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SOME RESULTS ON C-INVERSES OF A CORE MATRIX

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This paper contributes several new results on core matrices by presenting some characterizations involving c-inverses. Some relations to matrix partial orders are shown as well.

Key words : Core matrix; group inverse; right-sharp partial ordering; sharp partial ordering.

1. PREREQUISITES

The matrices under consideration are complex. We will denote the range, the nullspace, the conjugate transpose, and the rank of a matrix A by $R\,(A)$, $N\,(A)$, A^*, and $rk\,(A)$, respectively. We use I to denote an appropriate size identity matrix. We recall the following simple facts (see [2, p. 44]).

Lemma 1 — If matrices A and B are conformable for the product AB then $rk\,(AB) = rk\,(A)$ if and only if $R\,(AB) = R\,(A)$, and $rk\,(AB) = rk\,(B)$ if and only if $N\,(AB) = N\,(B)$.

We let A^- designate a generalized inverse of A, this being defined as a solution to the matrix equation $AXA = A$, relative to the unknown matrix X. Generalized inverses satisfying the condition $XAX = X$ are said to be reflexive. For future reference we state the following basic lemma which is a reformulation of Lemmas 2.2.2, 2.2.3, and 2.5.1 of [13].

Lemma 2 — A matrix G is a generalized inverse of A if and only if AG is a projector onto $R\,(A)$, or, equivalently, if and only if GA is a projector along $N\,(A)$. If G is a generalized inverse of A then $rk\,(G) \geq rk\,(A)$ with strict equality if and only if G is a reflexive generalized inverse.

We use A^+ to denote the Moore-Penrose inverse of A, that is, the unique reflexive generalized inverse of A satisfying furthermore the conditions $(AX)^* = AX$, $(XA)^* = XA$. It follows that AA^+ and A^+A are the orthogonal projectors onto $R(A)$ and $R(A^*)$, respectively.

We will mainly focus on square matrices. A square matrix A is said to be core if $R(A)$ and $N(A)$ are complementary subspaces, which is equivalent to saying that $rk(A) = rk\left(A^2\right)$ (see e.g. [11, Theorem 2.2.23]). For a core matrix A, we designate as c-inverses all those generalized inverses of A which satisfy $R(A^- A) = R(A)$ ([11, Definition 6.4.1]). A c-inverse of A is denoted by A_c^-.

Supposing A to be a core matrix, we may consider the projector Q_A which projects a vector onto $R(A)$ along $N(A)$ (see e.g. [2, p. 59, Theorem 8]). We will repeatedly use the fact that Q_A^* is the projector onto $R(A^*)$ in the direction of $N(A^*)$. The following result is a ready consequence of Lemma 2.

Lemma 3 — Necessary and sufficient for a matrix G to be a c-inverse of A is

$$GA = Q_A. \tag{1}$$

Among the c-inverses, those having $R(A_c^-) = R(A)$ are called χ-inverses ([11, Definition 2.4.1]). As observed in [13, p. 73], a χ-inverse A_χ^- is always expressible in the form

$$A_\chi^- = A\left(A^2\right)^-. \tag{2}$$

The group inverse of a core matrix A is defined as the unique reflexive generalized inverse $A^\#$ which satisfies the commutative law $AA^\# = A^\# A$ ([11, Theorem 2.4.6]). Since A and $A^\#$ are generalized inverses of each other, it follows from Lemma 2 that

$$R(A) = R\left(A^\#\right) = R\left(A^\# A\right), \quad N(A) = N\left(A^\#\right) = N\left(A^\# A\right). \tag{3}$$

On account of Lemma 3, it further follows that a matrix G is a c-inverse of A if and only if

$$G = A^\# + U\left(I - AA^\#\right), \tag{4}$$

for some matrix U of appropriate size (see e.g. [13, Theorem 2.3.2]). It is also useful to know that

$$A^\# = A\left(A^3\right)^- A \tag{5}$$

for every choice of $\left(A^3\right)^-$ (see e.g. [13, (4.3.1)]). Moreover, Theorem 2.4.3 and Remark 2.4.14 in [11] allow us to pose the following proposition.

