4 — $\Delta(S_n) = 1/\sqrt{2},\ 1/2,\ 1/2\sqrt{2},\ \sqrt{3}/8$ and $1/8$ i.e. $\gamma_n = 2^{1/3},\ 4^{1/4},\ 8^{1/5},\ (\frac{64}{3})^{1/6}$ and $64^{1/7}$ for $n = 3,\ 4,\ 5, 6$ and 7 respectively.

For positive real numbers $X_1, \ldots, X_k$ we observe that

$$X_1 + \cdots + X_k \leq (k-1) + X_1 \cdots X_k \text{ if either all } X_i \leq 1 \text{ or all } X_i \geq 1. \tag{2.1}$$

Lemma 5 — Let $X_1, \ldots, X_n$ be positive real numbers, satisfying $X_1 X_2 \cdots X_n = 1$. Let

$$x_i = | X_i - 1 |, \;\; \alpha = \sum_{\substack{3 \le i \le n \\ X_i \le 1}} x_i.$$

Then the following hold

(i) If $X_i \ge 1$ for $i = 1, 3, 4, \ldots, n$ then
$\mathfrak{S}_1 = 4X_1 - \frac{2X_1^2}{X_2} + X_3 + \cdots + X_n \le n.$

(ii) If $X_1 \ge 1$ and $\alpha \le x_1 < 0.45$, then again $\mathfrak{S}_1 \le n$.

(iii) If $X_i \ge 1$ for $i = 1, 3, 5, 6, \ldots, n,$ and $X_i \le 4$ for $i \ge 5,$ then we have
$\mathfrak{S}_2 = 4X_1 - \frac{2X_1^2}{X_2} + 4X_3 - \frac{2X_3^2}{X_4} + X_5 + \cdots + X_n \le n.$

(iv) If $X_i \ge 1$ for $i = 1, 3, 5, 7, 8, \ldots, n,$ and $X_i \le 2^{3/2}$ for $i \ge 7,$ then we have
$\mathfrak{S}_3 = 4X_1 - \frac{2X_1^2}{X_2} + 4X_3 - \frac{2X_3^2}{X_4} + 4X_5 - \frac{2X_5^2}{X_6} + X_7 + \cdots + X_n \le n.$

PROOF : Using $X_1 X_2 \cdots X_n = 1$ and (2.1) we find that $\mathfrak{S}_1 = 4X_1 - 2X_1^3 X_3 \cdots X_n + X_3 + \cdots + X_n$. When $X_i \ge 1$ for $i = 1, 3, 4, \ldots, n$, $\mathfrak{S}_1$ is a decreasing function of each of X_i, so replacing each of these by 1 we get $\mathfrak{S}_1 \le n$. This proves (i).

Let $\beta = \sum_{\substack{3 \le i \le n \\ X_i \ge 1}} x_i$. Then $\mathfrak{S}_1 \le 4X_1 - 2X_1^3(1-\alpha)(1+\beta) + n - 2 - \alpha + \beta$. As the coefficient of β namely $1 - 2X_1^3(1-\alpha)$ is negative for $0 \le \alpha < 0.5$ and $\beta \ge 0$, we can replace β by 0 to get $\mathfrak{S}_1 \le 4X_1 - 2X_1^3(1-\alpha) + n - 2 - \alpha$. Further the coefficient of α namely $2X_1^3 - 1$ is positive, so we can replace α by x_1 to get $\mathfrak{S}_1 \le 3x_1 - 2(1+x_1)^3(1-x_1) + n + 2$ which is at most n for $x_1 \le 0.45$. This proves (ii).

Applying A.M-G.M inequality and using $X_1 X_2 \cdots X_n = 1$ we get $\mathfrak{S}_2 \le 4X_1 + 4X_3 + X_5 + \cdots + X_n - 4(X_1^3 X_3^3 X_5 \cdots X_n)^{\frac{1}{2}}$. Right side is a decreasing function of each of $X_5, \ldots, X_n$, so replacing each of these by 1 we get $\mathfrak{S}_2 \le 4X_1 + 4X_3 + n - 4 - 4(X_1^3 X_3^3)^{\frac{1}{2}}$ which is at most n for $X_1 \ge 1, X_3 \ge 1$. This proves (iii); the proof of (iv) is similar.

Lemma 6 — Let X_i be positive real numbers for $1 \le i \le m$ satisfying
$X_1 \ge 1, \;\; X_1 X_2 \cdots X_m = 1$. Let

$$x_i = | X_i - 1 |, \;\; \gamma = \sum_{\substack{4 \le i \le m \\ X_i \le 1}} x_i \quad \text{and} \quad \delta = \sum_{\substack{4 \le i \le m \\ X_i \ge 1}} x_i.$$

Suppose that either

(i) $X_i \geq 1$ for each i , $4 \leq i \leq m$ or
(ii) $\gamma \leq x_1 \leq 0.5$ or
(iii) $\delta \geq 2\gamma$ and $\gamma \leq 2x_1$ with $x_1 \leq 0.226$

then

$$4X_1 - X_1^4 X_4 \cdots X_m + X_4 + \cdots + X_m \leq m,$$

The simple proof similar to that given in Lemmas 8 and 10 of [6] is omitted.

3. Proof of Theorem 1

Let $\mathbb{L}$ be a lattice satisfying the hypothesis of Theorem 1. Suppose that there exists a closed sphere of radius $\sqrt{n}/2$ in $\mathbb{R}^n$ that contains no point of $\mathbb{L}$. We shall get a contradiction. Write $A = A_1^2,\ B = A_2^2,\ C = A_3^2, \ldots$. So we have $ABCD \cdots = 1$. Also we shall write $a = |A-1|, b = |B-1|$, $c = |C-1|, \ldots$.

We give some examples of inequalities that arise. Let $n = 7$ and $\mathbb{L}_i$, $1 \leqslant i \leqslant 4$, be lattices in $\mathbb{R}^1$ with basis (A_i) and $\mathbb{L}_5$ be a lattice in $\mathbb{R}^3$ with basis $(A_5, 0, 0), (a_{6,5}, A_6, 0), (a_{7,5}, a_{7,6}, A_7)$. Applying Lemma 2 repeatedly and using Lemma 1, we see that if $2\Delta(S_4)A_5^3 \geqslant A_5A_6A_7$ then any closed 7-sphere of radius

$$\left(\frac{1}{4}A_1^2 + \frac{1}{4}A_2^2 + \frac{1}{4}A_3^2 + \frac{1}{4}A_4^2 + A_5^2 - \frac{A_5^8\Delta(S_4)^2}{A_5^2A_6^2A_7^2}\right)^{1/2}$$

contains a point of $\mathbb{L}$. By our supposition this radius exceeds $\frac{1}{2}\sqrt{7}$. Since $\Delta(S_4) = 1/2$ and $A_1A_2 \cdots A_7 = 1$, this results in the conditional inequality:

$$\text{if } E^2 \geqslant FG \text{ then } A + B + C + D + 4E - E^4ABCD > 7. \tag{3.1}$$

