Indian J. Pure Appl. Math., **46**(3): 371-381, June 2015

DOI:10.1007/s13226-015-0129-4

CHARACTERIZATIONS OF COMMUTATIVE MAX RINGS AND SOME APPLICATIONS

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(*Received* 11 *May* 2013; *after final revision* 22 *October* 2013;
accepted 20 *August* 2014)

Several new characterizations are given for a commutative max ring of which many are categorical.

Key words : Max ring; nil-bounded; semi-Artinian ring; T-faithful.

1. INTRODUCTION

Rings are associative with an identity and modules are unitary. Ring homomorphisms are assumed to preserve the identity. Let R be a ring, M a nonzero (left) R-module. The module M is said to be a max module if any nonzero submodule of M has a maximal submodule. A ring R is called (left) max if all nonzero left R-modules are max modules. Bass [2] proved that a left perfect ring is a left max ring and it has no infinite set of orthogonal idempotents. The converse statement, often referred to as Bass conjecture, has been proved false; see [7] and [8]. However, Bass conjecture motivated the research on max rings, first for the case of commutative rings. If R is commutative, then by [7, 8] and [6] the following statements are equivalent:

1 - R is a max ring.

2 - $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is T-nilpotent.

3 - For any maximal ideal P of R, the local ring R_P is perfect.

4 - The local ring R_P is a max ring for any maximal ideal P of R.

Some characterizations of a (not necessarily commutative) max ring from [5, 7] and [8] are mentioned below:

i) R is a left max ring.

ii) $R/\mathrm{J}(R)$ is a left max ring and $\mathrm{J}(R)$ is left T-nilpotent.

iii) The injective envelop of any simple left R module is a left max module.

iv) There is a cogenerator in R-Mod which is a left max module.

Further information on max rings (also called Bass rings), and related topics may be found in [11] and [3]. In this paper we wish to contribute to the study of commutative max rings by offering several new characterizations of such rings, often in terms of the Hom and/or Tensor functors; see Theorem 2.7 below. Also as applications, a new characterization for a commutative semi-Artinian ring, and generalization to rings Morita equivalent to commutative max rings are given. We use standard notation, and refer the readers to [1] and [9] for background material. Let A and B be rings and $\varphi : A \to B$ be a ring homomorphism. If $M \in$ Mod-B and $N \in B$-Mod, then it is known that every B-module has an A -module structure induced by φ, and there exists a surjective additive homomorphism $M \otimes_A N \to M \otimes_B N$. We shall often make use of the implication $M \otimes_B N \neq 0 \Longrightarrow M \otimes_A N \neq 0$. A nonempty subset X of A is called ***nil-bounded*** whenever there exists a positive integer n such that $x^n = 0$ for all $x \in X$.

Standing Hypothesis

In the rest of this paper, R will denote a commutative ring. As usual, the prime radical of R is denoted by $\mathrm{N}(R)$.

2. The Characterization Theorem

We begin with a few preparatory results.

Lemma 2.1 — Let $\bar{R} = R/\mathrm{Soc}(R)$. Then $\mathrm{N}(R)$ is nil-bounded if and only if so is $\mathrm{N}(\bar{R})$.

PROOF : Denote the socle and the prime radical of R by S and N respectively. We claim that $\mathrm{N}(\bar{R}) = (N + S)/S$. For, let $\bar{x} = x + S \in \mathrm{N}(\bar{R})$ with $x^n \in S$ for some $n \geq 1$. Since a minimal ideal is generated by an idempotent or it has zero square, we can write $S = (N \cap S) \oplus K$, where K is the sum of minimal ideals each of which is generated by an idempotent, and $(N \cap S)^2 = 0$. Thus $x^{2n} \in K$, and consequently there is an idempotent e with $x^{2n} \in Re$. Clearly $x = re + t(1 - e)$ for some elements r and t in R. Hence $t^{2n}(1 - e) = 0$, thus $t(1 - e) \in N$, proving that $x \in N + S$, i.e., $\bar{x} \in (N + S)/S$. Evidently $(N + S)/S \subseteq \mathrm{N}(\bar{R})$, and so the claim is established. If now $\mathrm{N}(\bar{R})$ is nil bounded, since $(N \cap S)^2 = 0$ and $(N + S)/S \simeq N/(N \cap S)$, we see that N is also nil bounded. The converse is also clear by the claim. □

Proposition 2.2 — Consider the following statements.

