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ON THE DISTRIBUTION OF THE DISCRETE SPECTRUM OF NUCLEARLY PERTURBED OPERATORS IN BANACH SPACES

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Dedicated to Professor Kalyan B. Sinha on the occasion of his 70th birthday.

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Let Z_0 be a bounded operator in a Banach space X with purely essential spectrum and K a nuclear operator in X. We construct a holomorphic function the zeros of which coincide with the discrete spectrum of Z_0+K and derive a Lieb-Thirring type inequality. We obtain estimates for the number of eigenvalues in certain regions of the complex plane and an estimate for the asymptotics of the eigenvalues approaching to the essential spectrum of Z_0.

Key words : Eigenvalues; discrete spectrum; nuclear perturbations.

1. INTRODUCTION

In the present article we analyze the discrete spectrum of a linear, bounded operator $Z = Z_0 + K$, where Z_0 is a bounded operator with purely essential spectrum and K is a nuclear perturbation. If X is a Hilbert space and if K is in some Neumann-Schatten class this problem was well studied during the last years, see for instance for general non-selfadjoint operators Demuth *et al*., [6], for non-selfadjoint perturbations of selfadjoint operators Hansmann [16], for Schrödinger operators Abramov *et al*., [1], Davies and Nath [5], Frank [9], Frank *et al*., [10], Laptev and Safronov [21], Safronov [26], Hansmann [15] and for Jacobi operators Borichev *et al*., [2], Favorov and Golinskii [8], Golinskii and Kupin [13] and Hansmann and Katriel [17].

It turns out that we can generalize the theory known for Hilbert spaces if we can prove the so-called nuclear determinant, $\det\big(\mathbb{1} - K(z\mathbb{1} - Z_0)^{-1}\big)$ to be an analytic function on the resolvent set of Z_0.

In Section 2 we explain the method of the proof which uses substantially the behaviour of the zeros of holomorphic functions defined in the open unit disc.

In Section 3 we prove the holomorphy of $\lambda \mapsto d(\lambda) := \det\big(\mathbb{1} - KR_{Z_0}(\lambda)\big)$ and that λ_0 is a discrete eigenvalue of Z with algebraic multiplicity m if and only if λ_0 is a zero of d of order m.

In the last section we apply the results to the discrete Laplacian Δ_p in $l^p(\mathbb{Z})$. It turns out that

$$\sum_{z\in\sigma_{disc}(\Delta_p+K)} \frac{\operatorname{dist}\big(z,\sigma_{ess}(\Delta_p+K)\big)^{3+\tau}}{|z^2-4|} \le c(\tau)\|K\|_{\mathcal{N}}^2,$$

for arbitrary $K \in \mathcal{N}(l^p(\mathbb{Z}))$ with some $\tau > 0$, a constant $c(\tau) > 0$ and where $\|\cdot\|_{\mathcal{N}}$ denotes the norm in the space of nuclear operators in X. It is possible to give ananalogous estimate for the multiplication operator perturbed by nuclear integral operators on the space of continuous functions. Moreover, if $J \in \mathcal{N}(l^1(\mathbb{Z}))$ is a Jacobi operator we can derive

$$\sum_{z\in\sigma_{disc}(\Delta_1+J)} \operatorname{dist}(z,\sigma_{ess}(\Delta_1+J))^{1+\tau} \le c(\tau)\|J\|_{\mathcal{N}}^2,$$

where the constants has to be taken in the same way like before.

These inequalities can be used to estimate the number of the eigenvalues in certain parts of the complex plane or to estimate the possible asymptotics if the eigenvalues are approaching to the essential spectrum (see Chapter 5).

2. Objective and Motivation

Let X be a complex Banach space and Z_0 a bounded operator on X with purely essential spectrum ($\sigma_{ess}(Z) := \{\lambda \in \mathbb{C} : \lambda - Z$ is not a Fredholm operator$\}$, where an operator A is Fredholm if A has closed range and both, the kernel and the cokernel of A are finite dimensional) which is equal to an interval, i.e. $\sigma(Z_0) = \sigma_{ess}(Z_0) = [a,b]$. We denote by $(\lambda\mathbb{1} - Z_0)^{-1} =: R_{Z_0}(\lambda), \lambda \in \rho(Z_0) := (\sigma(Z_0))^c$ (resolvent set) the resolvent of Z_0.

We perturb Z_0 by a nuclear operator K and define

$$Z := Z_0 + K.$$

We are interested in the distribution of the discrete spectrum ($\sigma_{disc}(Z) := \{\lambda \in \mathbb{C} : \lambda$ is a discrete eigenvalue of $Z\}$, where an eigenvalue is discrete if it is isolated and its corresponding Riesz projection is of finite rank) of Z. For the sake of completeness we repeat here the definition of nuclear operators.

Definition 2.1 — Let K be a compact operator in $\mathcal{B}(X)$ (the space of linear bounded operators). K is called **nuclear** if there are sequences (not necessarily unique) $\{f_n\} \subseteq X$, $\{\phi_n\} \subseteq X^*$ (the dual of X) such that Kf can be represented by

$$Kf = \sum_{n=1}^{\infty} \langle \phi_n, f \rangle f_n$$

for all $f \in X$ and

$$\sum_{n=1}^{\infty} \|\phi_n\|_{X^*} \|f_n\|_X < \infty.$$

We denote this class by $\mathcal{N}(X)$

In $\mathcal{N}(X)$ a norm can be defined by

$$\|K\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|\phi_n\|_{X^*} \|f_n\|_X \ : \ Kf = \sum_{n=1}^{\infty} \langle \phi_n, f \rangle f_n \text{ for all } f \in X \right\}.$$

With this norm $\mathcal{N}(X)$ becomes a Banach ideal (see Pietsch [23], p. 64).

Examples 2.2 : (a) Let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard basis in $l^p(\mathbb{Z})$ with $1 \leq p \leq \infty$. Denote by ϕ_m the sequence $\phi_m = \{a_{mj}\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$ ($\frac{1}{p} + \frac{1}{q} = 1$). Assuming $\{\|\phi_m\|_q\}_{m\in\mathbb{Z}} \in l^1(\mathbb{Z})$, then the operator $K : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ defined by $Kf := \sum_{m\in\mathbb{Z}} \langle \phi_m, f \rangle e_m$ is nuclear. The corresponding infinite matrix is given by $(a_{mj})_{m,j\in\mathbb{Z}}$.

