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GENERALIZATIONS OF SOME HERMITE-HADAMARD-TYPE INEQUALITIES

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Recently some new Hermite-Hadamard-type inequalities for convex functions are established by Tseng *et al.* [Computers and Mathematics with Applications, **62**, 401-418, 2011]. In this paper, we give some generalizations of this result.

Key words : Hermite-Hadamard inequality; convex function.

1. Introduction

For a convex function $f : [a, b] \to \mathbb{R}$, it is known that [10] the following Hermite-Hadamard inequality holds

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(s)ds \leq \frac{f(a)+f(b)}{2}.$$

There have been lots of results that extend and generalize this well-known integral inequality [2]-[7], [9], [11], [14], [15], [17], [19]-[23]. Generalizations of this type of inequalities from the view point of operators can be found in [8] and [12]. Weighted Hermite-Hadamard inequalities are also of interested in the literature (see [1], [13], [18]).

This paper is closely related to a recent result [16]. In [16], by considering four points x, y, y', x' in $[a, b]$ such that $x < y \leq y' < x'$ and $x + x' = y + y'$, some new Hermite-Hadamard-type inequalities were proved. We give some extensions of these new inequalities from a weighted viewpoint.

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Throughout this paper, we suppose that f is a convex function on $[a,b]$ and $a \le x < y \le y' < x' \le b$. In addition, we impose the assumption $\lambda x + \lambda' x' = \lambda y + \lambda' y'$ for some $\lambda, \lambda' > 0$. When $\lambda = \lambda' = 1$, this reduces to the one imposed in [16].

We note that the analysis in [16] depends heavily on a lemma in [11]. Correspondingly, we have the following lemma which is the main tool for deriving our results.

Lemma 1.1 — Let $f : [a,b] \to \mathbb{R}$ be convex. Consider $a \le u < u' \le b$ and $v, v' \in [u, u']$ with $\lambda u + \lambda' u' = \lambda v + \lambda' v'$. Then

$$\lambda f(v) + \lambda' f(v') \le \lambda f(u) + \lambda' f(u').$$

PROOF : Note that

$$v = \frac{u'-v}{u'-u}u + \frac{v-u}{u'-u}u', \qquad v = \frac{u'-v'}{u'-u}u + \frac{v'-u}{u'-u}u'.$$

It follows from the convexity of f that

$$\lambda f(v) + \lambda' f(v') \le \lambda\left[\frac{u'-v}{u'-u}f(u) + \frac{v-u}{u'-u}f(u')\right] + \lambda'\left[\frac{u'-v'}{u'-u}f(u) + \frac{v'-u}{u'-u}f(u')\right]$$

$$= \frac{\lambda(u'-v)+\lambda'(u'-v')}{u'-u}f(u) + \frac{\lambda(v-u)+\lambda'(v'-u)}{u'-u}f(u') = \lambda f(u) + \lambda' f(u'). \qquad \square$$

As in [16], our main results are stated in terms of some functions which describe the average of f in a certain sense. In the next lemma, we give properties of these functions, some of which will be used in the next section.

Lemma 1.2 — The functions defined by

$$\mathcal{H}_1(t) = \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(ts+(1-t)y) + \lambda' f\left(t\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'} + (1-t)y'\right)ds,$$

$$\mathcal{P}_1(t) = \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(tx+(1-t)s) + \lambda' f\left(tx' + (1-t)\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right)ds,$$

$$\mathcal{G}_1(t) = \frac{1}{\lambda+\lambda'}\Big[\lambda f(tx+(1-t)y) + \lambda' f(tx'+(1-t)y')\Big],$$

are convex and increasing on $[0,1]$. In addition, the inequality $\mathcal{H}_1(t) \le \mathcal{G}_1(t) \le \mathcal{P}_1(t)$ holds.

PROOF : The convexity can be checked by direct calculation. To prove the monotonicity, let $0 \le t_1 < t_2 \le 1$.