We call this inequality (1, 1, 1, 1, 3), since it corresponds to the ordered partition (1, 1, 1, 1, 3) of 7 for the purpose of applying Lemma 2. Similarly the conditional inequality $(1, 1, 1, 1, 2)$ corresponding to the ordered partition $(1, 1, 1, 1, 2)$ of $n = 6$ is

$$\text{if } \ 2E \geq F \ \text{ then } \ A + B + C + D + 4E - \frac{2E^2}{F} > 6. \tag{3.2}$$

Since $4E - 2E^2/F \leq 2F$, the second inequality in (3.2) gives

$$A + B + C + D + 2F > 6. \tag{3.3}$$

One may remark here that the condition $2E \geq F$ is necessary only if we want to use inequality (3.2), but it is not necessary if we want to use the weaker inequality (3.3). This is so because if $2E < F$, using the partition $(1,1)$ in place of (2) for the relevant part, we get the upper bound $E+F$ which is clearly less than $2F$. We shall call inequalities of type (3.3) as weak inequalities and indicate it by the subscript $_w$ for example the inequality (3.3) is denoted by $(1,1,1,1,2)_w$. More examples of weak inequalities are (3.4)-(3.13).

In general, if $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is an ordered partition of n, then the conditional inequality arising from it, by using Lemmas 1 and 2, is also denoted by $(\lambda_1, \lambda_2, \ldots, \lambda_s)$. If the conditions in an inequality $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ are satisfied then we say that $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ holds.

For each $n, n \leq 7$, we discuss 2^{n-2} cases that arise depending upon whether $A_i > 1$ or $A_i \leq 1$ for $2 \leq i \leq n-1$. We list the cases and the inequalities used in the tables. If the case does not follow immediately from the inequalities, we also list relevant lemma from which it follows or the proposition where it is discussed. In three cases, where the list of inequalities is long, we list only the proposition in which the proof is given (Propositions 1, 4, 5). Sometimes, in these propositions, we have used the software Mathematica (7.0) to show that $f(x,y) < 0$ where $f(x,y)$ is some function by plotting its graph in given ranges of the variables.

Lemma 7 — Let $Y_i = A^2_{j+i}$ for some fixed j, $0 \leq j \leq n-3$ and for $1 \leq i \leq n$, the subscript $j+i$ being taken modulo n. Let

$$y_i = \mid Y_i - 1 \mid, \quad \eta = \sum_{\substack{4 \leq i \leq n \\ Y_i \leq 1}} y_i.$$

Then all the cases in which $Y_1 \geq 1, \;\; Y_2 > 1$, $Y_3 \leq 1$ and $\eta \leq y_1$ do not arise.

PROOF : Here we have, by Lemma 3, $y_1 \leq \frac{1}{2}$ and $y_2 \leq \frac{1}{3}$.

If $Y_1 \geq Y_2$, then the inequality $(\underbrace{1, \ldots, 1}_{j}, 3, 1, \ldots, 1)$ holds i.e.

$4Y_1 - Y_1^4 Y_4 \cdots Y_n + Y_4 + \cdots + Y_n > n$, which is not true by Lemma 6(ii) with $X_i = Y_i$ for all $i, 1 \leq i \leq n$ and $\gamma = \eta$.

If $Y_1 \leq Y_2$, we use the inequality $(\underbrace{1, \ldots, 1}_{j}, 1, 2, 1, \ldots, 1)$ which gives $Y_1 + 4Y_2 - 2Y_2^3 Y_4 \cdots Y_n + Y_4 + \cdots + Y_n > n$ which is not true by Lemma 5(ii) with $X_1 = Y_2$, $X_i = Y_{i+1}$ for $1 \leq i \leq n-1$, $X_n = Y_1$ and $\alpha = \eta$.

3.2 $n = 6$

The proof of Theorem 1 for $n = 6$ follows from the inequalities listed in Table 5 and Proposition 1.

Table 5

Case	A	B	C	D	E	F	Inequalities	Lemma/Proposition
1	$\geq$	$>$	$>$	$>$	$>$	$\leq$	$(1,1,1,1,2)$	Lemma 5(i)
2	$\geq$	$>$	$>$	$>$	$\leq$	$\leq$	$(1,1,1,3)$	Lemma 6(i)
3	$\geq$	$>$	$>$	$\leq$	$>$	$\leq$	$(1,1,2,2)$	Lemma 5(iii)
4	$\geq$	$>$	$>$	$\leq$	$\leq$	$\leq$		Proposition 1
5	$\geq$	$>$	$\leq$	$>$	$>$	$\leq$	$(1,2,1,2)$	Lemma 5(iii)
6	$\geq$	$>$	$\leq$	$>$	$\leq$	$\leq$	$(1,2,2,1)_w$	Lemma 7 with $Y_1 = A$
							$(3,1,1,1), (1,2,1,1,1)$	
7	$\geq$	$>$	$\leq$	$\leq$	$>$	$\leq$	$(1,2,1,2)_w$	Lemma 7 with $Y_1 = A$
							$(3,1,1,1), (1,2,1,1,1)$	
8	$\geq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$(1,2,1,1,1)_w$	Lemma 7 with $Y_1 = A$
							$(3,1,1,1), (1,2,1,1,1)$	
9	$\geq$	$\leq$	$>$	$>$	$>$	$\leq$	$(2,1,1,2)$	Lemma 5(iii)
10	$\geq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$(2,1,2,1)_w$	Lemma 7 with $Y_1 = C$
							$(1,1,3,1), (1,1,1,2,1)$	
11	$\geq$	$\leq$	$>$	$\leq$	$>$	$\leq$	$(2,2,2)_w$	
12	$\geq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$(2,2,1,1)_w$	
13	$\geq$	$\leq$	$\leq$	$>$	$>$	$\leq$	$(2,1,1,2)_w$	Lemma 7 with $Y_1 = D$
							$(1,1,1,3), (1,1,1,1,2)$,	
14	$\geq$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$(2,1,2,1)_w$	
15	$\geq$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$(2,1,1,2)_w$	
16	$\geq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$(2,1,1,1,1)_w$	

Proposition 1 — Case (4) i. e. $A \geq 1, B > 1, C > 1, D \leq 1, E \leq 1, F \leq 1$ does not arise.

PROOF : Recall that $a = |A-1|, \ldots, f = |F-1|$. Here by Lemma 3, $a \leq 1,\ b \leq 0.5,\ c \leq \frac{1}{3}$ and $F \geq \frac{4B}{9}$. Using weak inequalities $(1,1,2,2)_w$, $(2,2,2)_w$, $(1,1,2,1,1)_w$ and $(2,2,1,1)_w$ we get

$$a + b - 2d - 2f > 0, \tag{3.4}$$

$$2b - 2d - 2f > 0, \tag{3.5}$$

$$a + b - 2d - e - f > 0, \tag{3.6}$$

$$2b - 2d - e - f > 0. \tag{3.7}$$

Therefore $f < \frac{a+b}{2},\ f < b$, $e + f < a + b$ and $e + f < 2b$.