Proposition 4 — The group inverse $A^\#$ is the uniquely determined c-inverse satisfying the property $N\left(A^\#\right) = N(A)$.

In the following section we will be mostly concerned with the class of c-inverses. However, completely parallel results can be derived by working with a-inverses. In light of Lemma 3, an a-inverse of A may be defined by the equation $AX = Q_A$.

2. RESULTS CHARACTERIZING c-INVERSES

In the ensuing discussion, we assume that A and G are square matrices of the same order. The following three lemmas provide the most basic algebraic characterizations of c-inverses.

Lemma 5 — A matrix G is an A_c^- if and only if $GA^2 = A$.

PROOF : Necessity results immediately from Lemma 3. We pass to sufficiency. The equality $GA^2 = A$ clearly displays the fact A is core. But this means that $GA^2A^{\#} = AA^{\#}$, which justifies equality (1). ■

In Lemma 6 a two-condition characterization of the group inverse is derived.

Lemma 6 — A matrix G is the group inverse of A if and only if

$$GA^2 = A, \qquad G^2A = G. \tag{6}$$

PROOF : For the 'if' direction, notice that (6) allows one to assert that G is a c-inverse of A with the property $R(G^*) \subset R(A^*)$. As $rk(G) \geq rk(A)$, it follows at once that $N(G) = N(A)$. Thus, G meets the requirements for the group inverse stated in Proposition 4. The converse is obvious. ■

The content of Lemma 6 is that G is a group inverse of A if and only if G is a c-inverse of A and A^* is a c-inverse of G^*.

The next result characterizes the class of commuting inverses. The proof is straightforward.

Lemma 7 — A matrix G is a commuting generalized inverse of A if and only if

$$GA^2 = A, \qquad A^2G = A.$$

In what follows, we will assume that A and B are core matrices of the same size. We shall write $\{B_c^-\} \subset \{A_c^-\}$ meaning that every c-inverse of B is a c-inverse of A. Here and subsequently, we use $\subset$ for proper or improper containment.

One important concept later on will be that of the right-sharp ordering ([11, Definition 6.3.1]). We say that A is below B with respect to the right-sharp ordering, written as $A < \#B$, if

$$A^2 = BA \quad \text{and} \quad R(A^*) \subset R(B^*).$$

The right-sharp ordering was originally considered by Mitra [10, Section 5].

Next point to be discussed here concerns the equality $A^2 = BA$. Theorem 8 below indicates that this equality establishes a pre-order relation among core matrices, that is, a relation that is reflexive and transitive.

Prior to presenting our next result, we draw the reader's attention to the erroneous statement of Theorem 6.4.8 in [11], which goes as follows. If A and B are core matrices of the same size, then $A < \#B$ if and only if $\{B_c^-\} \subset \{A_c^-\}$.

A counterexample is provided by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is readily checked that every inverse A_c^- and every inverse B_c^- is expressible in the form $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$, where a, b are arbitrary complex numbers. Thus $\{B_c^-\} \subset \{A_c^-\}$ but $R(A^*)$ is not contained in $R(B^*)$, contrary to the claim in [11, Theorem 6.4.8].

The preceding example has the consequence of showing that the relation $\{B_c^-\} \subset \{A_c^-\}$ is not antisymmetric.

We proceed to establish the following theorem.

***Theorem* 8** — *Let A and B be core matrices of the same size. The following six statements are equivalent:*

(i) $\{B_c^-\} \subset \{A_c^-\}$

(ii) $A^\# A = B^\# A$

(iii) $A^\# A^\# = B^\# A^\#$

(iv) $A^\# = BA^\# A^\#$

(v) $A^2 = BA$

(vi) $AA_c^- = BA_c^-$ *for some* A_c^-.

PROOF : (i)⇒(ii): By Lemma 5, we derive the equality $A = B^\# A^2$, which postmultiplied by $A^\#$ leads to $AA^\# = B^\# A$, as required.

(ii)⇒(iii): Postmultiplying (ii) by $A^\# A^\#$ entails (iii).