Notice that

$$[t_1 s + (1-t_1)y] - [t_2 s + (1-t_2)y] = (t_1 - t_2)(s-y) > 0,$$

$$\left[t_1\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t_1)y'\right]-\left[t_2\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t_2)y'\right]=(t_1-t_2)\frac{\lambda(y-s)}{\lambda'}<0.$$

One can now apply Lemma 1.1 to get that $\mathcal{H}_1(t_1)<\mathcal{H}_1(t_2)$. This implies that $\mathcal{H}_1(t)$ is increasing. The corresponding conclusion for $\mathcal{P}_1(t)$ and $\mathcal{G}_1(t)$ can be proved similarly.

For $s\in[x,y]$, by Lemma 1.1, we have

$$\begin{aligned}&\lambda f(ts+(1-t)y)+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)\\ &\le \lambda f(tx+(1-t)y)+\lambda' f(tx'+(1-t)y')\\ &\le \lambda f(tx+(1-t)s)+\lambda' f\left(tx'+(1-t)\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\right)\end{aligned}$$

and the desired relation follows. □

Notice that when $\lambda=\lambda'=1$, the functions in Lemma 1.2 reduce to $H_1(t), P_1(t)$ and $G_1(t)$ defined in [16]. Similar extensions of the notations in [16] are used in the remaining of this paper. In the next section, we show that, for the functions which we introduce, the Hermite-Hadamard-type inequalities in [16] can be generalized.

2. Main Results

We are now ready to state and prove our main results. Throughout this paper, we denote $I=[x,y]$, $I'=[y',x']$, $I_1=\left[\frac{x+y}{2},y\right]$, $I_1'=\left[y',\frac{x'+y'}{2}\right]$, $I_2=\left[x,\frac{x+y}{2}\right]$ and $I_2'=\left[\frac{x'+y'}{2},x'\right]$. We first have the following:

***Theorem* 2.1** — *Denote*

$$A=\frac{1}{\lambda+\lambda'}\left(\lambda\frac{\int_I f(s)ds}{|I|}+\lambda'\frac{\int_{I'} f(s)ds}{|I'|}\right),\quad A'=\frac{1}{\lambda+\lambda'}\left(\lambda'\frac{\int_I f(s)ds}{|I|}+\lambda\frac{\int_{I'} f(s)ds}{|I'|}\right),$$

$$A_1=\frac{1}{\lambda+\lambda'}\left(\lambda\frac{\int_{I_1} f(s)ds}{|I_1|}+\lambda'\frac{\int_{I_1'} f(s)ds}{|I_1'|}\right).$$

Under our assumptions, the following inequalities hold:

$$\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}\le A_1\le\int_0^1\mathcal{H}_1(t)dt\le\frac{1}{2}\left(\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}+A\right),\tag{1}$$

$$\frac{1}{\lambda+\lambda'}\left(\lambda'\frac{\int_{J_1} f(s)ds}{|J_1|}+\lambda\frac{\int_{J_1'} f(s)ds}{|J_1'|}\right)\le\int_0^1\mathcal{H}_2(t)dt\le\frac{1}{2}\left[\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}+A'\right],\tag{2}$$

where $J_1 = [\frac{y'+x}{2}, \frac{y'+y}{2}]$, $J_1' = [\frac{y+y'}{2}, \frac{y+x'}{2}]$ *and*

$$\mathcal{H}_2(t) = \frac{1}{(\lambda+\lambda')(y-x)} \int_x^y \lambda' f(ts + (1-t)y') + \lambda f\left(t\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'} + (1-t)y\right) ds. \quad (3)$$

If f *is differentiable on* $[a,b]$, *for* $t \in [0,1]$, *we have*

$$0 \le A - \mathcal{H}_1(t) \le (1-t)\left[\frac{\lambda f(x) + \lambda' f(x')}{(\lambda+\lambda')} - A\right], \quad (4)$$

$$0 \le \frac{\lambda f(x) + \lambda' f(x')}{\lambda+\lambda'} - \mathcal{H}_1(t) \le \frac{\lambda'|I'|f'(x') - \lambda|I|f'(x)}{\lambda+\lambda'}, \quad (5)$$

$$0 \le \mathcal{H}_1(t) - \frac{\lambda f(y) + \lambda' f(y')}{\lambda+\lambda'} \le \frac{\lambda'|I'|f'(x') - \lambda|I|f'(x)}{\lambda+\lambda'}. \quad (6)$$