(1) R is a max ring.

(2) $M \otimes_R M \neq 0$, for all nonzero R-modules M.

(3) K.dim$(R) = 0$ and every nil-bounded subset of R is T-nilpotent.

Then (1)⇒(2)⇒(3). If further, N(R) is nil-bounded, then all the statements are equivalent.

PROOF : (1)⇒(2). Let M be a nonzero R-module and K be a maximal submodule of M. Then $M \otimes_R M \neq 0$, because, we have $M \otimes_R M \overset{epi}{\to} M/K \otimes_R M/K \simeq M/K$.

(2)⇒(3). We first show that K.dim$(R) = 0$, that is, any prime ideal P of R is maximal. Notice that if $A, B \in$ Mod-R', where R' is a factor ring of R, then $A \otimes_{R'} B \simeq A \otimes_R B$. Thus in particular every prime factor ring of R inherits the property stated in (2). We may then assume that R is a domain and $M \otimes_R M \neq 0$ for all nonzero R-modules M, in order to prove that R is a field. Let Q be the quotient field of R. Since Q is a divisible R-module and Q/R is a torsion R-module, $Q \otimes_R Q/R = 0$. So by the right exactness of tensor functor, $Q/R \otimes_R Q/R = 0$, yielding $R = Q$. Thus every prime ideal of R is maximal.

Now let H be a subset of R such that for a fixed n, we have $h^n = 0$ for all $h \in H$. Let $\{h_i\}_{i=1}^{\infty}$ be a sequence in H, F be a free R-module with basis $\{x_i\}_{i=1}^{\infty}$ and F' be the submodule of F generated by the elements $\{x_i - h_i x_{i+1}\}_{i=1}^{\infty}$. Let $K_i = F/F'$ for $i = 1, ..., n$. Then by (2), $K := K_1 \otimes_R ... \otimes_R K_n \neq 0$ provided that $F \neq F'$. Consider an arbitrary generator $\bar{x}_{i_1} \otimes ... \otimes \bar{x}_{i_n}$ of K. Without loss of generality suppose i_n is the largest index in $\{i_1, ..., i_n\}$. Since $\bar{x}_{i_k} = h_{i_n} ... h_{i_k} \bar{x}_{i_n+1}$ for all $i_k \in \{i_1, ..., i_n\}$,

$$\bar{x}_{i_1} \otimes ... \otimes \bar{x}_{i_n} = h_{i_n} ... h_{i_1} \bar{x}_{i_n+1} \otimes ... \otimes h_{i_n} \bar{x}_{i_n+1}$$

$$= h_{i_n-1} ... h_{i_1} \bar{x}_{i_n+1} \otimes ... \otimes (h_{i_n})^n \bar{x}_{i_n+1} = 0.$$

Hence $K = 0$, a contradiction unless $F = F'$. So there exist $r_1, ..., r_t \in R$ such that $x_1 = \sum_{i=1}^{t} r_i (x_i - h_i x_{i+1})$. It follows that $h_1 ... h_t = 0$ and so H is T-nilpotent.

Now suppose that N(R) is nil-bounded. We complete the proof by showing that (3)⇒(1). From (3) we have J(R) = N(R) which is T-nilpotent. On the other hand, the commutative reduced ring $R/$J(R) with zero Krull dimension is regular and hence a max ring [9, Theorem 3.71]. This implies that R is a max ring. □

Corollary 2.3 — Suppose that $\mathrm{N}(R/\mathrm{Soc}(R))$ is a finitely generated R-module. Then R is a max ring if and only if $M \otimes_R M \neq 0$, for all nonzero R-modules M.

PROOF : Use Lemma 2.1 and Proposition 2.2. □

Remark 2.4 : By the proof of Proposition 2.2 and [1, Remark 28.5], we have an alternative proof of Hamsher's characterization of commutative max rings (mentioned in the Introduction as (1)⟺(2)).

Lemma 2.5 — Suppose that $\mathrm{J}(R)$ is T-nilpotent. Then for any nonzero R-module M, there exists $x \in R$ such that xM is a nonzero $R/\mathrm{J}(R)$-module.