Claim (i): $B > C$

Suppose $B \leq C$. Therefore $f < b \leq c$. Apply the inequality $(1, 1, 3, 1)$ to get $A + B + 4C - C^4FAB + F > 6$, which is not true by Lemma 6(ii) for $\gamma = f \leq c = x_1$ and $c \leq \frac{1}{3}$.

Claim (ii): $e + f > b$, $d \leq \frac{a}{2},\ d \leq \frac{b}{2}$.

Suppose $e+f \leq b$. As $B > C$ by claim (i), therefore $(1, 3, 1, 1)$ holds i. e. $A+4B-B^4EFA+E + F > 6$, which is not true by Lemma 6(ii) with $\gamma = e + f \leq b = x_1 \leq 0.5$. Now (3.6) and (3.7) give $d \leq \frac{a}{2},\ d \leq \frac{b}{2}$.

Claim (iii): $b < 0.226$

Suppose $b \geq 0.226$. We first prove that $B^4FA > 2$. If $A \geq B$, $B^4FA \geq B^5F > (1+b)^5(1-b) > 2$ for $b \geq 0.226$. If $A \leq B$, $B^4FA > (1 + b)^4(1 + a)(1 - \frac{a+b}{2}) = \phi(a)$, say. The second derivative of $\phi(a)$ is negative, its minimum occurs at end point of a, hence $\phi(a) \geq \min\{\phi(0), \phi(b)\} > 2$ for $0.226 \leq b \leq 0.5$. Now apply $(1, 4, 1)$ to get $A+4B-\frac{1}{2}B^5FA+F > 6$. As $B^5A \geq B^5 \geq (1.226)^5 > 2$, the left side is a decreasing function of F, so we replace F by $\frac{4B}{9}$ to get $A + \frac{40B}{9} - \frac{1}{2}\frac{4}{9}B^6A > 6$, which is not true for $1.226 \leq B \leq 1.5$ and $1 \leq A \leq 2$. This gives a contradiction.

Final contradiction

Apply $(1, 2, 2, 1)$ with A.M.-G.M. inequality to get $A + 4B + 4D + F - 4\sqrt{B^3D^3AF} > 6$, i.e.

$$4 + a + 4b - 4d - f - 4\sqrt{(1 + b)^3(1 - d)^3(1 + a)(1 - f)} > 0.$$

Left side is an increasing function of f.

If $a > b$, replace f by $b - d$ from (3.5) to get $\psi(d) = 4 + a + 3b - 3d - 4\sqrt{(1 + b)^3(1 - d)^3(1 + a)(1 - b + d)} > 0$. As $\psi''(d) > 0$, and $0 \leq d \leq \frac{b}{2}$, we have $\psi(d) \leq \max\{\psi(0), \psi(\frac{b}{2})\}$ which is less than 0 for $0 \leq a \leq 1$ and $0 < b \leq \min\{0.226, a\}$. This gives a contradiction.

If $a \leq b$, replace f by $\frac{a+b}{2} - d$ from (3.4) and proceed as above using $0 \leq d \leq \frac{a}{2}$, $0 \leq a \leq b$ and $0 < b \leq 0.226$ to arrive at a contradiction.

3.3 *n* = *7*

The proof of Theorem 1 for $n = 7$ follows from the inequalities listed in Table 6 and Propositions 2-5.

Table 6

Case	A	B	C	D	E	F	G	Inequalities	Lemma/Proposition
1	$\geq$	$>$	$>$	$>$	$>$	$>$	$\leq$	$(1,1,1,1,1,2)$	Lemma 5(i)
2	$\geq$	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$(1,1,1,1,3)$	Lemma 6(i)
3	$\geq$	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$(1,1,1,2,2)$	Lemma 5(iii)
4	$\geq$	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$		Proposition 4
5	$\geq$	$\geq$	$\geq$	$\leq$	$\geq$	$\geq$	$\leq$	$(1,1,2,1,2)$	Lemma 5(iii)
6	$\geq$	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$(2,2,3),(1,1,2,3),(1,3,3)$	Proposition 2
7	$\geq$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$(2,3,2),(1,1,3,2)$ $(1,2,2,2)_w,(1,3,1,1,1)$	Proposition 2
8	$\geq$	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$		Proposition 5
9	$\geq$	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$(1,2,1,1,2)$	Lemma 5(iii)
10	$\geq$	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$(1,2,1,3),(3,1,\ldots,1)$ $(2,1,2,1,1)_w,(1,1,3,1,1)$	Proposition 3
11	$\geq$	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$(1,2,2,2)$	Lemma 5(iv)
12	$\geq$	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$(1,2,2,1,1)_w$ $(3,1,1,1,1),(1,2,1,\ldots,1)$	Lemma 7, $Y_1 = A$
13	$\geq$	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$(1,3,1,2),(3,1,\ldots,1)$ $(2,1,1,2,1)_w,(1,1,1,1,3)$	Proposition 3
14	$\geq$	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$(1,2,1,2,1)_w$ $(3,1,1,1,1),(1,2,1,\ldots,1)$	Lemma 7, $Y_1 = A$
15	$\geq$	$>$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$(1,2,1,1,2)_w$ $(3,1,1,1,1),(1,2,1,\ldots,1)$	Lemma 7, $Y_1 = A$
16	$\geq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$(1,2,1,\ldots,1)_w$ $(3,1,1,1,1),(1,2,1,\ldots,1)$	Lemma 7, $Y_1 = A$
17	$\geq$	$\leq$	$>$	$>$	$>$	$>$	$\leq$	$(2,1,1,1,2)$	Lemma 5(iii)