PROOF : We first prove (1). By Lemma 1.1, we have

$$\begin{aligned} \lambda f(y) + \lambda' f(y') &= \frac{1}{y-x}\int_x^y \lambda f(y) + \lambda' f(y') ds \\ &\le \frac{1}{y-x}\int_x^y \lambda f\Big(\frac{y+s}{2}\Big) + \lambda' f\Big(\frac{\lambda y + 2\lambda' y' - \lambda s}{2\lambda'}\Big) ds. \end{aligned}$$

This gives the first inequality in (1). The convexity of f yields

$$\begin{aligned} &\frac{2}{(\lambda+\lambda')(y-x)}\int_x^y\int_0^{\frac{1}{2}} \lambda f\Big(\frac{y+s}{2}\Big) + \lambda' f\left(\frac{\frac{\lambda y + 2\lambda' y' - \lambda s}{\lambda'}}{2}\right) dtds \\ &\le \frac{2}{(\lambda+\lambda')(y-x)}\int_x^y\int_0^{\frac{1}{2}} \frac{\lambda}{2}\Big[f(ty + (1-t)s) + f(ts + (1-t)y)\Big] \\ &\quad + \frac{\lambda'}{2}\left[f\left(t\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'} + (1-t)y'\right) + f\left(ty' + (1-t)\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right)\right] dtds, \end{aligned}$$

which is the second inequality in (1). By Lemma 1.1, for $s \in [x,y]$, we have

$$f(ty + (1-t)s) + f(ts + (1-t)y) \le f(s) + f(y),$$

$$\begin{aligned} &f\left(t\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'} + (1-t)y'\right) + f\left(ty' + (1-t)\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) \\ &\le f\left(\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) + f(y'). \end{aligned}$$

We can thus get that

$$\begin{aligned}
&\int_0^1 \mathcal{H}_1(t)dt \\
\le\ & \frac{1}{2}\Bigg(\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'} + \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(s)+\lambda' f\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)ds\Bigg) \\
=\ & \frac{1}{2}\Bigg(\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'} + \frac{1}{(\lambda+\lambda')(y-x)}\Big(\lambda\int_I f(s)ds + \frac{(\lambda')^2}{\lambda}\int_{I'} f(s)ds\Big)\Bigg) \\
=\ & \frac{1}{2}\Bigg(\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'} + \frac{\lambda\frac{\int_I f(s)ds}{|I|}+\lambda'\frac{\int_{I'} f(s)ds}{|I'|}}{\lambda+\lambda'}\Bigg).
\end{aligned}$$

The proof of (2) goes in the same spirit as that of (1) and can be concluded by the following:

$$\begin{aligned}
&\frac{2}{y-x}\int_x^y\int_0^{\frac{1}{2}} \frac{\lambda'}{\lambda+\lambda'} f\Big(\frac{y'+s}{2}\Big) + \frac{\lambda}{\lambda+\lambda'} f\Big(\frac{(\lambda+\lambda')y+\lambda' y'-\lambda s}{2\lambda'}\Big)dtds \\
\le\ & \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}} \frac{\lambda'}{\lambda+\lambda'}[f(ty'+(1-t)s)+f(ts+(1-t)y')] \\
&+\frac{\lambda}{\lambda+\lambda'}\Big[f\Big(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y\Big)+f\Big(ty+(1-t)\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)\Big]dtds \\
\le\ & \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}} \frac{\lambda'}{\lambda+\lambda'}(f(s)+f(y'))+\frac{\lambda}{\lambda+\lambda'}\Big(f(y)+f\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)\Big)dtds \\
=\ & \frac{1}{2(y-x)}\int_x^y \frac{1}{\lambda+\lambda'}(\lambda f(y)+\lambda' f(y'))+\frac{1}{\lambda+\lambda'}\Big(\lambda' f(s)+\lambda f\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)\Big)ds,
\end{aligned}$$

where the inequalities are consequences of Lemma 1.1.