We can conclude that every diagonal operator which is defined by an infinite matrix diag$(..., d_{-1}, d_0, d_1, ...)$ is nuclear if $\{d_n\}_{n\in\mathbb{Z}} \in l^1(\mathbb{Z})$.

(b) Every integral operator

$$K : C([\alpha, \beta]) \to C([\alpha, \beta]),\ (Kf)(t) := \int_{\alpha}^{\beta} k(t, s) f(s) ds$$

with continuous kernel k is nuclear and $\|K\|_{\mathcal{N}} \leq \int_{\alpha}^{\beta} \max_t |k(t, s)| ds$ (see Gohberg *et al.*, [12], Chapter 2 Theorem 2.2).

Remark 2.3 : If X is a Hilbert space $\mathcal{N}(X)$ coincides with the ideal of trace class operators. In this case we know that the eigenvalues are summable. However, there are Banach spaces and nuclear operators with non summable eigenvalues (see e.g. Gohberg *et al*., [12] p. 102).

In general one has the following estimate:

Let $\{\lambda_n(K)\}$ be the eigenvalues of the nuclear operator K, then

$$\sum_{n=1}^{\infty} |\lambda_n(K)|^2 \leq \|K\|_{\mathcal{N}}^2, \tag{2.1}$$

(see e.g. Pietsch [23], p. 160).

Example 2.4 : If X_1 and X_2 are compatible[1] Banach spaces and if K_1 and K_2 are consistent[2] compact operators acting in X_1 and X_2 then (see [4] p. 109).

$$\sigma(K_1) = \sigma(K_2).$$

We know that for $1 \leq p_1, p_2 < \infty$ the spaces $l^{p_1}(\mathbb{N})$ and $l^{p_2}(\mathbb{N})$ are compatible.

Now let K_1 be an operator on $l^1(\mathbb{N})$ and K_2 be an operator on $l^2(\mathbb{N})$ and let K_1 and K_2 be consistent. If the eigenvalues of K_1 are square summable the same is true for K_2. Now let K_2 be an operator defined on $l^2(\mathbb{N})$ which is consistent to a nuclear operator K_1 defined on $l^1(\mathbb{N})$. Then K_2 is not automatically a Hilbert-Schmidt operator or a trace-class operator.

To check this we define the infinite matrix

$$(a_{km})_{k,m\in\mathbb{N}} := \begin{pmatrix} 2^{-1} & 2^{-1} & 2^{-1} & \dots \\ 2^{-2} & 2^{-2} & 2^{-2} & \dots \\ 2^{-3} & 2^{-3} & 2^{-3} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and define with this matrix the operators K_1 and K_2.

For K_1 the nuclear norm is $\|K_1\|_{\mathcal{N}} = \sum_{k=1}^{\infty} \sup_m |a_{km}|$ (see [12], Chapter V Theorem 2.1). So we have

$$\|K_1\|_{\mathcal{N}} = \sum_{k=1}^{\infty} 2^{-k} = 1$$

such that K_1 is in fact a nuclear operator.

[1]Two Banach spaces X_1 and X_2 are called compatible if $X_1 \cap X_2$ is dense in X_1 and X_2.

[2]The operators K_1 and K_2 are called consistent, if they coincide on $X_1 \cap X_2$.

K_2 is a Hilbert-Schmidt operator on $l^2(\mathbb{N})$ iff the sum $\sum_{j=1}^{\infty} \|K_2 e_j\|_2$ is finite, where (e_j) is the orthonormal standard basis in $l^2(\mathbb{N})$ (see [12], Chapter IV Theorem 7.1). In the present example

$$\sum_{j=1}^{\infty} \|K_2 e_j\|_2^2 = \sum_{j=1}^{\infty} \|(2^{-k})\|_2^2 = \infty,$$

that means K_2 is not a Hilbert-Schmidt operator and hence not a trace class operator.

Because every nuclear operator K is compact $\sigma_{ess}(Z) = \sigma_{ess}(Z_0)$ (see [11] Chapter XI Theorem 4.2) and the spectrum of Z is the disjoint union of $\sigma_{ess}(Z)$ and $\sigma_{disc}(Z)$.

We are interested in estimates of the form

$$\sum_{\lambda \in \sigma_{disc}(Z)} \frac{(\text{dist}(\lambda, [a,b]))^{\alpha}}{|b-\lambda|^{\beta}|a-\lambda|^{\beta}} \leq C(\alpha, \beta, a, b)\|K\|_{\mathcal{N}}^{\gamma}$$

with positive exponents $\alpha, \beta, \gamma \in \mathbb{R}$.

Instead of studying $\sigma_{disc}(Z)$ directly we define a holomorphic function in $\mathbb{C} \setminus [a,b]$ the zeros of which coincide with $\{\lambda_n(Z)\}$. Then we study the behaviour of the zeros of holomorphic functions in the unit disc. Finally we transform the problem back and can analyze $\sigma_{disc}(Z)$. The function we have in mind is the determinant of $\mathbb{1} - KR_{Z_0}(\lambda)$.

Definition 2.5 — The **nuclear determinant** (or also called regularized determinant see [12] Chapter IX) of a nuclear operator K in X is given by

$$\det(\mathbb{1} - K) := \prod_{n=1}^{\infty} \big(1 - \lambda_n(K)\big) \exp\big(\lambda_n(K)\big)$$

where $\{\lambda_n(K)\}$ are again the eigenvalues of K.

The determinant has some important properties used in this article which we summarize here.

Lemma 2.6 — Let $K \in \mathcal{N}(X)$. Then

(i) $|\det(\mathbb{1} - K)| \leq \exp\big(\frac{1}{2}\|K\|_{\mathcal{N}}^2\big)$, which implies the existence of the determinant.

(ii) $\det(\mathbb{1} - K) = 0$ iff $\lambda_n(K) = 1$ for some $n \in \mathbb{N}$.

(iii) $\det(\mathbb{1} - K) = 0$ iff $\mathbb{1} - K$ is not invertible.

PROOF : (ii) and (iii) are obvious. (i) follows by the inequality

$$|(1-z)\exp(z)| \leq \exp\left(\frac{1}{2}|z|^2\right)$$

which holds for all $z \in \mathbb{C}$ (see for instance Nevanlinna [22] p. 225).