(iii)⇒(iv): It follows from equations (3) that $A^\# A^\# = B^\# A^\#$ ensures $R(A) \subset R(B)$. Hence, premultiplying (iii) by B yields (iv).

(iv)⇒(v): Postmultiplying (iv) by A^3 gives (v).

(v)⇒(vi): As $A^2 = BA$, we may take any χ-inverse $A\left(A^2\right)^-$ as A_c^- with the desired property.

(vi)⇒(i): Let B_c^- be an arbitrary but fixed c-inverse of B. Having (vi), we clearly may claim that $R(A) \subset R(B)$ and $B_c^- AA_c^- A = B_c^- BA_c^- A$. Thus, as a consequence of Lemma 3, $B_c^- A = Q_A$, implying that B_c^- is a c-inverse of A. ■

In comment, we mention that one may reexpress condition (vi) in Theorem 8 as the identity $AA^{\#} = BA^{\#}$, while condition (iv) translates into the condition that B is a c-inverse of $A^{\#}$.

We record the following consequences of the equality $A^2 = BA$.

$$(B - A) B_c^- A = B_c^- (B - A) A = 0. \tag{7}$$

For further considerations we introduce some more notations.

As usual, we write $A <^- B$ to express that

$$A^- A = A^- B \quad \text{and} \quad AA^- = BA^-$$

for some generalized inverse A^-. We remark that an alternative formulation can be given by considering a different generalized inverse in each of the two identities above ([11, Definition 3.3.1, Theorem 3.3.2]). If $A <^- B$ then A is said to be below B under the minus order. The minus order was introduced by Hartwig [7] and Nambooripad [12]. There are many equivalent characterizations of the minus order. We recall two results which we will need in the sequel ([8, Theorem 2.2], [11, Remark 3.3.12]).

Lemma 9 — For a pair of matrices A and B of the same size, the following three statements are equivalent:

(i) $A <^- B$

(ii) every B^- is a generalized inverse of A

(iii) $R(A) \subset R(B)$, $R(A^*) \subset R(B^*)$, $A = AB^- A$ for some B^-.

For a pair of core matrices A, B the notation $A <^{\#} B$ indicates that A is below B under the sharp order in the following sense

$$A^{\#} A = A^{\#} B \quad \text{and} \quad AA^{\#} = BA^{\#}.$$

The sharp order was introduced and characterized in [9]. By means of Lemma 2.2 in [9] the sharp order can equivalently be defined by declaring that

$$A^2 = AB = BA.$$

The sharp order has been the subject of a recent study in [3, Section 5], [5, Section 5] and [6].

Here is a lemma giving conditions under which $B^{\#}$ is a commuting generalized inverse of A ([11, Theorem 4.2.8]) or $B^{\#} = A^{\#}$.

Lemma 10 — Let A and B be core matrices of the same size. Then

(i) $B^{\#}$ is a commuting generalized inverse of A if and only if $A^2 = BA = AB$

(ii) $B^{\#}$ is the group inverse of A if and only if $A^2 = BA$ and $R(A^*) = R(B^*)$.

PROOF : (i) In light of Lemma 7, we see that what is required is that $B^{\#}A^2 = A$ and $A^2B^{\#} = A$, what, by referring to part (ii)⇔(v) of Theorem 8, can be shown to be equivalent to $A^2 = BA$ and $A^2 = AB$, respectively. Hence the result holds.

(ii) By means of Lemma 6 and the proof of point (i) of the lemma, $B^{\#}$ satisfies all the requirements that make it the group inverse of A if and only if $A^2 = BA$ and $B^{\#}B^{\#}A = B^{\#}$, the latter condition being equivalent to $BA = B^2$. Obviously, $A^2 = BA = B^2$ entails $R(A^*) = R(B^*)$. Conversely, if $A^2 = BA$ and $R(A^*) = R(B^*)$, then, on account of condition (ii) in Theorem 8, $BA = B^2B^{\#}A = B^2A^{\#}A = B^2$, establishing point (ii) of the lemma. ■

It is worth making a few remarks about part (ii) of the preceding lemma.