To prove (4), note that the first inequality follows from Lemma 1.2 and $\mathcal{H}_1(1)=A$. If f is differentiable, by the convexity of f, for $s\in[x,y]$ and $t\in[0,1]$, we have

$$f(s)-f(ts+(1-t)y)\le(1-t)(s-y)f'(s), \tag{7}$$

$$\begin{aligned}
&f\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)-f\Big(t\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)+(1-t)y'\Big) \\
&\le(1-t)\Big(\frac{\lambda(y-s)}{\lambda'}\Big)f'\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big).
\end{aligned} \tag{8}$$

On the other hand, integration by part gives

$$\begin{aligned}
&\frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda(s-y)f'(s)+\lambda'\frac{\lambda(y-s)}{\lambda'}f'\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)ds \\
=\ & \frac{\lambda f(x)+\lambda' f(x')}{(\lambda+\lambda')}-\frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(s)+\lambda' f\Big(\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\Big)ds.
\end{aligned}$$

The second inequality in (4) follows by adding (7) to (8) and integrating the inequalities from x to y. Similarly, we have

$$f(x)-f(y)\le (x-y)f'(x),\quad f(x')-f(y')\le (x'-y')f'(x').$$

Thus

$$\frac{\lambda f(x)+\lambda' f(x')}{\lambda+\lambda'}-\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}\le \frac{\lambda}{\lambda+\lambda'}(x-y)f'(x)+\frac{\lambda'}{\lambda+\lambda'}(x'-y')f'(x').$$

Note that, by Lemma 1.2, we have

$$\frac{\lambda f(y)+\lambda' f(y')}{(\lambda+\lambda')}=\mathcal{H}_1(0)\le\mathcal{H}_1(t)\le\mathcal{H}_1(1),$$

$$\mathcal{P}_1(0)\le\mathcal{P}_1(t)\le\mathcal{P}_1(1)=\frac{\lambda f(x)+\lambda' f(x')}{(\lambda+\lambda')}$$

and $\mathcal{H}_1(1)=\mathcal{P}_1(0)$. This gives the last two inequalities in the theorem. □

One can easily see that, when $\lambda=\lambda'=1$, Theorem 2.1 reduces to Theorem 1 in [16]. We will generalize Theorem 2-4 of [16] in a similar way. These are given in the following theorems.

Remark 2.1 : If we define

$$\widetilde{\mathcal{H}}_2(t)=\frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(ts+(1-t)y')+\lambda' f\big(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y\big)ds,$$

we can show that

$$\begin{aligned} f(\frac{y+y'}{2}) &\le \frac{1}{\lambda+\lambda'}(\lambda\frac{\int_{J_1}f(s)ds}{|J_1|}+\lambda'\frac{\int_{J_1'}f(s)ds}{|J_1'|}) \\ &\le \int_0^1\mathcal{H}_2(t)dt\le\frac{1}{2}[\frac{\lambda' f(y)+\lambda f(y')}{\lambda+\lambda'}+A] \end{aligned}$$

which gives a complete generalization of the inequalities for $H_2(t)$ in [16]. However, by using the definition (3), we can obtain a generalization of Theorem 4 in [16], while this cannot be achieved by using $\widetilde{\mathcal{H}}_2(t)$.

***Theorem* 2.2** — *Denote* $A_2=\dfrac{1}{\lambda+\lambda'}\Big(\lambda\frac{\int_{I_2}f(s)ds}{|I_2|}+\lambda'\frac{\int_{I_2'}f(s)ds}{|I_2'|}\Big)$. *We have*

$$A\le A_2\le\int_0^1\mathcal{P}_1(t)dt\le\frac{1}{2}\Big(\frac{\lambda f(x)+\lambda' f(x')}{\lambda+\lambda'}+A\Big). \tag{9}$$

If f is differentiable on $[a,b]$, then

$$0 \le t\left[A - \frac{\lambda f(y)+\lambda' f(y')}{(\lambda+\lambda')}\right] \le \mathcal{P}_1(t) - A,$$

$$0 \le \mathcal{P}_1(t) - \frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'} \le \frac{\lambda'|I'|f'(x') - \lambda|I|f'(x)}{\lambda+\lambda'},$$

$$0 \le \frac{\lambda f(x)+\lambda' f(x')}{\lambda+\lambda'} - \mathcal{P}_1(t) \le \frac{\lambda'|I'|f'(x') - \lambda|I|f'(x)}{\lambda+\lambda'},$$

$$0 \le \mathcal{P}_1(t) - \mathcal{H}_1(t) \le \frac{\lambda'|I'|f'(x') - \lambda|I|f'(x)}{\lambda+\lambda'}.$$