Hence we obtain, using (2.1),

$$\begin{aligned} |\det(\mathbb{1} - K)| &\leq \prod_{n=1}^{\infty} \exp\left(\frac{1}{2}|\lambda_n(K)|^2\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{2}|\lambda_n(K)|^2\right) \\ &\leq \exp\left(\frac{1}{2}\|K\|_{\mathcal{N}}^2\right). \end{aligned}$$

■

Let Z_0 be as mentioned above and $Z = Z_0 + K$, $K \in \mathcal{N}(X)$, such that $\sigma_{ess}(Z) = \sigma_{ess}(Z_0) = [a, b]$.

Take $\lambda_0 \in \rho(Z_0)$. Then

$$(\lambda_0 \mathbb{1} - Z)R_{Z_0}(\lambda_0) = \mathbb{1} - KR_{Z_0}(\lambda_0).$$

The operator $\mathbb{1} - KR_{Z_0}(\lambda_0)$ is not invertible iff $\lambda_0 \in \sigma_{disc}(Z)$. Because $\mathcal{N}(X)$ is an ideal $KR_{Z_0}(\lambda_0) \in \mathcal{N}(X)$. Therefore the determinant

$$\det\big(\mathbb{1} - KR_{Z_0}(\lambda)\big)$$

is well defined for any $\lambda \in \rho(Z_0)$. Denote by

$$d(\cdot) := \det\big(\mathbb{1} - KR_{Z_0}(\cdot)\big),$$

i.e. the map $\rho(Z_0) \ni \lambda \mapsto \det\big(\mathbb{1} - KR_{Z_0}(\lambda)\big)$. The complex number λ_0 is a zero of d iff $\lambda_0 \in \sigma_{disc}(Z)$. Denoting the zero set of d by $\mathcal{Z}(d)$ it follows that $\mathcal{Z}(d) = \sigma_{disc}(Z)$.

Remark 2.7 : It is possible to extend the domain of d to $\rho(Z_0) \cup \{\infty\}$ by setting $d(\infty) := 1$. This definition makes sense, since $\lim_{\lambda \to \infty} KR_{Z_0}(\lambda) = 0$ (see e.g. Kato [20], p. 176) and $\det(\mathbb{1} - 0) = 1$.

Thus we are able to analyze $\sigma_{disc}(Z)$ by studying the zeros of the function d defined on $\mathbb{C} \setminus [a, b]$.

We will follow the strategy used in [6], that is based on Jensen's identity (see Rudin [25], p. 307) for the zeros of holomorphic functions in the open unit disc $\mathbb{D}$. Let ϕ be the conformal map from $\mathbb{D} \setminus \{0\}$ to $\mathbb{C} \setminus [a, b]$ given by

$$\phi(w) = \frac{b - a}{4}\big(w + w^{-1} + 2\big) + a. \tag{2.2}$$

Then the new function h, given by

$$h(w) := \begin{cases} (d \circ \phi)(w), & w \in \mathbb{D} \setminus \{0\} \\ 1, & w = 0 \end{cases}$$

is defined on $\mathbb{D}$.

Let $\mathcal{Z}(h)$ be the set of zeros of h. If we can show that h is a holomorphic function and since $|h(0)| = 1$

$$\sum_{w \in \mathcal{Z}(h), |w| \leq r} \log \left| \frac{r}{w} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta \tag{2.3}$$

with $0 < r < 1$. If h is a holomorphic function and if there is a proper estimate for $\log |h(re^{i\theta})|$ such that the left hand side in (2.3) gives an effective sum over the zeros of h, then we can derive from this sum a new sum over the discrete spectrum of Z.

For instance if, in the simplest case,

$$\log |h(w)| \leq \frac{C_0}{\left(1 - |w|\right)^{\alpha}}, \quad w \in \mathbb{D}, \tag{2.4}$$

with $\alpha > 0$ and C_0 a positive constant, then

$$\sum_{w \in \mathcal{Z}(h)} \left(1 - |w|\right)^{\alpha + \tau + 1} \leq C(\alpha, \tau) C_0 \tag{2.5}$$

for any $\tau > 0$ (see [6], Theorem 3.3.1).
(2.5) implies

$$\sum_{\lambda \in \sigma_{disc}(Z)} \left(1 - |\phi^{-1}(\lambda)|\right)^{\alpha + \tau + 1} \leq C(\alpha, \tau) C_0. \tag{2.6}$$

Obviously by Lemma 2.6 we know, since $\mathcal{N}(X)$ is a Banach ideal and $R_{Z_0}(\phi(w))$ is bounded, that

$$\log |h(w)| \leq \frac{1}{2} \|K\|_{\mathcal{N}}^2 \|R_{Z_0}(\phi(w))\|^2. \tag{2.7}$$

If we can estimate the resolvent in a similar way that finally (see Golinskii and Kupin [13], Hansmann and Katriel [17])

$$\log |h(w)| \leq C_0 \frac{|w|^2}{|w-1|^2 |w+1|^2 (1 - |w|)^2} \tag{2.8}$$

then for any $\tau > 0$

$$\sum_{\lambda \in \sigma_{disc}(Z)} \frac{(1-|\phi^{-1}(\lambda)|)^{3+\tau}}{|\phi_1^{-1}(\lambda)|^{1+\tau}} |(\phi_1^{-1}(\lambda))^2 - 1|^{1+\tau} \leq c(\tau) C_0.$$

From this estimate we are able to derive

$$\sum_{\lambda \in \sigma_{disc}(Z)} \frac{\big(\mathrm{dist}(\lambda, [a,b])\big)^{3+\tau}}{|\lambda - a||\lambda - b|} \leq c(\tau) \cdot C_0 \tag{2.9}$$

(see [6], proof of Theorem 4.2.2).

The next part of this article is the proof of the holomorphy of d and an assertion about the connection between the algebraic multiplicity of discrete eigenvalues of $Z_0 + K$ and the order of the zeros of the perturbation determinant $\det(\mathbb{1} - KR_{Z_0}(\cdot))$. For the holomorphy of d we can also refer to [19], where you have to use the fact, that every nuclear operator is a p-summable operator and can be approximated by finite rank operators in the nuclear norm.