Remark 11 : Particular note should be made of the fact that under the hypothesis $A^2 = BA$ the condition $R(B^*) = R(A^*)$ entails $A^2 = AB$. Indeed, as $R(B^*) = R(A^*)$, we have $AB = ABAA^{\#} = A^2$. From the proof of part (ii), it is apparent that one may reexpress the condition $A^2 = BA$, $R(A^*) = R(B^*)$ as the identity $A^2 = BA = B^2$. Point (ii) of Lemma 10 is equivalent to saying that two core matrices A and B are equal if and only if $A < \#B$ and $rk(A) = rk(B)$, which was also observed in [11, Remark 6.3.5].

We continue with a result concerning the group inverse of AB.

Lemma 12 — If A and B are core matrices such that $A^2 = BA$, then

(i) $(AB)^{\#}$ exists

(ii) both the product $A_c^- B_c^-$ and the product $B_c^- A_c^-$ are c-inverses of AB irrespective of the choices of A_c^- and B_c^-

(iii) $(AB)^{\#} = B^{\#}A^{\#}$ if and only if $AB = A^2$

(iv) $(AB)^{\#} = A^{\#}B^{\#}$ if and only if $AB = A^3B^{\#}$.

PROOF (i) To confirm this claim notice that $rk(AB) \geq rk(ABA) = rk(A)$.

(ii) Assertion (ii) follows immediately from Lemma 5 upon recalling from Theorem 8 that $A^2 = BA$ gives $B_c^- A = Q_A$.

(iii), (iv) By the statement above, we know that $B^{\#}A^{\#}$ and $A^{\#}B^{\#}$ are c-inverses of AB. The proof proceeds by considering equations (ii) and (iii) of Theorem 8 and then applying the second equation of Lemma 6. ■

At this point it is appropriate to make the following observation. The condition $AB = A^2$ forces $R(A^*) \subset R(B^*)$, what in turn implies that $A^2 = A^2BB^{\#} = A^3B^{\#}$. We have the following consequence.

Corollary 13 — Under the hypothesis of Lemma 12, the condition $(AB)^{\#} = B^{\#}A^{\#}$ entails $(AB)^{\#} = A^{\#}B^{\#}$.

We can now complement Theorem 5.3 in [3] and Theorem 4.2.14 in [11] by the following corollary.

Corollary 14 — If A and B are core matrices such that $A^2 = BA$, then

$$(AB)^{\#} = A^{\#}B^{\#} = B^{\#}A^{\#} = \left(A^{\#}\right)^2$$

if and only if $AB = A^2$.

It is appropriate here to mention a result reported in [4, Theorem 3.1] according to which $(AB)^{\#}$ exists and $(AB)^{\#} = B^{\#}A^{\#}$ if and only if $R(AB) = R(BA)$ and $N(AB) = N(BA)$.

The following theorem offers five equivalent statements for the partial order relation $A < \#B$. A correction of Theorem 6.4.8 in [11] is made by replacing the condition $\{B_c^-\} \subset \{A_c^-\}$ with stronger ones as stated in parts (ii) through (iv). Two relevant results from [11, Theorems 6.3.13(ii), 6.3.6] are rederived here for completeness' sake.

***Theorem* 15** — *Let A and B be core matrices of the same size. The following six statements are equivalent:*

(i) $A^2 = BA$ *and* $R(A^*) \subset R(B^*)$

(ii) $A^2 = BA$ *and* $B^{\#}$ *is a generalized inverse of* $B - A$

(iii) $A^2 = BA$ *and* $B^{\#} - A^{\#}$ *is a generalized inverse of* $B - A$

(iv) $A^2 = BA$ *and* $A_c^- A = A_c^- B$ *for some* A_c^-

(v) $A <^- B$ *and* $R(BA) = R(A)$

(vi) *there exists* A_c^- *such that* $(B - A)A_c^- = A_c^-(B - A) = 0$.