Case	A	B	C	D	E	F	G	Inequalities	Lemma/Proposition
18	$\geq$	$\leq$	$>$	$>$	$>$	$\leq$	$\leq$	$(2,2,3),(2,1,1,3)$	Proposition 2
								$(2,2,1,2)_w,(1,1,1,3,1)$	
19	$\geq$	$\leq$	$>$	$>$	$\leq$	$>$	$\leq$	$(2,1,2,2)$	
20	$\geq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$\leq$	$(2,2,1,2)_w,(1,1,1,3,1)$	Lemma 6(ii)
21	$\geq$	$\leq$	$>$	$\leq$	$>$	$>$	$\leq$	$(2,2,1,2)$	Lemma 5(iv)
22	$\geq$	$\leq$	$>$	$\leq$	$>$	$\leq$	$\leq$	$(2,2,2,1)_w$	
23	$\geq$	$\leq$	$>$	$\leq$	$\leq$	$>$	$\leq$	$(2,2,1,2)_w$	
24	$\geq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$(2,2,1,1,1)_w$	
25	$\geq$	$\leq$	$\leq$	$>$	$>$	$>$	$\leq$	$(3,2,2),(3,1,1,2),(3,1,3)$	Proposition 2
26	$\geq$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$(2,1,1,2,1)_w$	Lemma 7, $Y_1=D$
								$(1,1,1,3,1),(1,\dots,1,2,1)$	
27	$\geq$	$\leq$	$\leq$	$>$	$\leq$	$>$	$\leq$	$(2,1,2,2)_w$	
28	$\geq$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$(2,1,2,1,1)_w$	
29	$\geq$	$\leq$	$\leq$	$\leq$	$>$	$>$	$\leq$	$(2,1,1,1,2)_w$	Lemma 7, $Y_1=E$
								$(1,1,1,1,3),(1,\dots,1,2)$	
30	$\geq$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$(2,1,1,2,1)_w$	
31	$\geq$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$(2,1,1,1,2)_w$	
32	$\geq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$(2,1,1,1,1,1)_w$	

Proposition 2 — Cases 6, 7, 18 and 25 do not arise.

PROOF : We illustrate the proof of Case 6 where $A \geq 1, B > 1, C > 1, D \leq 1, E > 1$, $F \leq 1, G \leq 1$.

Subcase (i) : $A \geq B$. Here $(2,2,3)$ holds i.e. $2B+4C-\frac{2C^2}{D}+4E-E^4ABCD > 7$. Applying A.M.-G.M. inequality to $\frac{C^2}{D}+E^4ABCD$ and noting that $\frac{C^2}{D} > C$, we get $2B+3C+4E-2(E^4C^3AB)^{\frac{1}{2}} > 7$. As $A \geq B$, we can replace A by B to get $2B+3C+4E-2(E^4C^3B^2)^{\frac{1}{2}} > 7$. Left side of this inequality is a decreasing function of C, therefore we can replace C by 1 to get $2B+4E-2E^2B > 4$ which is clearly not true for $B > 1, E > 1$.

Subcase (ii) : $A < B, B < E^4C^3$. The inequality $(1,1,2,3)$ with A.M.-G.M gives $A+B+3C+4E-2(E^4ABC^3)^{\frac{1}{2}} > 7$. Left side is a decreasing function of A as $1 \leq A < B$, so we can replace A by 1 to get $B+3C+4E-2(E^4BC^3)^{\frac{1}{2}} > 6$. Further left side is a decreasing function of B for $B < E^4C^3$, therefore we can replace B by 1 to get $3C+4E-2(E^4C^3)^{\frac{1}{2}} > 5$ which is clearly not true for $E > 1$ and $C > 1$.

Subcase (iii) : $A < B, B \geq E^4C^3$. This gives $B \geq C$. Therefore $(1,3,3)$ holds, which using A.M.-G.M. inequality gives $A + 4B + 4E - 2B^2E^2\sqrt{A} > 7$. Left side is a decreasing function of A for $1 \leq A < B$, so replacing A by 1 we get $2B + 2E - B^2E^2 > 3$ which is clearly not true for $B > 1$ and $E > 1$.

In Case 25, we distinguish the subcases $D \geq E$; $D < E, E < A^4F^3$ and $D < E, E \geq A^4F^3$ and proceed as in Case 6.

In Case 7, we distinguish the subcases $A \geq B$; $A < B, B < C^4F^3$ and $A < B, B \geq C^4F^3$ and proceed as in Case 6 except in Subcase (iii) where we use the weak inequality $(1,2,2,2)_w$ to get $e + g < \frac{a}{2} + c < \frac{b}{2} + \frac{b}{4} < b$. Then use $(1,3,1,1,1)$ and apply Lemma 6(ii) with $\gamma = e + g < x_1 = b \leq 0.5$ to get a contradiction.

In Case 18, we distinguish the subcases $C \geq D$; $C < D, D < E^4B^3$ and $C < D, D \geq E^4B^3$ and proceed as in Case 6 except in Subcase (iii) where we use the weak inequality $(2,2,1,2)_w$ to get $b + g < \frac{c}{2} + e < \frac{d}{2} + \frac{d}{4} < d$. Then use $(1,1,1,3,1)$ and apply Lemma 6(ii) with $\gamma = b + g < x_1 = d \leq 0.5$ to get a contradiction.

Proposition 3 — Cases 10 and 13 do not arise.

PROOF : We illustrate the proof of Case 10 where $A \geq 1, B > 1, C \leq 1, D > 1, E > 1, F \leq 1, G \leq 1$.

Subcase (i) : $\max(A, D) < E^4B^3$. Here we apply $(1,2,1,3)$ and get $A + 3B + D + 4E - 2(E^4ADB^3)^{\frac{1}{2}} > 7$. Left side is a symmetric function of A and D. Suppose, therefore without loss of generality that $A \leq D$. Now left side is a decreasing function of A, so we can replace A by 1 to get $3B + D + 4E - 2(E^4DB^3)^{\frac{1}{2}} > 6$. Further left side is a decreasing function of D for $D < E^4B^3$, so we can replace D by 1 to get $3B + 4E - 2(E^4B^3)^{\frac{1}{2}} > 5$ which is clearly not true for $B > 1$ and $E > 1$.

Subcase (ii) : $\max(A, D) \geq E^4B^3$. If $\max(A, D) = A$, we get $a \geq 4e + 3b \geq b$, therefore $(3,1,1,1,1)$ holds which gives $4A - A^4DEFG + D + E + F + G > 7$. Also the weak inequality $(2,1,2,1,1)$ gives $2b - c + 2e - f - g > 0$ which further gives $f + g < 2b + 2e \leq a$. Apply Lemma 6(ii) with $\gamma = f + g < x_1 = a \leq 0.5$ to get a contradiction. If $\max(A, D) = D$, we get $d \geq 4e + 3b \geq e$, therefore $(1,1,3,1,1)$ holds which gives $A + B + C + 4D - D^4GABC + G > 7$. Also the weak inequality $(2,1,2,1,1)$ gives $2b - c + 2e - f - g > 0$ which further gives $c + g < 2b + 2e \leq d$. Apply Lemma 6(ii) with $\gamma = c + g < x_1 = d \leq 0.5$ to get a contradiction.

In Case 13, we distinguish the subcases $\max(A, E) < B^4F^3$ and $\max(A, E) \geq B^4F^3$ and proceed as in Case 10.

Proposition 4 — Case 4 where $A \geq 1, B > 1, C > 1, D > 1, E \leq 1, F \leq 1, G \leq 1$ does not arise.