In the final sections we consider nuclear perturbations of the discrete Laplacian in $l^p(\mathbb{Z})$ and certain perturbations by integral operators in $C[\alpha, \beta]$. It turns out that an estimate like (2.9) can be verified.

3. Holomorphy of Nuclear Determinants and Zeros of the Perturbation Determinant

***Theorem* 3.1** — *Let $\lambda \mapsto K(\lambda)$ be an analytic map in $\Omega \subseteq \mathbb{C}$ with values in $\mathcal{N}(X)$. Then d, defined by*

$$d(\lambda) := \det(\mathbb{1} - K(\lambda)),$$

is a holomorphic function in Ω

Proof : At first let us note, that $\mathcal{N}^2(X) := \{AB : A, B \in \mathcal{N}(X)\}$ together with the quasi-norm $\|K\|_{\mathcal{N}^2} = \inf_{A,B:AB=K} \|A\|_{\mathcal{N}} \|B\|_{\mathcal{N}}$ creates a quasi-Banach ideal[3] (see Pietsch [23] p.27).

Since the eigenvalues of every operator in $\mathcal{N}^2(X)$ are summable[4], the space $\mathcal{N}^2(X)$ supports an unique continuous trace, which is spectral (see [23] p. 180). This assertion is equivalent to $\mathcal{N}^2(X)$

[3] A quasi-Banach ideal is an ideal which is complete according to a quasi-norm, where the only difference to a norm is, that instead of the triangle inequality there has to be a $c > 1$ s.t. $\|x + y\|_{\mathcal{N}^2} \leq c(\|x\|_{\mathcal{N}^2} + \|y\|_{\mathcal{N}^2})$ for all $x, y \in \mathcal{N}^2(X)$.

[4] $\mathcal{N}(X) \subseteq \Pi_2(X)$ (the space of 2-summable operators), hence $\mathcal{N}^2(X) \subseteq \Pi_2^2(X)$. Since every Operator in Π_2^2 has summable eigenvalues, the same is true for every operator in $\mathcal{N}^2(X)$ (see e.g. [19] p. 85-86).

supports an unique continuous determinant $\tilde{\det}$ which is spectral (see [23], p. 210), i.e.

$$\tilde{\det}(\mathbb{1} - K) = \prod_{\lambda \in \sigma(K)} (1 - \lambda), \text{ for all } K \in \mathcal{N}^2(X).$$

Following [23] p. 193, $\lambda \mapsto \tilde{\det}\big(\mathbb{1} - \tilde{K}(\lambda)\big)$ is holomorphic on the domain $\Omega \subseteq \mathbb{C}$, if $\lambda \mapsto \tilde{K}(\lambda)$ is analtytic on the domain Ω.

Now let $\lambda \mapsto K(\lambda) \in \mathcal{N}(X)$. We define

$$\begin{aligned}
\big(\mathbb{1} - K(\lambda)\big) \exp\big(K(\lambda)\big) &= \big(\mathbb{1} - K(\lambda)\big) \left(\sum_{j=0}^{\infty} \frac{1}{j!} K(\lambda)^j \right) \\
&= \mathbb{1} + \underbrace{\sum_{j=2}^{\infty} \frac{1}{j!} K(\lambda)^j - \sum_{j=1}^{\infty} \frac{1}{j!} K(\lambda)^{j+1}}_{\in \mathcal{N}^2(X)} \\
&= \mathbb{1} - \Big(\underbrace{\sum_{j=1}^{\infty} \frac{1}{j!} K(\lambda)^j - \sum_{j=2}^{\infty} \frac{1}{j!} K(\lambda)^{j+1}}_{=: \tilde{K}(\lambda)} \Big) \\
&= \mathbb{1} - \tilde{K}(\lambda).
\end{aligned}$$

Using the spectral mapping-theorem for bounded operators (see e.g. [11] p. 16), we also see

$$\sigma\big(\tilde{K}(\lambda)\big) = \{1 - (1 - \mu)\exp(\mu) | \, \mu \in \sigma(K(\lambda))\}.$$

Finally we have the equation

$$\begin{aligned}
\tilde{\det}\big(\mathbb{1} - \tilde{K}(\lambda)\big) &= \prod_{\tilde{\mu} \in \sigma\left(\tilde{K}(\lambda)\right)} (1 - \tilde{\mu}) \\
&= \prod_{\mu \in \sigma\left(K(\lambda)\right)} \Big(1 - \big(1 - (1 - \mu)\exp(\mu)\big)\Big) = \prod_{\mu \in \sigma\left(K(\lambda)\right)} (1 - \mu)\exp(\mu) \\
&= \det\big(\mathbb{1} - K(\lambda)\big).
\end{aligned}$$

Since the left hand side of the previous equation depends holomorphically on λ, the same is true for the right hand side. ■

We can apply the general result in Theorem 3.1 to our initial problem of Section 2.

***Theorem* 3.2** — *Let $Z_0 \in \mathcal{B}(X)$, X Banach space, and $Z = Z_0 + K$, $K \in \mathcal{N}(X)$. Then the determinant*

$$d(\cdot) = \det\left(\mathbb{1} - KR_{Z_0}(\cdot)\right)$$

is holomorphic on $\mathbb{C} \setminus [a, b]$.

Therefore $h = d \circ \phi$, with ϕ from (2.2) is a holomorphic function in $\mathbb{D}$.

Moreover there is the following connection between the algebraic multiplicity $m_\lambda(Z)$ of any eigenvalue λ of Z and the order $o_\lambda(d)$ of any zero of d.

$$\lambda \in \sigma_d(Z) \; with \; m_\lambda(Z) = m \Leftrightarrow \lambda \in \mathcal{Z}(h) \; with \; o_\lambda(d) = m.$$

PROOF : Since $R_{Z_0}(\cdot)$ is analytic on $\rho(Z_0)$, d is holomorphic on $\rho(Z_0)$.

In Section 2 we have already mentioned, that the zeros of d coincide with the discrete spectrum of Z. So only we have to show that the algebraic multiplicity of any discrete eigenvalue of Z is equal to its order as a zero of d.

For this lets fix an eigenvalue $\lambda_0 \in \sigma_{disc}(Z)$ and an $\epsilon > 0$, such that $B_\epsilon(\lambda_0) \cap \sigma_{disc}(Z) = \{\lambda_0\}$.