PROOF : We only need to consider the case $JM \neq 0$ where $J = \mathrm{J}(R)$. Thus there exists an element $x_1 \in J$ such that $M_1 := x_1M \neq 0$. If $JM_1 \neq 0$, choose $x_2 \in J$ with $M_2 := x_2M_1 \neq 0$, and so $x_2x_1M \neq 0$. Continuing this process, by the T-nilpotence of J, we obtain an element $x := x_n ... x_1$ in J with $xM \neq 0$ but $JxM = 0$. □

Lemma 2.6 — Let $N \leq {}_RM$ such that $M/N \otimes_R M/N \neq 0$. Then there is an element $\overline{m} \in M/N$ such that $\mathrm{ann}_R(\overline{m})M$ is a proper submodule of M.

PROOF : Assume the contrary statement that $\mathrm{ann}_R(\overline{m})M = M$ for any $\overline{m} \in M/N$. We reach a contradiction to $M/N \otimes_R M/N \neq 0$, by showing that $M \otimes_R M/N = 0$. Consider an arbitrary generator $a \otimes \bar{b}$ where $a \in M$ and $\bar{b} = b + N \in M/N$. We can write $a = \sum_{i=1}^{t} x_i a_i$ where $x_i \in \mathrm{ann}_R(\bar{b})$ and $a_i \in M$ for $i = 1, ..., t$. Now

$$a \otimes \bar{b} = (\sum_{i=1}^{t} x_i a_i) \otimes \bar{b} = \sum_{i=1}^{t} (a_i \otimes x_i \bar{b}) = 0,$$

since $x_i b \in N$ for each $i = 1, ..., t$. □

Let B be a ring and M be a B-module. The full subcategory of B-modules whose objects are all B-modules subgenerated by M will be denoted by $\sigma[M]$. Also set:

$$\mathfrak{F}_M = \{I \mid I \text{ is an ideal of } B \text{ such that } I/\mathrm{ann}_B(M) \leq_e B/\mathrm{ann}_B(M) \text{ and } IM \neq M\}.$$

***Theorem* 2.7** — *The following are equivalent statements for a commutative ring R.*

(1) *R is a max ring.*

(2) *For all R-modules M and N, if $M \otimes_R N = 0$, then Hom$_R(N, M) = 0$.*

(3) *If I is an ideal in R and $IM = M$ for $M \in R$-Mod, then $IN = N$ for all $N \leq M$.*

(3′) *If I is an ideal in R and $IM = M$ for $M \in R$-Mod, then $IN = N$ for all $N \in \sigma[M]$.*

(4) *For any R-module M, ann$_R(X)M \neq M$ where $0 \neq X$ is a nonempty subset of M.*

(4′) *For any R-module M and $0 \neq N \in \sigma[M]$, ann$_R(N)M \neq M$.*

(5) *If an R-module M has a submodule N such that Hom$_R(N, S) \neq 0$ for some simple R-module S, then Hom$_R(M, S) \neq 0$.*

(5′) *For every R-module M, if Hom$_R(N, S) \neq 0$ for some nonzero $N \in \sigma[M]$, where S is a simple R-module, then Hom$_R(M, S) \neq 0$.*

(6) *For any nonzero R-module M, Rej$_R(M, S)$ is a proper submodule of M, for every simple R-module S in $\sigma[M]$.*

(7) *For any R-module M, if Soc$(M) \neq 0$, then $M \otimes_R$Soc$(M) \neq 0$.*

(8) *For any R-module M, $M \otimes_R N \neq 0$ for all nonzero submodules N of M.*

(9) *For all R-modules $N \leq M$ and K, if $N \otimes_R K \neq 0$, then $M \otimes_R K \neq 0$.*

(10) *The set $\mathfrak{F}_M$ has a maximal member for every non semisimple R-module M.*

(11) *For any nonzero R-module M, Hom$_R(M/\mathrm{J}(M), M) \neq 0$.*

(12) *For any nonzero R-module M, there exists $N < M$ such that Soc$(R/I) \neq 0$ with $I =$ ann$_R(M/N)$.*