Proof : The proof of (9) is similar to that of (1). We summarize it as follows.

$$\begin{aligned}
&\frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f(s) + \lambda' f\left(\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) ds \\
\le\ & \frac{2}{(\lambda+\lambda')(y-x)}\int_x^y \int_0^{\frac{1}{2}} \lambda f\left(\frac{x+s}{2}\right) + \lambda' f\left(\frac{\lambda x + 2\lambda' x' - \lambda s}{2\lambda'}\right) dtds \\
\le\ & \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \int_0^{\frac{1}{2}} \lambda f(tx+(1-t)s) + \lambda f((1-t)x+ts) \\
& +\lambda' f\left(tx' + (1-t)\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) + \lambda' f\left((1-t)x' + t\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) dtds \\
\le\ & \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \int_0^{\frac{1}{2}} \lambda(f(s)+f(x)) + \lambda'\left[f\left(\frac{\lambda y + \lambda' y' - \lambda s}{\lambda'}\right) + f(x')\right] dtds,
\end{aligned}$$

where all the inequalities can be obtained by Lemma 1.1.

For the remaining of the theorem, notice that by the convexity of f, we have

$$f(tx+(1-t)s) - f(s) \ge t(x-s)f'(s),$$

$$\begin{aligned}
& f\left(tx' + (1-t)\left(\frac{\lambda x + \lambda' x' - \lambda s}{\lambda'}\right)\right) - f\left(\frac{\lambda x + \lambda' x' - \lambda s}{\lambda'}\right) \\
\ge\ & t\left(\frac{\lambda(s-x)}{\lambda'}\right) f'\left(\frac{\lambda x + \lambda' x' - \lambda s}{\lambda'}\right).
\end{aligned}$$

We can follow the proof of (4)-(6) to get the conclusion. □

Our next task is to give a generalization of Theorem 3 in [16].

Theorem 2.3 — *Under our assumptions, the following inequalities hold:*

$$A_1 \le \frac{\lambda f\left(\frac{y+x}{2}\right) + \lambda' f\left(\frac{x'+y'}{2}\right)}{(\lambda+\lambda')} \le \int_0^1 \mathcal{G}_1(t)dt \le \frac{1}{2}\left(\frac{\lambda f(x)+\lambda' f(x')}{\lambda+\lambda'} + \frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}\right), \quad (10)$$

$$\begin{aligned}\frac{1}{\lambda+\lambda'}\left(\lambda\frac{\int_{J_2} f(s)ds}{|J_2|}+\lambda'\frac{\int_{J_2'} f(s)ds}{|J_2'|}\right) &\leq \frac{\lambda f\left(\frac{x'+y}{2}\right)+\lambda' f\left(\frac{x+y'}{2}\right)}{\lambda+\lambda'}\\ &\leq \int_0^1 \mathcal{G}_2(t)dt \\ &\leq \frac{1}{2}\left[\frac{\lambda f(x')+\lambda' f(x)}{\lambda+\lambda'}+\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}\right],\end{aligned} \tag{11}$$

where $J_2=[\frac{x'+x}{2},\frac{x'+y}{2}]$, $J_2'=[\frac{x+y'}{2},\frac{x+x'}{2}]$ *and* $\mathcal{G}_2(t)=\frac{\lambda f(tx'+(1-t)y)+\lambda' f(tx+(1-t)y')}{\lambda+\lambda'}$.

Moreover, one has

$$0\leq \mathcal{H}_1(t)-\frac{\lambda f(y)+\lambda' f(y')}{\lambda+\lambda'}\leq \mathcal{G}_1(t)-\mathcal{H}_1(t), \tag{12}$$

and

$$0\leq \mathcal{P}_1(t)-\mathcal{G}_1(t)\leq \frac{\lambda f(x)+\lambda' f(x')}{\lambda+\lambda'}-\mathcal{P}_1(t). \tag{13}$$