Next we choose a sequence $(K_n) \subseteq \mathcal{F}(X)$ with

$$\|K - K_n\| \to 0 \text{ as } n \to \infty. \tag{3.1}$$

For $Z_n := Z_0 + K_n$ (3.1) implies

$$\|Z - Z_n\| \to 0 \text{ as } n \to \infty. \tag{3.2}$$

The statement in Gohberg *et al.*, [11] Chapter II Theorem 4.2. and (3.2) implies that there is $N_\epsilon \in \mathbb{N}$ with

$$\sum_{\mu \in \sigma_{disc}(Z_n) \cap B_\epsilon(\lambda_0)} m_\mu(Z_n) = m_{\lambda_0}(Z) \text{ for all } n \geq N_\epsilon. \tag{3.3}$$

Associated to $\{Z_n\}$ we have a sequence of holomorphic funtions

$$d_n(\lambda) := \det\left(\mathbb{1} - K_n R_{Z_0}(\lambda)\right) \text{ for } \lambda \in \rho(Z_0)$$

with $d_n(\lambda) = 0$ iff $\lambda \in \sigma_{disc}(Z_n)$. Since $K_n \in \mathcal{F}(X)$ we know that $K_n R_{Z_0}(\lambda) \in \mathcal{F}(X)$. So for every $\lambda \in \rho(Z_0)$ we can consider $K_n R_{Z_0}(\lambda)$ as a Hilbert-Schmidt operator. Following Hansmann [14] p. 20-22, we deduce

$$\mu \in \sigma_{disc}(Z_n) \text{ with } m_\mu(Z_n) = m \Leftrightarrow d_n(\mu) = 0 \text{ and } o_\mu(d_n) = m. \tag{3.4}$$

Using (3.1) we can conclude that $\|KR_{Z_0}(\lambda) - K_nR_{Z_0}(\lambda)\| \to 0$ locally uniformly. This result implies that $d_n \to d$ locally uniformly.

Thus we can find $N \geq N_\epsilon$ such that

$$|d_n(\lambda) - d(\lambda)| \leq |d(\lambda)|$$

for all $\lambda \in \partial B_\epsilon(\lambda_0)$ and for $n \geq N$. Rouche's Theorem (see e.g. [25], p. 225) provides

$$\sum_{\mu \in \mathcal{Z}(d_n) \cap B_\epsilon(\lambda_0)} o_\mu(d_n) = o_{\lambda_0}(d) \text{ for all } n \geq N.$$

Now using this formula, equation (3.3) and equivalence (3.4) we receive

$$o_{\lambda_0}(d) = m_{\lambda_0}(Z).$$

On the other hand, if λ_0 is a zero of d, we already know that λ_0 is a discrete eigenvalue of Z. Hence, by the previous argumentation, the algebraic multiplicity of λ_0 as an eigenvalue of Z is equal to the order of λ_0 as a zero of d. ■

Due to the first assertion in Theorem 3.2 Jensen's identity (see (2.3)) holds for $h = d \circ \phi$ and we can apply the theory developed in [6]. In particular we can use the following result (see Hansmann and Katriel [17] Theorem 4), which is an extension of a theorem by Borichev *et al*., [2].

***Theorem* 3.3** — *Let $h : \mathbb{D} \to \mathbb{C}$ be holomorphic and*

$$|h(w)| \leq \exp\left(\frac{C_0|w|^\gamma}{(1-|w|)^\alpha \prod_{j=1}^N |w-\xi_j|^{\beta_j}}\right), \quad w \in \mathbb{D}, h(0) = 1$$

with $|\xi_j| = 1, \xi_i \neq \xi_j$, for $i \neq j, \alpha > 0, \beta_j \geq 0, \gamma, C_0 \geq 0$. Then we have for $\epsilon, \tau > 0$

$$\sum_{w \in \mathcal{Z}(h)} \frac{(1-|w|)^{\alpha+\tau+1}}{|w|^{(\gamma-\epsilon)_+}} \prod_{j=1}^N |w-\xi_j|^{(\beta_j-1+\tau)_+} \leq C(\alpha,\beta,\xi,\epsilon,\tau)C_0,$$

where $C(\alpha,\beta,\xi,\epsilon,\tau) > 0$ denotes a constant depending on $\alpha,\beta,\xi,\epsilon,\tau$; $(x)_+ := \max(x,0)$ for $x \in \mathbb{R}$.

This theorem will be very useful for the examples in the next sections. (see (4.3) and (5.1)).

4. The Discrete Laplacian on $l^p(\mathbb{Z})$

For illustration we will apply the results from Sections 2 and 3 to the discrete Laplacian on $l^p(\mathbb{Z})$, $1 \leq p \leq \infty$, where the case $p = \infty$ has to be stressed, since $l^\infty(\mathbb{Z})$ is not compatible to $l^2(\mathbb{Z})$. This operator $\Delta_p : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ is given by

$$(\Delta_p f)(n) := f(n-1) + f(n+1), \quad f \in l^p(\mathbb{Z}).$$

Δ_p is a bounded operator on $l^p(\mathbb{Z})$, $p \in [0, \infty]$. It can be rewritten as

$$\Delta_p f = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \end{pmatrix} \tag{4.1}$$

for $f \in l^p(\mathbb{Z})$, $f = \begin{pmatrix} \vdots \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \end{pmatrix}$.

$\Delta_p \in \mathcal{B}(l^p(\mathbb{Z}))$ follows by

$$\|\Delta_p f\|_p \leq 2\|f\|_p, \; f \in l^p(\mathbb{Z}).$$

Proposition 4.1 — The resolvent-set of Δ_p is $\mathbb{C} \setminus [-2, 2]$ (hence $\sigma(\Delta_p) = \sigma_{ess}(\Delta_p) = [-2, 2]$) and the resolvent for $z \in \rho(\Delta_p)$ is given by

$$R_{\Delta_p}(z) = (z\mathbb{1} - \Delta_p)^{-1} := \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \dots & b_{-1}(z) & b_0(z) & b_1(z) & b_2(z) & \dots \\ \dots & b_{-2}(z) & b_{-1}(z) & b_0(z) & b_1(z) & \dots \\ \dots & b_{-3}(z) & b_{-2}(z) & b_{-1}(z) & b_0(z) & \dots \\ & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{4.2}$$

with

$$b_k(z) := \left(\frac{z \pm \sqrt{z^2 - 4}}{2} \right)^{|k|} \frac{1}{\sqrt{z^2 - 4}} \text{ for } k \in \mathbb{Z} \text{ and } z \in \rho(\Delta_p) = \mathbb{C} \setminus [-2, 2].$$

The sign of $\sqrt{z^2-4}$ should be chosen, such that the inequality $|z \pm \sqrt{z^2-4}| < 2$ is fulfilled.