PROOF : (1)⇒(2). Suppose Hom$_R(N, M) \neq 0$ in order to show that $M \otimes_R N \neq 0$. It is enough to prove that $M' \otimes_R S \neq 0$ for some homomorphic images M' of M and S of N. Let f be a nonzero element of Hom$_R(N, M)$. Then by (1), there exists a maximal submodule K of $f(N)$, hence $S := f(N)/K$ embeds essentially in some factor M' of M. As observed before, $R/\mathrm{J}(R)$ is a regular ring and $\mathrm{J}(R)$ is T-nilpotent. By Lemma 2.5, there exists $x \in R$ such that xM' is a nonzero $R/\mathrm{J}(R)$-module. Since S is an $R/\mathrm{J}(R)$-submodule of xM', we have

$$S \simeq S \otimes_{R/J(R)} S \hookrightarrow xM' \otimes_{R/J(R)} S.$$

It follows that $xM' \otimes_R S \neq 0$, from which by the epimorphism $M' \otimes_R S \to xM' \otimes_R S$, we deduce that $M' \otimes_R S \neq 0$. Therefore $M \otimes_R N \neq 0$.

(2)⇒(3). Let I be a proper ideal of R such that $IN \neq N$ for some $N \leq M$. Since Hom$_R(N/IN, M/IN) \neq 0$, by assumption $N/IN \otimes_R M/IN \neq 0$, hence $IM \neq M$.

(3)⇒(4). Let $X \neq \varnothing$ be a nonzero subset of an R-module M and $I = \text{ann}_R(X)$. We have $\text{ann}_R(X) = \text{ann}_R(N)$ where N is the submodule of M generated by X. Hence by (3), $IM \neq M$.

(4)⇒(5). Suppose that $\text{Hom}_R(N, R/P) \neq 0$ for some maximal ideal P of R. Hence $PN \neq N$ and so $P \leq \text{ann}_R(N/PN)$. Thus by hypothesis $PM' \neq M'$, where $M' = M/PN$. Therefore M/PM is a nonzero vector space over R/P and so $\text{Hom}_R(M, R/P) \neq 0$.

(5)⇒(6). It is enough to show that $\text{Hom}_R(M, S) \neq 0$ for all simple S in $\sigma[M]$. We have $S \leq M^{(I)}/W$ for some set I and $W \leq M^{(I)}$. Since $\text{Hom}_R(S, S) \neq 0$, by hypothesis, $\text{Hom}_R(M^{(I)}/W, S) \neq 0$. It follows that $\text{Hom}_R(M, S) \neq 0$.

(6)⇒(7). Let S be a simple submodule of M. By (6), $\text{Rej}_R(M, S) \neq M$, so there exists a submodule $W < M$ such that $M/W \simeq S$. Since $S \otimes_R S \simeq S$, we have $M \otimes_R S \neq 0$, and so, $M \otimes_R \text{Soc}(M) \neq 0$.

(7)⇒(8). Suppose that N is a submodule of M and let $0 \neq x \in N$. Let K be a maximal submodule of Rx. Then Rx/K essentially embeds in suitable factor modules M' of M and N' of N. Hence by (7), $M' \otimes_R Rx/K$ and $N' \otimes_R Rx/K$ are nonzero vector spaces over R/P where $R/P \simeq Rx/K$. Thus $(M' \otimes_R Rx/K) \otimes_R (N' \otimes_R Rx/K) \neq 0$ and it follows that $M \otimes_R N \neq 0$.

(8)⇒(9). Let N be a submodule of M such that $N \otimes_R K \neq 0$ for some R-module K. Let S be an essential simple submodule of some factor of $N \otimes_R K$. Thus by (8) and the right exactness of tensor functor, $(N \otimes_R K) \otimes_R S \neq 0$. Since $(N \otimes_R S) \otimes_R (K \otimes_R S) \simeq (N \otimes K) \otimes_R S$, we have N/PN and K/PK are nonzero vector spaces over R/P where $S \simeq R/P$. Thus there exists a nonzero factor module W of K such that $W \leq N/PN$. Hence W can be essentially embedded in some factor M' of M, and so, $M' \otimes_R W \neq 0$ by the hypothesis. It follows that $M \otimes_R K \neq 0$.