PROOF : The equivalences (i)⇔(ii)⇔(iii) follow from Theorem 8 and the remark following it by noting that if $A^2 = BA$, then $(B - A)A^{\#} = 0$ and $(B - A)B^{\#}(B - A)$ reduces to $B - AB^{\#}B$.

(i)⇒(iv): We only need to confirm the second claim in item (iv). Referring to part (i)⇔(v) of Theorem 8, it is seen that $A^2 = BA$ entails $AB_c^- A = A$. Further, having $R(A^*) \subset R(B^*)$, we may assume that $AB_c^- B = A$. Consequently, it follows that $A_c^- = A^{\#} AB_c^-$ meets our requirement.

(iv)⇒(v): As already mentioned, $A^2 = BA$ ensures $AB_c^- A = A$ and $R(A) = R(BA) \subset R(B)$. Further, $A_c^- A = A_c^- B$ yields $R(A^*) \subset R(B^*)$. Hence, from conditions (iii) of Lemma 9, we get $A <^- B$, as desired.

(v)⇒(vi): By means of Lemma 9, another way of stating $A <^- B$ is to say that every B^- is a generalized inverse of A. Taken with Lemma 2, this implies that AB^- is a projector onto $R(A)$, and therefore combining this assertion with the conditions $R(A) = R(BA) \subset R(B)$ yields $BA = AB_c^- BA = A^2$. Further, as $A <^- B$, we clearly may claim that $A^- A = A^- B$ for some generalized inverse A^-. Recalling that any χ-inverse can be factored into a product of type (2), it is readily checked that $A_c^- = A\left(A^2\right)^- AA^-$ satisfies the conditions laid down in item (vi).

(vi)⇒(i): To establish the statement observe that the condition $A_c^- B = A_c^- A$ implies $R(A^*) \subset R(B^*)$, whereas the condition $BA_c^- = AA_c^-$, as stated by Theorem 8, is equivalent to $BA = A^2$.

We shall here combine the foregoing considerations, by stating some equivalent conditions under which the pre-order relation $A^2 = BA$ yields the sharp order.

Theorem 16 — *Let A and B be core matrices of the same size such that $A^2 = BA$. Then the following five statements are equivalent:*

(i) $AB = A^2$

(ii) $B_c^- (B - A) B_c^-$ *is a c-inverse of $B - A$ for every variant of B_c^-*

(iii) $B^{\#}$ *is a commuting generalized inverse of A*

(iv) $(AB)^{\#} = B^{\#} A^{\#}$

(v) $AA^{\#} B = A$.

PROOF : The equivalence (i)⇔(v) can easily be established by employing similar reasoning as in the proof of Theorem 8. The properties (i)⇔(iii) and (i)⇔(iv) were announced in Lemma 10 and Lemma 12, respectively. We confirm that (ii)⇔(v). Let $H = B_c^- (B - A) B_c^-$. In view of the equalities laid down in (7) our hypothesis $A^2 = BA$ guarantees that $HA = HBA = 0$. Hence $H(B - A)^2 = B - AA^{\#} B$. Consequently, by Lemma 5, H is a c-inverse of $B - A$ if and only if $AA^{\#} B = A$, as required. ■

We give a remark related to Theorem 16.

Remark 17 : Theorem 16 provides an extension of Theorem 6.3.15 in [11].

The equivalence (i)⇔(v) holds without the restriction $A^2 = BA$, however the properties (i) and (ii) are in general independent of each other as can be seen by taking

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which satisfy the condition in (i) but not the statement in (ii), and

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for which the statement in (ii) is fulfilled but the condition in (i) does not hold.

The implication (i), (ii)⇒ $A^2 = BA$ is seen to be false under consideration of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An addition to Theorem 16 is given by the following two statements.

Lemma 18 — Let A and B be square matrices of the same size. If A is core, then any two of the following three statements imply the third:

$$BA^{\#}B = A, \qquad BA = A^2, \qquad AB = A^2.$$

PROOF : The result is easily confirmed by writing the first condition as

$$A = BA\left(A^3\right)^{-} AB.$$

■

***Theorem* 19** — *Let A and B be core matrices of the same size such that $BA^{\#}B = A$. Then the following four statements are equivalent:*

(i) $BA = A^2$

(ii) $B^{\#}A$ is idempotent

(iii) $A <^{-} B$

(iv) $B^{\#} - A^{\#}$ is a generalized inverse of $B - A$.