PROOF : As $\gamma_7 = 64^{\frac{1}{7}}$ by Lemma 4, we get $A \leq 64^{\frac{1}{7}} < 1.82$. Also, we have, by Lemma 3, $b \leq 1,\ c \leq 0.5,\ d \leq \frac{1}{3}$. Using weak inequalities $(1,1,1,2,2)_w,\ (2,2,1,2)_w,\ (1,2,2,2)_w,$ $(1,1,2,2,1)_w$, $(2,2,2,1)_w$ and $(1,1,2,1,2)_w$ we get

$$a + b + c - 2e - 2g > 0, \tag{3.8}$$

$$2b + 2d - e - 2g > 0, \tag{3.9}$$

$$a + 2c - 2e - 2g > 0, \tag{3.10}$$

$$a + b + 2d - 2f - g > 0, \tag{3.11}$$

$$2b + 2d - 2f - g > 0, \tag{3.12}$$

$$a + b + 2d - e - 2g > 0. \tag{3.13}$$

Claim (i) : $D^4ABC < 2$ and hence $D^4 < 2$, $EFG > \frac{1}{2}$.

Suppose $D^4ABC \geq 2$. Then the inequality $(1,1,1,4)$ holds which gives

$$\phi(A,B,C,D) = A + B + C + 4D - \frac{1}{2}D^5ABC > 7. \tag{3.14}$$

The coefficient of C in ϕ namely $1 - \frac{1}{2}D^5AB$ may be positive or negative, therefore the maximum can occur at the end points of C. Hence $\phi(A,B,C,D) \leq \max\{\phi(A,B,1,D),\ \phi(A,B,1.5,D)\}$. Similarly the maximum can occur at end points of A and B. Therefore $\phi(A,B,C,D) \leq$ $\max\{\phi(1,1,1,D),\ \phi(1,1,1.5,D), \phi(1,2,1,D),\ \phi(1,2,1.5,D), \phi(1.82,1,1,D),$ $\phi(1.82,1,1.5,D), \phi(1.82,2,1,D),\ \phi(1.82,2,1.5,D)\}$. This can be easily seen to be less than 7 for $1 < D \leq \frac{4}{3}$. This gives a contradiction to (3.14), therefore $D^4ABC < 2$.

As $ABC > 1$ and $EFG = \frac{1}{ABCD} > \frac{D^3}{2}$, we get the other results in the claim.

Claim (ii) : $A \leq \sqrt{2}$.

Suppose $A^2 \geq 2$, then $A^4EFG \geq 4 \times 0.5 = 2$. Then the inequality $(4,1,1,1)$ holds which gives $\phi(A,y) = 4A - \frac{1}{2}A^5y + 2 + y > 7$, where $y = EFG \geq 0.5$. This is not true for $A \geq 1$.

Claim (iii) : $g > 2d$.

Suppose $g \leq 2d$. The inequality $(1,1,1,3,1)$ holds i. e. $A+B+C+4D-D^4GABC+G>7$. This is not true by Lemma 6(iii) as $\gamma = g \leq \frac{a+b+c}{2} = \frac{\delta}{2}$ from (3.8) and $x_1 = d < 2^{\frac{1}{4}}-1 < 0.226$. Hence $g > 2d$.

Claim (iv) : $d < 0.1$.

Suppose $d \geq 0.1$. Then from (3.8) and Claim (iii), $D^4ABC \geq (1+d)^4(1+a+b+c) > (1+d)^4(1+2g) > (1+d)^4(1+4d) > 2$ for $d \geq 0.1$. This contradicts Claim (i).

Claim (v) : $b > 0.145$.

Suppose $b \leq 0.145$. Apply $(1,2,2,2)$ with A.M.-G.M. inequality to get $A+4B+4D+4F-6BDF\sqrt[3]{A} > 7$, i.e.

$$6+a+4b+4d-4f-6(1+b)(1+d)(1-f)(1+a)^{\frac{1}{3}} > 0. \tag{3.15}$$

Left side is an increasing function of f.

If $a > b$, we get from (3.12) and Claim (iii) that $f < b$. Therefore we can replace f by b in equation (3.15) to get $\psi_1(d) = 6+a+4d-6(1+b)(1+d)(1-b)(1+a)^{\frac{1}{3}} > 0$. As $\psi_1(d)$ is a decreasing function of d we can replace d by 0 to get $6+a-6(1+b)(1-b)(1+a)^{\frac{1}{3}} > 0$. which is not true for $a > b$ and $0 < b \leq 0.145$. This gives a contradiction.

If $a \leq b$, we get from (3.11) and Claim (iii) that $f < \frac{a+b}{2}$. Therefore we can replace f by $\frac{a+b}{2}$ in equation (3.15) to get $\psi_2(d) = 6-a+2b+4d-6(1+b)(1+d)(1-\frac{a+b}{2})(1+a)^{\frac{1}{3}} > 0$. As $\psi_2(d)$ is a decreasing function of d we can replace d by 0 to get $6-a+2b-6(1+b)(1-\frac{a+b}{2})(1+a)^{\frac{1}{3}} > 0$, which is not true for $0 < b \leq 0.145$ and $0 \leq a \leq b$. This gives a contradiction.

Claim (vi) : $B \leq \sqrt{2}$, in fact $B \leq 1.3196$ if $A \geq B$.

We have $B^4FGA \geq B^4 \cdot \frac{1}{2} \cdot A \geq \begin{cases} B^5/2 > 2 & \text{if } B > 1.3196 \text{ and } A \geq B \\ B^4/2 > 2 & \text{if } B > \sqrt{2} \text{ and } A < B. \end{cases}$ Then the inequality $(1,4,1,1)$ holds which gives $A+4B-\frac{1}{2}B^5Az+1+z > 7$ where $z = FG \geq \frac{1}{2}$. As the coefficient of z is negative, we can replace z by $\frac{1}{2}$ to get $\phi(A,B) = A+4B-\frac{1}{4}B^5A+1+\frac{1}{2} > 7$. This is not true in both the cases $A \geq B > 1.3196$ as well as in $B > \sqrt{2}, A < B$. Hence the claim.

Claim (vii) : $c < 0.203$.

Suppose $c \geq 0.203$.

Case (i): $a \geq b$.

We have from (3.9) and Claim (iv) that $g < b+d < b+0.1$. Also $0.145 < b \leq 0.3196$ here.

Therefore $C^4GAB > C^4(1-b-0.1)(1+b)^2 > 2$, as $(1-b-0.1)(1+b)^2$ attains its minimum at the end points of b. Then the inequality $(2,4,1)$ holds which gives $2B+4C-\frac{1}{2}C^5GAB+G>7$. We can replace A by B and G by $1-b-0.1$ to get $4c+b-0.1-\frac{1}{2}(1+c)^5(1-b-0.1)(1+b)^2>0$ which is not true for $0.145 < b \leq 0.3196$ and $c \geq 0.203$.

Case (ii) : $a < b$.