Proof : The inequality (10) can be concluded by

$$\begin{aligned}&\frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f\left(\frac{y+s}{2}\right)+\lambda' f\left(\frac{\lambda y+\lambda' y'+\lambda' y'-\lambda s}{2\lambda'}\right)ds\\ &\leq \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda f\left(\frac{y+x}{2}\right)+\lambda' f\left(\frac{x'+y'}{2}\right)ds\\ &\leq \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}} \frac{\lambda[f(tx+(1-t)y)+f(ty+(1-t)x)]}{\lambda+\lambda'}\\ &\qquad+\frac{\lambda'[f(tx'+(1-t)y')+f(ty'+(1-t)x')]}{\lambda+\lambda'}\,dt\,ds\\ &\leq \frac{1}{y-x}\int_x^y\int_0^{\frac{1}{2}}\frac{\lambda[f(x)+f(y)]+\lambda'[f(x')+f(y')]}{\lambda+\lambda'}\,dt\,ds,\end{aligned}$$

where we have used Lemma 1.1 for the first and the third inequalities, while the second one is obtained by the convexity of f.

Now we turn to prove (11). We have

$$\begin{aligned}&\frac{1}{y-x}\int_x^y \frac{\lambda}{\lambda+\lambda'}f\left(\frac{x'+s}{2}\right)+\frac{\lambda'}{\lambda+\lambda'}f\left(\frac{(\lambda+\lambda')x+\lambda' x'-\lambda s}{2\lambda'}\right)ds\\ &\leq \frac{1}{y-x}\int_x^y \frac{1}{\lambda+\lambda'}\left[\lambda f\left(\frac{x'+y}{2}\right)+\lambda' f\left(\frac{x+y'}{2}\right)\right]ds\\ &\leq \frac{1}{\lambda+\lambda'}\int_0^{\frac{1}{2}} \lambda[f(ty+(1-t)x')+f(tx'+(1-t)y)]\\ &\qquad+\lambda'[f(ty'+(1-t)x)+f(tx+(1-t)y')]dt\\ &\leq \frac{1}{\lambda+\lambda'}\int_0^{\frac{1}{2}} \lambda[f(x')+f(y)]+\lambda'[f(x)+f(y')]dt,\end{aligned}$$

where the first and the third inequalities are once again consequences of Lemma 1.1, while the middle is by convexity.

To prove (12), notice that

$$\int_x^y f(ts+(1-t)y)ds = \int_x^y f(t(x+y-s)+(1-t)y]ds,$$

$$\int_x^y f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)ds = \int_x^y f\left(t\frac{\lambda' y'-\lambda x+\lambda s}{\lambda'}+(1-t)y'\right)ds.$$

Thus

$$\begin{aligned}
&2\mathcal{H}_1(t)\\
&= \frac{1}{(\lambda+\lambda')(y-x)}\left[\lambda\int_x^y f(ts+(1-t)y)ds + f(t(x+y-s)+(1-t)y)ds\right.\\
&\qquad \left.+\lambda'\int_x^y f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)ds + f\left(t\frac{\lambda' y'-\lambda x+\lambda s}{\lambda'}+(1-t)y'\right)ds\right]\\
&\le \frac{1}{(\lambda+\lambda')(y-x)}\int_x^y \lambda[f(tx+(1-t)y)+f(y)]+\lambda'[f(y')+f(tx'+(1-t)y')]ds,
\end{aligned}$$

which is a consequence of Lemma 1.1. The inequality (13) can be proved similarly. □

Finally we conclude this paper with the following theorem.

Theorem 2.4 — *Let* $\mathcal{L}_1(t) = \frac{1}{2}\left(\frac{\int_I f(s;t)ds}{|I|} + \frac{\int_{I'} f(s;t)ds}{|I'|}\right)$,

$$\begin{aligned}
\mathcal{F}_1(t) = \frac{1}{2(\lambda+\lambda')}&\left[\frac{1}{|I|}\int_I\left(\lambda\frac{\int_I f(s,u;t)du}{|I|}+\lambda'\frac{\int_{I'} f(s,u;t)du}{|I'|}\right)ds\right.\\
&\left.+\frac{1}{|I'|}\int_{I'}\left(\lambda\frac{\int_I f(s,u;t)du}{|I|}+\lambda'\frac{\int_{I'} f(s,u;t)du}{|I'|}\right)ds\right],
\end{aligned}$$

where

$$f(s;t) = \frac{\lambda f(tx+(1-t)s)+\lambda' f(tx'+(1-t)s)}{\lambda+\lambda'}, \quad f(s,u;t) = f(ts+(1-t)u).$$