(9)⇒(10). Suppose that R_M is not semisimple and let N be a proper essential submodule of M. Pick a nonzero element x in M which is not in N. Then, from $R\bar{x} \otimes_R R\bar{x} \simeq R\bar{x}$, by using (9) we get $M/N \otimes_R M/N \neq 0$. Hence by Lemma 2.6, $\text{ann}_R(\overline{m})M \neq M$ for some $\overline{m} \in M/N$. Therefore $\mathfrak{F}_M \neq \varnothing$. Now let $I \in \mathfrak{F}_M$, and note that $M' := M/IM \neq 0$. Let $0 \neq y \in M'$, with $Ry \simeq R/K$. Clearly $I \leq K \leq P$ for some maximal ideal P. Now $Ry \otimes_R R/P \simeq R/K \otimes_R R/P \simeq R/(K+P) = R/P \neq 0$. Thus by (9), $M/PM \simeq M \otimes_R R/P \neq 0$. It follows that P is a maximal member of $\mathfrak{F}_M$.

(10)⇒(1). Let M be an R-module that is not semisimple and I be a maximal member of $\mathfrak{F}_M$. If $M' := M/IM$ is not semisimple, then by the hypothesis $\mathfrak{F}_{M'}$ has a maximal member I' such that $0 \neq I'M' \neq M'$. Therefore $I' \in \mathfrak{F}_M$ and $I < I'$, a contradiction. Thus M/IM is a nonzero semisimple R-module, hence M has a maximal submodule.

(1)⇒(11). By (1), we know that $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is T-nilpotent. For every nonzero R-module M we have $\mathrm{J}(M) = \mathrm{J}(R)M$, and by Lemma 2.5, there exists $x \in R$ such that xM is a nonzero $R/\mathrm{J}(R)$-module. Now multiplication by x defines a nonzero homomorphism $f_x : M \to M$ such that $\mathrm{J}(M) = \mathrm{J}(R)M \leq \mathrm{Ker} f_x$. Thus f_x induces a nonzero R-homomorphism $\bar{f} : M/\mathrm{J}(M) \to M$.

(11)⇒(12). Let M be an R-module and assume that $\mathrm{Hom}_R(M/\mathrm{J}(M), M) \neq 0$. Thus $\mathrm{J}(M) \neq M$, which implies that M has a maximal submodule, say N. Hence $\mathrm{Soc}(R/I) = R/I$ where $I = \mathrm{ann}_R(M/N)$.

(12)⇒(1). Suppose that M is a nonzero R-module. Then by (12), there exists $N < M$ such that $\mathrm{Soc}(R/I) \neq 0$ for $I = \mathrm{ann}_R(M/N)$. Let $M' = M/N$ and $S = A/I$ be a minimal ideal of R/I. Since $A \not\subseteq I$, $AM \not\subseteq N$ and so $SM' \neq 0$. Now $\mathrm{ann}_R(S) := P$ is a maximal ideal of R and SM' is a vector space over R/P. On the other hand, if $0 \neq x \in S$ then $xM' = SM'$ and so the multiplication by x yields a non zero homomorphism from M' to SM'. It follows that M' and hence M has a maximal submodule, proving that R is a max ring.

Finally we note that (3)⇔(3′), (4)⇔(4′) and (5)⇔(5′). Thus the proof is complete. □

Corollary 2.8 — Let M be a module over a max ring R. Then any nonsingular simple R-module in $\sigma[M]$ is injective and isomorphic to a direct summand of M.

PROOF : Suppose that S is a nonsingular simple R-module in $\sigma[M]$. Then by Theorem 2.7(6), $\mathrm{Hom}_R(M, S) \neq 0$. Let f be a nonzero element of $\mathrm{Hom}_R(M, S)$. Since S is nonsingular, $K := \mathrm{Ker} f$ is not an essential submodule of M. Thus K is a direct summand of M, because K is a maximal submodule of M. It follows that S is isomorphic to a direct summand of M. Let $E = \mathrm{E}(M)$, the injective hull of M. Since $S \in \sigma[\mathrm{E}(M)]$, we have by the above reasoning that S isomorphic to a direct summand of $\mathrm{E}(M)$, hence it is injective. □

The following result is already well known. It is also immediate from our characterizations of max rings. A ring B is called left semi-Artinian if $\mathrm{Soc}(M) \neq 0$ for all nonzero left B-module M.

Corollary 2.9 — Every commutative semi-Artinian ring is a max ring.