From (3.13) and Claim (iii), we have $2d < g < \frac{a+b}{2} + d$ which gives $d < \frac{a+b}{2}$. Also $d < 0.1$. Here we use

$$g < \begin{cases} \frac{a+b}{2} + \frac{a+b}{2} & \text{if} \quad a+b \leq 0.2 \\ \frac{a+b}{2} + 0.1 & \text{if} \quad a+b \geq 0.2. \end{cases}$$

If $a+b \leq 0.2$, $C^4GAB > (1.203)^4(1-a-b)(1+a+b) > 2$ for $c \geq 0.203$.

If $a+b \geq 0.2$, i. e. $a \geq \max\{0, 0.2-b\}$, one finds that $C^4GAB > (1.203)^4(1-\frac{a+b}{2}-0.1)(1+a)(1+b) = \psi(a)$, say. The second derivative $\psi''(a)$ is negative, so $\psi(a) \geq \min\{\psi(\max(0, 0.2-b)), \psi(b)\} > 2$ for $0.145 < b \leq \sqrt{2}-1$. Hence $C^4GAB > 2$ in both the cases. Therefore the inequality $(1,1,4,1)$ holds i.e.

$$\phi(g) = a + b + 4c - g - \frac{1}{2}(1+c)^5(1-g)(1+a)(1+b) > 0.$$

$\phi(g)$ is an increasing function of g. If $a+b \leq 0.2$, $\phi(g) < 4c - \frac{1}{2}(1+c)^5(1-a-b)(1+a+b) < 4c - \frac{1}{2}(1+c)^5(1-0.2)(1+0.2) < 0$, for $c \geq 0.203$.

If $a+b \geq 0.2$, $\phi(g) < \frac{a+b}{2} - 0.1 + 4c - \frac{1}{2}(1+c)^5(1-\frac{a+b}{2}-0.1)(1+a)(1+b) = \psi(c)$, say. One finds that $\psi(c)$ is a decreasing function of c, therefore $\psi(c) \leq \psi(0.203) = \frac{a+b}{2} - 0.1 + 4(0.203) - \frac{1}{2}(1.203)^5(1-\frac{a+b}{2}-0.1)(1+a)(1+b)$ which is atmost 0 for $0.145 < b \leq \sqrt{2}-1$ and $0 \leq a < b$.

This gives a contradiction in both the cases. Hence $c < 0.203$.

Claim (viii) : $A > B$.

Suppose $A \leq B$. The inequality $(1,2,2,1,1)$ with A.M.-G.M. gives $A+4B+4D+F+G-4\sqrt{B^3D^3AFG} > 7$. Left side is a decreasing function of F as $\sqrt{G} \geq G \geq \frac{1}{2}$. Replacing F by $1-\frac{a+b}{2}-d+\frac{g}{2}$, from (3.11), we get

$$\begin{aligned} \phi(g) = 4 + a + 4b + 4d - \tfrac{a+b}{2} - d + \tfrac{g}{2} - g - 4\sqrt{(1+b)^3}\sqrt{(1+d)^3} \\ \times \sqrt{(1+a)(1-g)}\sqrt{1 - \tfrac{a+b}{2} - d + \tfrac{g}{2}} > 0. \end{aligned} \tag{3.16}$$

The second derivative $\phi''(g)$ is positive and $0 \leq g < \frac{a}{2} + c < \frac{a}{2} + 0.203$ from (3.10) and Claim (vii). Therefore $\phi(g) \leq \max\{\phi(0), \phi(\frac{a}{2}+0.203)\}$. Let $\phi(0) = \psi_1(d)$ and $\phi(\frac{a}{2}+0.203) = \psi_2(d)$.

One finds that $\psi_1''(d) > 0$ and $\psi_2''(d) > 0$ and $0 < d < 0.1$. Therefore $\psi_i(d) \le \max\{\psi_i(0), \psi_i(0.1)\}$ for $i = 1, 2$. Now one finds that $\psi_1(0) = 4 + a + 4b - \frac{a+b}{2} - 4\sqrt{(1+b)^3}\sqrt{(1+a)}\sqrt{1 - \frac{a+b}{2}} < 0$, $\psi_1(0.1) = 4 + a + 4b + 4(0.1) - \frac{a+b}{2} - 0.1 - 4\sqrt{(1+b)^3}\sqrt{(1.1)^3}\sqrt{(1+a)}\sqrt{1 - \frac{a+b}{2} - 0.1} < 0$, $\psi_2(0) = 4+a+4b-\frac{a+b}{2}-\frac{a}{4}-\frac{0.203}{2}-4\sqrt{(1+b)^3}\sqrt{(1+a)}\sqrt{(1-\frac{a}{2}-0.203)}\sqrt{1-\frac{a+b}{2}+\frac{a}{4}+\frac{0.203}{2}} < 0$, $\psi_2(0.1) = 4+a+4b+4(0.1)-\frac{a+b}{2}-0.1-\frac{a}{4}-\frac{0.203}{2}-4\sqrt{(1+b)^3}\sqrt{(1.1)^3}\sqrt{(1+a)(1-\frac{a}{2}-0.203)}\times \sqrt{1-\frac{a+b}{2}-0.1+\frac{a}{4}+\frac{0.203}{2}} < 0$ for $0 \le a \le b$ and $0.145 < b \le \sqrt{2}-1$. This gives a contradiction.

Claim (ix) : $A > 1.32$.

Suppose $A \le 1.32$. Working as in Claim (viii) and replacing F by $1 - b - d + \frac{g}{2}$, from (3.12), we get instead of (3.16)

$$\begin{aligned}\phi(g) = 4 + a + 4b + 4d - b - d + \tfrac{g}{2} - g - 4\sqrt{(1+b)^3}\sqrt{(1+d)^3}\sqrt{(1+a)} \\ \times \sqrt{1-g}\sqrt{1 - b - d + \tfrac{g}{2}} > 0.\end{aligned} \tag{3.17}$$

The second derivative $\phi''(g)$ is positive and $0 \le g < \frac{a}{2} + c < \frac{a}{2} + 0.203$ from (3.10) and Claim (vii). Therefore $\phi(g) \le \max\{\phi(0), \phi(\frac{a}{2} + 0.203)\}$. Let $\phi(0) = \psi_1(d)$ and $\phi(\frac{a}{2} + 0.203) = \psi_2(d)$. One finds that $\psi_1''(d) > 0$ and $\psi_2''(d) > 0$ and $0 < d < 0.1$. Now one finds that $\psi_i(d) \le \max\{\psi_i(0), \psi_i(0.1)\} < 0$ for $i = 1, 2$ and $0.145 < b < a \le 0.32$. This gives a contradiction.