Then $\mathcal{L}_1(t)$ *is convex on* $[0,1]$ *and*

$$\frac{\mathcal{H}_1(t)+\mathcal{H}_2(t)}{2} \le \mathcal{F}_1(t) \le \mathcal{L}_1(1-t), \tag{14}$$

$$0 \le \mathcal{F}_1(t) - \frac{1}{2}(\mathcal{H}_1(t)+\mathcal{H}_2(t)) \le \mathcal{L}_1(1-t) - \mathcal{F}_1(t). \tag{15}$$

PROOF : The convexity of $\mathcal{L}_1$ can be checked by direct calculation. To prove (14), notice that Lemma 1.1 implies

$$\begin{aligned}
&\lambda f(ts+(1-t)y) + \lambda' f(ts+(1-t)y')\\
&\le \lambda f(ts+(1-t)u) + \lambda' f\left(ts+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right),
\end{aligned}$$

$$\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y\right)+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)$$
$$\leq \lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)u\right)+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right).$$

Thus the inequalities can be concluded by

$$\begin{aligned}
&\frac{1}{2}(\mathcal{H}_1(t)+\mathcal{H}_2(t))\\
&=\frac{1}{2(\lambda+\lambda')(y-x)}\int_x^y\left[\lambda f(ts+(1-t)y)+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)\right]\\
&\quad+\left[\lambda' f(ts+(1-t)y')+\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y\right)\right]ds\\
&\leq\frac{1}{2(\lambda+\lambda')(y-x)^2}\int_x^y\int_x^y\left[\lambda f(ts+(1-t)u)+\lambda' f\left(ts+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right)\right]du\\
&\quad+\int_x^y\left[\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)u\right)\right.\\
&\qquad\left.+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right)\right]duds\\
&=\mathcal{F}_1(t)\\
&\leq\frac{1}{2(\lambda+\lambda')(y-x)}\int_x^y[\lambda f(ts+(1-t)x)+\lambda' f(ts+(1-t)x')]\\
&\quad+\left[\lambda f\left((1-t)x+t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\right)+\lambda' f\left((1-t)x'+t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}\right)\right]ds\\
&=\mathcal{L}_1(1-t),
\end{aligned}$$

where the last inequality once again follows from Lemma 1.1.

For (15), note that

$$\begin{aligned}
&\int_x^y\left[\lambda f(ts+(1-t)u)+\lambda' f\left(ts+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right)\right]du\\
&=\int_x^{\frac{x+y}{2}}\Big[\lambda f(ts+(1-t)u)+\lambda f(ts+(1-t)(x+y-u))\\
&\quad+\lambda' f\left(ts+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right)+\lambda' f\left(ts+(1-t)\frac{-\lambda x+\lambda' y'+\lambda u}{\lambda'}\right)\Big]du\\
&\leq\int_x^{\frac{x+y}{2}}[\lambda f(ts+(1-t)x)+\lambda f(ts+(1-t)y)\\
&\qquad+\lambda' f(ts+(1-t)y')+\lambda' f(ts+(1-t)x')]du\\
&=\frac{y-x}{2}[\lambda f(ts+(1-t)x)+\lambda f(ts+(1-t)y)\\
&\qquad+\lambda' f(ts+(1-t)y')+\lambda' f(ts+(1-t)x')].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_x^y \Bigg[\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)u\right)\\&\qquad+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)\frac{\lambda y+\lambda' y'-\lambda u}{\lambda'}\right)\Bigg]du\\&\le \frac{y-x}{2}\Bigg[\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)x\right)+\lambda f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y\right)\\&\quad+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)y'\right)+\lambda' f\left(t\frac{\lambda y+\lambda' y'-\lambda s}{\lambda'}+(1-t)x'\right)\Bigg].\end{aligned}$$

These imply $\mathcal{F}_1(t) \le \frac{1}{2}[\mathcal{L}_1(1-t)+\frac{1}{2}(\mathcal{H}_1(t)+\mathcal{H}_2(t))]$ and our result follows. □

We remark that, under our formulation, we cannot generalize all parts of the theorems in [16]. It will be our future study to get full generalizations of these theorems and further extensions of the inequalities involved.

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