PROOF : Evident by Theorem 2.7 (12). □

3. SOME APPLICATIONS

Let B be an arbitrary ring. In [10], a B-module M is called a weak generator for a class $\mathcal{C}$ of B-modules if $\mathrm{Hom}_B(M, N) \neq 0$ for all nonzero $N \in \mathcal{C}$. Let R be a commutative ring as before and M be an R-module. If M is a weak generator for $\sigma[M]$, then we say that M is ***Hom-faithful***. Dually, M

is called *Tensor-faithful* if $M \otimes_R N \neq 0$ for all nonzero $N \in \sigma[M]$. In this last section we give two applications of Theorem 2.7.

***Theorem* 3.1** — *A ring R is max (resp. semi-Artinian) if and only if all nonzero R-modules are Tensor-faithful (resp. Hom-faithful).*

PROOF : We note that statement (8) of Theorem 2.7 is equivalent to "all nonzero R-modules are Tensor-faithful". It remains to prove the semi-Artinian case. Let R be a semi-Artinian ring, M be a nonzero R-module and $0 \neq N \in \sigma[M]$. Then by Corollary 2.9, R is a max ring and by Theorem 2.7(6), $\text{Hom}_R(M, S) \neq 0$ for a simple submodule S of N. This shows that M is Hom-faithful. Conversely, suppose that all nonzero R-modules are Hom-faithful. Clearly by Theorem 2.7 (6), R is a max ring and so $R/\text{J}(R)$ is regular and $\text{J}(R)$ is T-nilpotent. By Lemma 2.5, every nonzero R-module M contains a nonzero submodule N which is an $R/\text{J}(R)$-module. We prove that $R/\text{J}(R)$ is semi-Artinian, hence N contains a simple submodule, and consequently R will be semi-Artinian. We may suppose that $\text{J}(R) = 0$, hence R is a regular ring. We will show that every nonzero cyclic R-module contains an injective submodule from which the proof will be completed by [4, 15.11]. Suppose that $0 \neq R/I$ is a cyclic R-module. Let Q be the injective hull of the R/I-module R/I. By the hypothesis there exists $0 \neq f \in \text{Hom}_{R/I}(Q, R/I)$. Since R/I is a nonsingular R/I-module, Ker$f$ is a closed R/I-submodule in the injective module $Q_{R/I}$. It follows that Ker$f$ is a direct summand of Q. Thus R/I contains an injective R/I-module which is injective as an R-module by [9, Corollary (3.6 A)], as desired. □

Corollary 3.2. — For a ring R, the following statements are equivalent;

(1) R is a local max ring.

(2) For all nonzero R-modules M and N, $M \otimes_R N \neq 0$.

PROOF : (1)⇒(2). Let M and N be nonzero R-modules. Since R is a local max ring, there exist maximal submodules K and L of M and N respectively, such that $M/K \simeq N/L$. Thus $M/K \otimes_R N/L \neq 0$ and hence $M \otimes_R N \neq 0$.

(2)⇒(1). By Theorem 3.1, R is a max ring. Now if I and P are two maximal ideals of R, the isomorphism $R/I \otimes_R R/P \simeq R/(I + P)$ implies that $I = P$. □

It is evident that Theorem 2.7 heavily depends on the assumption that the base ring is commutative. However, some of the statements of Theorem 2.7 are easily seen to be categorical. Thus they may be equivalent conditions for being max in the noncommutative case. Below, we prove that some of the characterizations in Theorem 2.7 carry over to rings Morita equivalent to commutative max rings. But let us first recall some rudiments of Morita theory. If two rings A and B are Morita equivalent, with category equivalence $\alpha : A\text{-Mod} \to B\text{-Mod}$, we write $A \overset{\alpha}{\approx} B$. There is an isomorphism θ_α between the lattices of two sided ideals of A and B, given by $\theta_\alpha(I) = \mathrm{ann}_B(\alpha(A/I))$ where I is an ideal in A. Also α induces an equivalence $A/I \overset{\alpha_I}{\approx} B/\theta_\alpha(I)$.

Lemma 3.3 — Suppose that $A \overset{\alpha}{\approx} B$. Then the following hold.

(1) For any ideal I of A and any A-module M, if $IM = M$, then $\theta_\alpha(I)\alpha(M) = \alpha(M)$.