Final Contradiction

We are left with $A > 1.32$, $B \le 1.3196$ and $C < 1.203$. Therefore $A^2 > BC$, so the inequality $(3, 3, 1)$ holds. After applying A.M.-G.M. inequality we get $4A + 4D + G - 2A^2D^2\sqrt{G} - 7 > 0$. Left side of this inequality is a quadratic in $\sqrt{G}$. Since $A^4D^4 - 4A - 4D + 7 > 0$, we have

$$\sqrt{G} < A^2D^2 - (A^4D^4 - 4A - 4D + 7)^{\frac{1}{2}} = \alpha \ \text{(say)}. \tag{3.18}$$

Using AM-GM inequality in $(1, 2, 2, 2)$, we get $A + 4B + 4D + 4F - 6BDFA^{\frac{1}{3}} > 7$ which gives $F < (A + 4B + 4D - 7)(6BDA^{\frac{1}{3}} - 4)^{-1}$. Substituting this upper bound of F in the inequality $(2, 2, 2, 1)$, we get

$$G > 7 - 2B - 2D - 2F > 7 - 2B - 2D - \frac{2(A + 4B + 4D - 7)}{6BDA^{\frac{1}{3}} - 4} = \beta \ \text{(say)}. \tag{3.19}$$

From (3.18) and (3.19) we have $\beta < \alpha^2$. On simplifying we get

$$\phi(B) = A^4D^4 - 2A + B - D + \tfrac{A+4B+4D-7}{6BDA^{\frac{1}{3}}-4} - A^2D^2\{A^4D^4 - 4A - 4D + 7\}^{\frac{1}{2}} > 0. \tag{3.20}$$

One can see that $\phi(B)$ is an increasing function of B. From Claim (vi), we have $B \leq 1.3196$. Therefore $\phi(B) \leq A^4D^4 - 2A + 1.3196 - D + \frac{A+4(1.3196)+4D-7}{6(1.3196)DA^{\frac{1}{3}}-4} - A^2D^2\{A^4D^4 - 4A - 4D + 7\}^{\frac{1}{2}} < 0$ for $1.32 < A \leq \sqrt{2}$ and $1 < D < 1.1$. This contradicts (3.20). Hence the result.

Proposition 5 — Case 8 where $A \geq 1, B > 1, C > 1, D \leq 1, E \leq 1, F \leq 1, G \leq 1$ does not arise.

PROOF : As in Proposition 4, $A < 1.82$. Also, we have, by Lemma 3, $b \leq 0.5$, $c \leq \frac{1}{3}$. Using weak inequalities $(1,1,2,2,1)_w$, $(2,2,2,1)_w$, $(1,1,2,1,2)_w$, $(2,2,1,2)_w$, $(1,1,2,1,1,1)_w$ and $(2,2,1,1,1)_w$ we get

$$a + b - 2d - 2f - g > 0, \tag{3.21}$$

$$2b - 2d - 2f - g > 0, \tag{3.22}$$

$$a + b - 2d - e - 2g > 0, \tag{3.23}$$

$$2b - 2d - e - 2g > 0, \tag{3.24}$$

$$a + b - 2d - e - f - g > 0, \tag{3.25}$$

$$2b - 2d - e - f - g > 0. \tag{3.26}$$

Claim (i) : $B > C$

Suppose $B \leq C \leq \frac{4}{3}$. The inequality $(1,1,3,1,1)$ holds which gives

$$1 + a + b + 4c - (1+c)^4(1-f)(1-g)(1+a)(1+b) - f - g > 0. \tag{3.27}$$

Left side of (3.27) is an increasing function of f.

If $A > B$, we use (3.22) to get $f < b - \frac{g}{2}$. Replace f by $b - \frac{g}{2}$ to get $\phi_1(g) = 1 + a + 4c - (1+c)^4(1 - b + \frac{g}{2})(1-g)(1+a)(1+b) - \frac{g}{2} > 0$. The second derivative of $\phi_1(g)$ is positive and $0 \leq g < b$ from (3.24). Therefore $\phi_1(g) \leq \max\{\phi_1(0), \phi_1(b)\}$. Now since $c \geq b$, we get $\phi_1(0) \leq 1 + a + 4b - (1+b)^4(1-b)(1+a)(1+b) < 0$ and $\phi_1(b) \leq 1 + a + 4b - (1+b)^4(1 - \frac{b}{2})(1-b)(1+a)(1+b) - \frac{b}{2} < 0$ for $0 < b \leq \frac{1}{3}$ and $b < a < 0.82$. This gives a contradiction.

If $A \leq B$, replace f by $\frac{a+b}{2} - \frac{g}{2}$ (from (3.21)) in (3.27) to get $\phi_2(g) = 1 + \frac{a+b}{2} + 4c - (1+c)^4(1 - \frac{a+b}{2} + \frac{g}{2})(1-g)(1+a)(1+b) - \frac{g}{2} > 0$. From (3.23), $0 \leq g < \frac{a+b}{2}$. Therefore $\phi_2(g) \leq \max\{\phi_2(0), \phi_2(\frac{a+b}{2})\}$. Now $\phi_2(0) = 1 + \frac{a+b}{2} + 4c - (1+c)^4(1 - \frac{a+b}{2})(1+a)(1+b) \leq 1 + \frac{a+b}{2} + 4b - (1+b)^4(1 - \frac{a+b}{2})(1+a)(1+b) < 0$ and $\phi_2(\frac{a+b}{2}) = 1 + \frac{a+b}{2} + 4c - (1+c)^4(1 - \frac{a+b}{2} + \frac{a+b}{4})(1 - \frac{a+b}{2})(1+a)(1+b) - \frac{a+b}{4} < 1 + \frac{a+b}{4} + 4b - (1+b)^4(1 - \frac{a+b}{4})(1 - \frac{a+b}{2})(1+a)(1+b) < 0$ for $0 < b \leq \frac{1}{3}$ and $0 \leq a \leq b$.

Claim (ii) : $B < 1.25$

Suppose $b \geq 0.25$.

Case(i) : $A > B$ We use here $f < b - \frac{g}{2}$, $g < b$ and find that $B^4FGA > (1+b)^4(1-\frac{b}{2})(1-b)(1+a) > 2$ for $a > b \geq 0.25$. Then $(1,4,1,1)$ holds which gives $A + 4B + F + G - 0.5B^5FGA > 7$. Left side is a decreasing function of F, so we replace F by $1 - b + \frac{g}{2}$ to get $\psi_1(g) = a + 3b - \frac{g}{2} - \frac{1}{2}(1+b)^5((1 - b + \frac{g}{2})(1-g)(1+a) > 0$. As $\psi_1''(g) > 0$ we get $\psi_1(g) \leq \max\{\psi_1(0), \psi_1(b)\} < 0$ for $0.25 \leq b < a < 0.82$.