PROOF : Let $C = (B - A)\left(B^{\#} - A^{\#}\right)(B - A)$. Assuming $BA^{\#}B = A$, we may claim that $R(A) \subset R(B)$, $R(A^*) \subset R(B^*)$ and $BA^{\#} = AB^{\#}$, $A^{\#}B = B^{\#}A$. Hence $C = B - 4A + 3AB^{\#}A$.

(i)⇒(ii): The statement follows from part (ii)⇔(v) of Theorem 8.

(ii)⇒(iii): As $R(A) \subset R(B)$, premultiplying the equality $\left(B^{\#}A\right)^2 = B^{\#}A$ by B yields $AB^{\#}A = A$. Recalling part (i)⇔(iii) of Lemma 9, we find that $A <^- B$.

(iii)⇒(iv): From part (i)⇔(ii) of Lemma 9, it is known that $A <^- B$ ensures $AB^{\#}A = A$, which leads to $C = B - A$, as needed.

(iv)⇒(i): It is clear that (iv) entails $C = B - A$, thus showing that $AB^{\#}A = A$. Having this, and postmultiplying the equality $BA^{\#}B = A$ by $B^{\#}A^2$ yields $BA = A^2$, and the proof is completed. ■

We next present a result characterizing the matrix equation $AXA^- = A^-$.

***Theorem* 20** — *Let A and B be square matrices of the same size. Then*

$$ABA^- = A^- \tag{8}$$

if and only if $A^-$is a χ-inverse of A and B is a generalized inverse of A.

PROOF : First, let (8) hold. It is at once apparent that $R(A^-) \subset R(A)$. Hence, A^- is guaranteed to be a χ-inverse of A. Postmultiplying (8) by A^2 yields $ABA = A$, as claimed. To confirm the "if" statement, recall that any A_χ^- admits representation (2). Then $ABA_\chi^- = ABA\left(A^2\right)^- = A\left(A^2\right)^- = A_\chi^-$, whence the assertion follows. ■

A specialization of Theorem 20 yields Theorem 5 in [1]. Namely, Theorem 20 has the consequence of showing that $ABA^+ = A^+$ if and only if B is a generalized inverse of A and $A^+A = AA^{\#}$, which occurs if and only if $R(A) = R(A^*)$ (i.e. A is a range hermitian matrix).

We continue with some results concerned with the reverse order rule $(AB)_c^- = B_c^- A_c^-$ for c-inverses of two core matrices. As a preliminary, consideration is given to a matrix product AB involving rectangular matrices of appropriate sizes.

The following useful facts are well-known and easy to establish (see e.g. [13, Lemma 2.2.6]).

Lemma 21 — Suppose that matrices A and B are conformable for the product AB. Then

(i) the following two statements are equivalent:

(a) $rk(AB) = rk(B)$

(b) $N(A) \cap R(B) = \{0\}$.

(ii) the following three statements are equivalent:

(a) $rk(AB) = rk(A) = rk(B)$

(b) $N(A)$ and $R(B)$ are complementary subspaces

(c) $B\,(AB)^{-}\,A$ is the projector onto $R\,(B)$ along $N\,(A)$ irrespective of the choice of generalized inverse $(AB)^{-}$.

We give a list of equivalent statements for the product AB to be core.

***Theorem* 22** — *Suppose that matrices A and B are conformable for the product ABA. Then the following four statements are equivalent:*

(i) AB is core

(ii) $N\,(A) \cap R\,(BAB) = \{0\}$ and $N\,(B) \cap R\,(AB) = \{0\}$

(iii) there exist generalized inverses A^{-} and B^{-} such that

$$A^{-}ABAB = BAB \quad \textit{and} \quad B^{-}BAB = AB \tag{9}$$

(iv) $B^{-}A^{-}ABAB = AB$ for some generalized inverses A^{-} and B^{-}.