(2) For any A-module K, $\theta_\alpha(\mathrm{ann}_A(K)) = \mathrm{ann}_B(\alpha(K))$.

(3) If I is an ideal in A essential as a left ideal, then the ideal $\theta_\alpha(I)$ is essential as a left ideal in B.

(4) For any A-module M, if $I \in \mathfrak{F}_M$, then $\theta_\alpha(I) \in \mathfrak{F}_{\alpha(M)}$ and θ_α induces a lattice isomorphism between $\mathfrak{F}_M$ and $\mathfrak{F}_{\alpha(M)}$.

(5) For any A-module M, there exists a surjective B-homomorphism
$$\alpha(M)/\mathrm{J}(\alpha(M)) \to \alpha(M/\mathrm{J}(M)).$$

PROOF : (1) Note that $IM = M$ if and only if $\mathrm{Hom}_A(M, X) = 0$ for any $X \in A/I$-Mod.

(2) Let $L = \mathrm{ann}_A(K)$. Then (2) is obtained by the fact that α_L preserves faithful modules.

(3) Consider the following exact sequence in B-Mod

$$0 \to \alpha(I) \to \alpha(A) \to \alpha(A/I) \to 0,$$

in which the monomorphism is essential because I is left essential in A. Thus $\alpha(A/I)$ is singular as a left B-module. Since $B/\theta_\alpha(I)$ is a direct summand of $(\alpha(A/I))^n$ for some n, we deduce that $B/\theta_\alpha(I)$ is also singular.

(4) Let $I \in \mathfrak{F}_M$. First note that if $L := \mathrm{ann}_A(M)$ and $\alpha_L = \varepsilon$, then $\theta_\varepsilon(I/L) = \theta_\alpha(I)/\theta_\alpha(L)$. Then as $I/\mathrm{ann}_A(M) \leq_e A/\mathrm{ann}_A(M)$, by (2) and (3) we have

$$\theta_\alpha(I)/\mathrm{ann}_B(\alpha(M)) = \theta_\alpha(I)/\theta_\alpha(L) = \theta_\varepsilon(I/L) \leq_e B/\theta_\alpha(L) = B/\mathrm{ann}_B(\alpha(M)).$$

By assumption $IM \neq M$, and so by (1), $\theta_\alpha(I)\alpha(M) \neq \alpha(M)$. Thus $\theta_\alpha(I) \in \mathfrak{F}_{\alpha(M)}$.

(5) Consider the following exact sequence in B-Mod,

$$0 \to \alpha(\mathrm{J}(M)) \to \alpha(M) \to \alpha(M/\mathrm{J}(M)) \to 0.$$

Thus there exists $K \leq \alpha(M)$ such that $\alpha(M)/K \simeq \alpha(M/\mathrm{J}(M))$. Since $\alpha(M/\mathrm{J}(M))$ is J-semisimple, $\mathrm{J}(\alpha(M)) \leq K$. Hence $\alpha(M)/\mathrm{J}(\alpha(M)) \stackrel{epi}{\to} \alpha(M/\mathrm{J}(M))$, as desired. □

Combining Theorem 2.7 and Lemma 3.3, we have:

***Theorem* 3.4** — *Let B be a ring Morita equivalent to a commutative ring. Then the following statements are equivalent.*

(1) *B is a left max ring.*

(2) *If I is an ideal in B and $M \in B$-Mod, then $IM = M \implies IN = N$ for all $N \leq M$.*

(3) *For any ${}_BM$, ann$_B(N)M \neq M$ where $0 \neq N \leq M$.*

(4) *For any nonzero left B-module M, Rej$_B(M, S)$ is a proper submodule of M, for every simple B-module S in $\sigma[M]$.*

(5) *The set $\mathfrak{F}_M$ has a maximal member for every non semisimple B-module M.*

(6) *For any nonzero left B-module M, Hom$_B(M/\mathrm{J}(M), M) \neq 0$.*

(7) *For any nonzero left B-module M, there exists $N \leq M$ such that for $I = $ ann$_B(M/N)$ we have Soc$({}_B(B/I)) \neq 0$.*

If any of the above statements holds, then the right hand version for each of (1)-(7) is valid, and they are all equivalent. □

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