Case(ii) : $A \leq B$ We use here $f < \frac{a+b}{2} - \frac{g}{2}$, $g < \frac{a+b}{2}$ and find that that $B^4FGA > (1+b)^4(1 - \frac{a+b}{4})(1 - \frac{a+b}{2})(1+a) > 2$ for $b \geq 0.25$ and $0 \leq a \leq b$. Working as in case (i), we find that $\psi_2(g) = a + 3.5b - 0.5a - \frac{g}{2} - \frac{1}{2}(1+b)^5((1 - \frac{a+b}{2} + \frac{g}{2})(1-g)(1+a) > 0$. As $\psi_2''(g) > 0$, one gets $\psi_2(g) \leq \max\{\psi_2(0), \psi_2(\frac{a+b}{2})\} < 0$ for $0.25 \leq b \leq 0.5$ and $0 \leq a \leq b$. This gives a contradiction.

Claim (iii) : $e + f + g > b$ and hence $d < \frac{a}{2}$, $d < \frac{b}{2}$

Suppose $e + f + g \leq b$. As $B > C$, we have $B^2 > CD$. therefore $(1,3,1,1,1)$ holds i. e. $A + 4B - B^4EFGA + E + F + G > 7$. Apply Lemma 6(ii) with $\gamma = e + f + g \leq x_1 = b \leq 0.5$ to get a contradiction. Now (3.25) and (3.26) gives $d < \frac{a}{2}$ and $d < \frac{b}{2}$.

Final Contradiction

Case(i) : $A \leq B$

From (3.21) and (3.23) we have $2f + g < a + b - 2d$, $g < \frac{a+b}{2} - d$. Adding these two we get $f + g < \frac{3(a+b)}{4} - \frac{3d}{2}$. Apply $(1,2,2,1,1)$ with A.M.-G.M. inequality to get $A + 4B + 4D + F + G - 4\sqrt{B^3D^3AFG} > 7$ which gives $4 + a + 4b - 4d - (f+g) - 4\sqrt{(1+b)^3(1-d)^3}\sqrt{(1+a)(1-f-g)} > 0$ which further implies $\theta_1(d) = 4 + a + 4b - 4d - \frac{3(a+b)}{4} + \frac{3d}{2} - 4\sqrt{(1+b)^3(1-d)^3}\sqrt{(1+a)(1 - \frac{3(a+b)}{4} + \frac{3d}{2})} > 0$. As $\theta_1''(d) > 0$, and $0 \leq d < \frac{a}{2}$ we have $\theta_1(d) \leq \max\{\theta_1(0), \theta_1(\frac{a}{2})\}$. Now $\theta_1(0) = 4 + a + 4b - \frac{3(a+b)}{4} - 4\sqrt{(1+b)^3}\sqrt{(1+a)(1 - \frac{3(a+b)}{4})} < 0$ and $\theta_1(\frac{a}{2}) = 4 - a + 4b - \frac{3(a+b)}{4} + \frac{3a}{4} - 4\sqrt{(1+b)^3(1-\frac{a}{2})^3}\sqrt{(1+a)(1 - \frac{3(a+b)}{4} + \frac{3a}{4})} < 0$ for $0 \leq a \leq b < 0.25$.

Case(ii): $A > B$

From (3.22) and (3.24) we have $2f + g < 2b - 2d$, $g < b - d$. Adding these two we get $f + g < \frac{3b}{2} - \frac{3d}{2}$. Applying $(1,2,2,1,1)$ with A.M.-G.M. inequality and working as in Case (i) we get $\theta_2(d) = 4 + a + 4b - 4d - \frac{3b}{2} + \frac{3d}{2} - 4\sqrt{(1+b)^3(1-d)^3}\sqrt{(1+a)(1 - \frac{3b}{2} + \frac{3d}{2})} > 0$. As $\theta_2''(d) > 0$, and $0 \leq d < \frac{b}{2}$ we have $\theta_2(d) \leq \max\{\theta_2(0), \theta_2(\frac{b}{2})\}$. Now $\theta_2(0) = 4 + a + 4b - \frac{3b}{2} - 4\sqrt{(1+b)^3}\sqrt{(1+a)(1 - \frac{3b}{2})} < 0$ and $\theta_2(\frac{b}{2}) = 4 + a + 2b - \frac{3b}{2} + \frac{3b}{4} - 4\sqrt{(1+b)^3(1-\frac{b}{2})^3}\sqrt{(1+a)(1 - \frac{3b}{2} + \frac{3b}{4})} < 0$ for $0 < b <$

$\min\{a, 0.25\}$ and $b < a < 0.82$. This gives a contradiction.

Remark : The proof of Case 8 for $A \geq B$ is much simpler than our earlier proof of the same, see Proposition 15 of [6].

4. INVESTIGATIONS UNDER THE CONDITIONS $A_1 < 1$

PROOF OF THEOREM 2 : Let $k > 1$ be any real number. Consider the reduced lattice $\mathbb{L}_k$ corresponding to the K-Z reduced form

$$\frac{1}{k}x_1^2 + x_2^2 + \cdots x_{n-1}^2 + kx_n^2.$$

It is clear that the square of the covering radius of $\mathbb{L}_k$ is

$$\frac{1}{4}(\frac{1}{k} + n - 2 + k) > \frac{n}{4}.$$

PROOF OF THEOREM 3 : For $n = 3$, the only case is $A_1 < 1, A_2 > 1, A_3 \leq 1$ and the result follows from the weak inequality $(1, 2)_w$.

For $n = 4$, the cases and the inequalities used to get the result are listed below :

Case	A	B	C	D	Inequalities	Lemma
1	$<$	$>$	$>$	$\leq$	$(1,3),(1,1,2)$	Lemma 7 with $Y_1 = B$
2	$<$	$>$	$\leq$	$\leq$	$(1,2,1)_w$	
3	$<$	$\leq$	$>$	$\leq$	$(1,1,2)_w$	

Let $n \geq 8$.

Let k be any real number satisfying $(\frac{4}{3})^{\frac{1}{n-1}} < k \leq \frac{4}{3}$. Let $\mathbb{L}_k$ be the K-Z reduced lattice corresponding to the quadratic form

$$\frac{4}{3k^{n-1}}x_1^2 + k\left\{x_2^2 + x_3^2 + \cdots + x_{n-2}^2 + (x_{n-1} + \frac{1}{2}x_n)^2 + \frac{3}{4}x_n^2\right\}.$$

Considering the covering of the point $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, 0, \frac{1}{2})$ we see that square of the covering radius of $\mathbb{L}_k$ is

$$\geq \tfrac{1}{4}\left\{\tfrac{4}{3k^{n-1}} + k\left\{n - 3 + \tfrac{1}{4} + \tfrac{3}{4}\right\}\right\}$$
$$= \tfrac{1}{4}\left\{\tfrac{4}{3k^{n-1}} + k(n-2)\right\} = f(k), \text{ say.}$$

Since $f(\frac{4}{3}) > \frac{n}{4}$ when $n \geq 8$, it follows that for k near $\frac{4}{3}$, $\mathbb{L}_k$ has the covering radius $> \sqrt{n}/2$.

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