Moreover, $R_{\Delta_p}(z)$ is a bounded operator and

$$\|R_{\Delta_p}(z)\|_{l^p} \leq \frac{1}{|z^2-4|^{1/2}} \frac{2+|z \pm \sqrt{z^2-4}|}{2-|z \pm \sqrt{z^2-4}|},$$

$z \in \rho(\Delta_p) = \mathbb{C} \setminus [-2,2]$.

PROOF : Let $B_p(z)$, with $z \in \mathbb{C} \setminus [-2,2]$, be the right hand side of (4.2), then we have (see e.g. Kato [20], p. 143)

$$\begin{aligned}
\|B_p(z)f\|_p &\leq \Big(\sum_{k=-\infty}^{\infty} |b_k(z)| \Big)^{1-\frac{1}{p}} \Big(\sum_{k=-\infty}^{\infty} |b_k(z)| \Big)^{\frac{1}{p}} \|f\|_p \\
&= \Big(\sum_{k=-\infty}^{\infty} |b_k(z)| \Big) \|f\|_p = \frac{1}{|z^2-4|^{\frac{1}{2}}} \Big(2 \frac{1}{1-\left| \frac{z \pm \sqrt{z^2-4}}{2} \right|} - 1 \Big) \|f\|_p \\
&= \frac{1}{|z^2-4|^{\frac{1}{2}}} \Big(\frac{2+|z \pm \sqrt{z^2-4}|}{2-|z \pm \sqrt{z^2-4}|} \Big) \|f\|_p.
\end{aligned}$$

And a direct calculation shows

$$B_p(z)(\Delta_p - z)f = f = (\Delta_p - z)B_p(z)f$$

such that $\mathbb{C} \setminus [-2,2] \subseteq \rho(\Delta_p)$ and $B_p(z) = R_{\Delta_p}(z)$.

To show $\mathbb{C} \setminus [-2,2] = \rho(\Delta_p)$, we show that for $\lambda_0 \in [-2,2]$

$$\|R_{\Delta_p}(z)\| \overset{z \to \lambda_0}{\to} \infty.$$

We will show this in case of $p = \infty$:

For $z \in \mathbb{C} \setminus [-2,2]$ we define $f := \big(\frac{|b_k(z)|}{b_k(z)} \big)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$, then $\|f\|_\infty = 1$.

Since $|\lambda_0 \pm \sqrt{\lambda_0^2 - 4}| = 2$ for $\lambda_0 \in [-2,2]$, we have

$$\begin{aligned}
\|R_{\Delta_p}(z)\| \geq \|R_{\Delta_p}(z)f\|_\infty = \sum_{k=-\infty}^{\infty} |b_k(z)| = \\
\frac{1}{|z^2-4|^{\frac{1}{2}}} \Big(\frac{2+|z \pm \sqrt{z^2-4}|}{2-|z \pm \sqrt{z^2-4}|} \Big) \overset{z \to \lambda_0}{\to} \infty.
\end{aligned}$$

For $p \neq \infty$ we can show this in a similar way if we take $f = (\delta_{i,k})_{k \in \mathbb{Z}}$, where $\delta_{i,k} = 1$ if $i = k$ and 0 else. ■

Δ_p plays the role of Z_0 in the sections above. Now we add a nuclear perturbation and study the discrete spectrum of the perturbed operator.

Let $K \in \mathcal{N}(l^p(\mathbb{Z}))$ and denote

$$Z = \Delta_p + K.$$

Since K is compact we have

$$\sigma_{ess}(Z) = \sigma_{ess}(\Delta_p) = [-2, 2].$$

According to Corollary 3.2 a holomorphic function is given by

$$d(\lambda) = \det\big(\mathbb{1} - KR_{\Delta_p}(z)\big),$$

d defined on $\mathbb{C} \setminus [-2, 2]$. In order to use the method from Section 2 we take the conformal map (see (2.2))

$$\phi(w) = w + w^{-1}.$$

ϕ maps $\mathbb{D} \setminus \{0\}$ to $\mathbb{C} \setminus [-2, 2]$.

Denote $h := d \circ \phi$, then (see (2.7))

$$\log|h(w)| \leq \frac{1}{2}\|K\|_{\mathcal{N}}^2 \|R_{\Delta_p}(\phi(w))\|^2.$$

For the norm of the resolvent we obtain

$$\begin{aligned}
\|R_{\Delta_p}(w + w^{-1})\| &\leq \frac{1}{|(w + w^{-1})^2 - 4|^{\frac{1}{2}}} \left(\frac{2 + |w + w^{-1} \pm \sqrt{(w + w^{-1})^2 - 4}|}{2 - |w + w^{-1} \pm \sqrt{(w + w^{-1})^2 - 4}|} \right) \\
&= \frac{1}{|w - w^{-1}|} \left(\frac{2 + |w + w^{-1} \pm \sqrt{(w - w^{-1})^2}|}{2 - |w + w^{-1} \pm \sqrt{(w - w^{-1})^2}|} \right) \\
&= \frac{|w|}{|w^2 - 1|} \left(\frac{2 + 2|w|}{2 - 2|w|} \right) \\
&\leq \frac{2|w|}{|w - 1||w + 1|(1 - |w|)}, \quad w \in \mathbb{D} \setminus \{0\}.
\end{aligned}$$

Hence

$$\log|h(w)| \leq 2\|K\|_{\mathcal{N}}^2 \frac{|w|^2}{(1 - |w|)^2 |w - 1|^2 |w + 1|^2}. \tag{4.3}$$

There is a holomorphic extension for h to $\mathbb{D}$ realized by $h(0) := d(\infty) := 1$. Using Theorem 3.3 with $\epsilon = 1 - \tau$

$$\sum_{w \in \mathcal{Z}(h)} \frac{(1-|w|)^{3+\tau}}{|w|^{1+\tau}} |w^2-1|^{1+\tau} \leq C(\tau)\|K\|_{\mathcal{N}}^2$$

with $0 < \tau < 1$.