PROOF : (i)⇒(ii): Since AB is core, we have $rk\,(ABAB) = rk\,(BAB) = rk\,(AB)$, and the rest follows easily from part (i) of Lemma 21.

(ii)⇒(iii): In view of the assumed properties, there is a linear space K such that the disjoint union $K \oplus R\,(BAB) \oplus N\,(A)$ spans the whole space. Therefore we can define the projector P onto $K \oplus R\,(BAB)$ along $N\,(A)$. If G is an arbitrary but fixed generalized inverse of A, then $PGA = P$ and any one of generalized inverses $A^{-} = PG + U\,(I - AG)$, where U is an arbitrary matrix, fulfills requirement (9). By varying K we can generate all generalized inverses satisfying (9). The second statement can be obtained in much the same way.

(iii)⇒(iv): All one needs to do is premultiply the first equation by B^{-}.

(iv)⇒(i): It is evident that (iv) guarantees that AB is core. ■

By referring to point (iv) of Theorem 22, we can say that AB is core if and only if it has a c-inverse expressible as $B^{-}A^{-}$.

We continue with a relevant result.

Lemma 23 — Under the assumption of Theorem 22, if the product AB is core then $R\,(BAB)$ and $N\,(ABA)$ are complementary subspaces, and $P = B\,(AB)^{\#}\,A$ is the associated projector onto $R\,(BAB)$ along $N\,(ABA)$.

PROOF : Let $C = AB$. As C is core, it follows that $rk\,(CA) = rk\,(BC) = r\left(C^{3}\right)$. Now part (ii) of Lemma 21 implies that $N\,(CA)$ and $R\,(BC)$ are complementary and $P = BC\left(C^{3}\right)^{-}CA$ is the associated oblique projector. The argument is completed by recalling equation (5). ■

We assume again that A and B are core matrices of the same size. The theorem below establishes a relation between the inverse order rule $(AB)_c^- = B_c^- A_c^-$ and some range inclusions.

***Theorem* 24** — *Let A, B be two core matrices of the same size. Then $B_c^- A_c^-$ is a c-inverse of AB for every choice of A_c^- and B_c^- if and only if*

$$R\,(BAB) \subset R\,(A) \quad and \quad R\,(AB) \subset R\,(B)\,. \tag{10}$$

PROOF : As follows from Lemma 5, $B_c^- A_c^-$ is a c-inverse of AB if and only if

$$B_c^- AA^{\#} BAB = AB. \tag{11}$$

For necessity of (10), assume that (11) holds for every B_c^-. It follows that

$$B^{\#} AA^{\#} BAB = AB, \tag{12}$$

which in turn implies $R\,(AB) \subset R\,(B)$. Having this, and recalling the expression given in (4), one sees that the underlying assumption forces

$$\left(I - BB^{\#}\right) AA^{\#} BAB = 0.$$

As a result, $R\left(AA^{\#}BAB\right) \subset R\,(B)$. Hence, premultiplying (12) by B proves that $AA^{\#}BAB = BAB$, which yields $R\,(BAB) \subset R\,(A)$, as desired. Sufficiency of the conditions revealed in (10) follows by direct verification of criterion (11). ■

We mention without proof a few conditions under which both inclusions in (10) hold.

Proposition 25 — Under the assumption of Theorem 24, any one of the following conditions implies (10): $BA = A^2$, $R\,(AB) = R\,(BA)$.

The following matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

show that the sufficient conditions in the proposition above are not necessary, while

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

indicate that none of the conditions constituting (10) is necessary for a product AB to be core.

In closing, we complement Theorem 24 by the following assertion about a-inverses.

***Theorem* 26** — *Let A, B be two core matrices of the same size. Then $B_a^- A_a^-$ is an a-inverse of AB irrespective of the choices of A_a^- and B_a^- if and only if*

$$N(B) \subset N(ABA) \quad and \quad N(A) \subset N(AB).$$

PROOF : The proof proceeds by redoing the proof of Theorem 24. ■

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