For transforming these estimate to an estimate for $\sigma_{disc}(Z)$ we use the following relations ([6], p. 130).

Lemma 4.2 — Let $z = w + w^{-1}$, $w \in \mathbb{D} \setminus \{0\}$. Then we have

$$\frac{1}{2}\frac{|w^2-1|(1-|w|)}{|w|} \leq \text{dist}(z,[-2,2]) \leq \frac{1+\sqrt{2}}{2}\frac{|w^2-1|(1-|w|)}{|w|}$$

and

$$\left|\frac{w^2-1}{w}\right|^2 = |z^2-4|.$$

***Theorem* 4.3** — *Let $Z = \Delta_p + K$ be in $l^p(\mathbb{Z})$ with $K \in \mathcal{N}(l^p(\mathbb{Z}))$, $1 \leq p \leq \infty$. Then we get for $\tau > 0$*

$$\sum_{z \in \sigma_{disc}(Z)} \frac{\text{dist}(z,[-2,2])^{3+\tau}}{|z^2-4|} \leq C(\tau)\|K\|_{\mathcal{N}}^2. \tag{4.4}$$

PROOF : Let $w \in \mathbb{D} \setminus \{0\}$, $z = w + w^{-1}$. By Lemma 4.2 we obtain

$$\begin{aligned}(1-|w|)^{3+\tau}\left|\frac{w^2-1}{w}\right|^{1+\tau} &= \left(\frac{(1-|w|)|w^2-1|}{|w|}\right)^{3+\tau}\left|\frac{w}{w^2-1}\right|^2 \\ &\geq \left(\frac{2}{1+\sqrt{2}}\right)^{3+\tau}\frac{\text{dist}(z,[-2,2])^{3+\tau}}{|z^2-4|}.\end{aligned}$$

■

Remark 4.4 : Whenever $1 \leq p < \infty$ the Banach space $l^p(\mathbb{Z})$ is compatible to $l^2(\mathbb{Z})$. Having a nuclear operator K_1 on $l^p(\mathbb{Z})$ and a Hilbert-Schmidt operator K_2 on $l^2(\mathbb{Z})$ which are consistent we know (see Davies [4] p. 109) that the spectra of K_1 and K_2 coincide. Moreover in this case $K_1 R_{\Delta_p}(z)$ is nuclear in $l^p(\mathbb{Z})$, $K_2 R_{\Delta_2}(z)$ is Hilbert-Schmidt in $l^2(\mathbb{Z})$, both operators are consistent and $1 \in \sigma(K_1 R_{\Delta_p}(z))$ iff $1 \in \sigma(K_2 R_{\Delta_2}(z))$ and this means that the zero set of $\det(1 - K_1 R_{\Delta_p}(z))$ coincides with the zero set of $\det(1 - K_2 R_{\Delta_2}(z))$ (compare with the discussion after Lemma 2.6). That means we only have to consider the Hilbert space case which was already studied by Borichev *et*

al., [2] and Hansmann, Katriel [17]. Recall that $l^\infty(\mathbb{Z})$ and $l^2(\mathbb{Z})$ are not compatible, and the assertion in Theorem 4.3 holds also for $p = \infty$. Nevertheless there are nuclear operators on $l^p(\mathbb{Z})$ which are not consistent to a Hilbert-Schmidt operator in $l^2(\mathbb{Z})$ (see Example 2.4).

Remark 4.5 : With (4.4) it is possible to give an estimate for the number of eigenvalues in certain regions of the complex plane.

Define

$$M := \{z \in \mathbb{C} : |\mathrm{Re}(z)| > 2, r < |z+2| < R\}$$

with $R > r > 0$ (since the operator norm is a bound for the spectrum, it makes sense to set $\|Z\| - 2 \geq R$). Then for $\lambda \in M$ the inequalities

$$\mathrm{dist}(\lambda, [-2,2]) > r,\ \frac{1}{|\lambda+2|} > \frac{1}{R},\ \frac{1}{|\lambda-2|} > \frac{1}{R+4}$$

are valid.

By (4.4) we obtain for every $\tau > 0$

$$\begin{aligned} &\sum_{\lambda \in \sigma_{disc}(Z) \cap M} \frac{r^{3+\tau}}{R(R+4)} \\ &\leq \sum_{z \in \sigma_{disc}(Z) \cap M} \frac{\mathrm{dist}(z, \sigma_{ess}(Z))^{3+\tau}}{|z^2-4|} \\ &\leq \sum_{z \in \sigma_{disc}(Z)} \frac{\mathrm{dist}(z, \sigma_{ess}(Z))^{3+\tau}}{|z^2-4|} \leq \|K\|_{\mathcal{N}}^2 C(\tau). \end{aligned}$$

We can conclude

$$\#\big(\sigma_{disc}(Z) \cap M\big) \leq \frac{R(R+4)}{r^{3+\tau}} C(\tau) \|K\|_{\mathcal{N}}^2.$$

$$N := \{z \in \mathbb{C} : -2 \leq \mathrm{Re}(z) \leq 2,\ r < |\mathrm{Im} z| < R\}$$

with $\|Z\| \geq R > r > 0$ we receive for every $\lambda \in N$ the inequalities

$$\mathrm{dist}(\lambda, [-2,2]) > r,\ \frac{1}{|\lambda+2|} > \frac{1}{\sqrt{16+R^2}},\ \frac{1}{|\lambda-2|} > \frac{1}{\sqrt{16+R^2}}.$$

By the same argumentation we get for every $\tau > 0$

$$\sum_{\lambda \in \sigma_{disc}(Z) \cap N} \frac{r^{3+\tau}}{16+R^2} \leq C(\tau) \|K\|_{\mathcal{N}}.$$

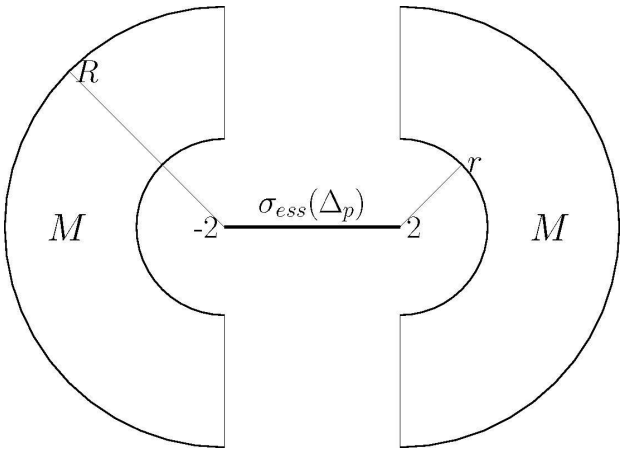

Figure 1: Region M

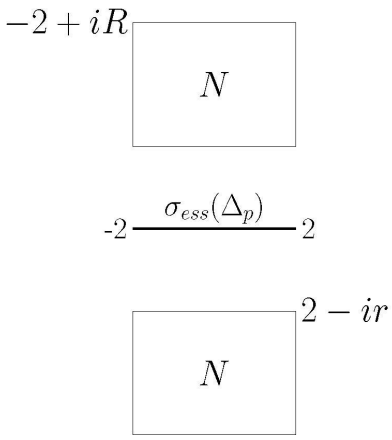

Figure 2: Region N

Hence for the number of eigenvalues in N

$$\#\big(\sigma_{disc}(Z)\cap N\big) \leq \frac{16+R^2}{r^{3+\tau}}\|K\|_{\mathcal{N}}^2 C(\tau).$$

Remark 4.6 : If we take a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq \sigma_{disc}(Z)$ converging to some $\lambda_0 \in [-2,2]$ we can consider the following cases (restricting the sequence to a subsequence):

(i.a) $\lambda_0 = -2$ and $\mathrm{Re}(\lambda_k) \leq -2$. (i.b) $\lambda_0 = 2$ and $\mathrm{Re}(\lambda_k) \geq 2$.

(ii.a) $\lambda_0 = -2$ and $\mathrm{Re}(\lambda_k) > -2$. (ii.b) $\lambda_0 = 2$ and $\mathrm{Re}(\lambda_k) < 2$.

(iii) $\lambda_0 \in (-2,2)$.

Because of the symmetry it is sufficient to consider only the cases (i.a), (ii.a) and (iii).

In case (i.a) the estimate (4.4) implies for $\tau > 0$.

$$\sum_{k=1}^{\infty} |\lambda_k + 2|^{2+\tau} < \infty,$$

in (ii.a)

$$\sum_{k=1}^{\infty} \frac{|\mathrm{Im}(\lambda_k)|^{3+\tau}}{|\lambda_k+2|} < \infty$$

and finally in (iii)

$$\sum_{k=1}^{\infty} |\mathrm{Im}(\lambda_k)|^{3+\tau} < \infty.$$

One disadvantage of Theorem 4.3 is, that it comes from (4.3), which depends to strong on the behaviour of the resolvent of Δ_p, but not really on the nuclear operator K.

However, if we concentrate on the class of Jacobi operators, we can derive a better estimate.

We call an operator $J \in \mathcal{B}\big(l^1(\mathbb{Z})\big)$ Jacobi if it is defined by

$$Jf := \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & d_{-1} & c_{-1} & & & \\ & & a_0 & d_0 & c_0 & & \\ & & & a_1 & d_1 & c_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} f, \text{ for all } f \in l^1(\mathbb{Z}).$$

Then J is nuclear, if and only if $(a_k), (d_k), (c_k) \in l^1(\mathbb{Z})$. In this case we have

$$\|J\|_{\mathcal{N}} = \sum_{k\in\mathbb{Z}} \sup\{|a_k|, |d_k|, |c_k|\} < \infty.$$

***Theorem* 4.7** — *Let $Z = \Delta_1 + J$ be in $l^1(\mathbb{Z})$ with $J \in \mathcal{N}(l^1(\mathbb{Z}))$ a Jacobi operator. Then we get for $\tau > 0$*

$$\sum_{z\in\sigma_{disc}(Z)} \operatorname{dist}(z, [-2, 2])^{1+\tau} \leq C(\tau)\|J\|^2_{\mathcal{N}}. \tag{4.5}$$

PROOF : It is possible to estimate the nuclear norm of $JR_\Delta(\lambda)$:

$$\begin{aligned} \|JR_\Delta(\lambda)\|_{\mathcal{N}} &= \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} |a_k b_j(\lambda) + d_k b_{j-1}(\lambda) + c_k b_{j-2}(\lambda)| \\ &\leq \sum_{k\in\mathbb{Z}} \max\{|a_k|, |d_k|, |c_k|\} \sup_{j\in\mathbb{Z}} |3b_j(\lambda)| \\ &\leq \sum_{k\in\mathbb{Z}} \max\{|a_k|, |d_k|, |c_k|\} 3 \frac{1}{|\sqrt{\lambda^2 - 4}|} \\ &= \frac{3\|J\|_{\mathcal{N}}}{|\sqrt{\lambda^2 - 4}|}. \end{aligned}$$

This implies that

$$\log |\underbrace{\det(\mathbb{1} - KR_\Delta(\lambda))}_{:=d(\lambda)}| \leq \frac{\frac{9}{2}\|J\|^2_{\mathcal{N}}}{|\lambda^2 - 4|}.$$

So the holomorphic function $h := d \circ \phi$ satisfies

$$\log |h(w)| \leq \frac{\frac{9}{2}\|J\|^2_{\mathcal{N}}|w|^2}{|w - 1|^2|w + 1|^2}, \text{ for all } w \in \mathbb{D}. \tag{4.6}$$

Using Theorem 3.3 we can derive for the zeros of h

$$\sum_{w\in\mathcal{Z}(h)} \frac{(1-|w|)^{1+\tau}}{|w|^{1+\tau}}|w^2-1|^{1+\tau} \le C(\tau)\|J\|^2_{\mathcal{N}}.$$

The left inequality in Lemma 4.2 gives us the desired inequality. ■

Remark 4.8 : Similar to Remark 4.5 we can derive upper bounds for the number of discrete eigenvalues in the set

$$M_R := \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \sigma_{ess}(\Delta)) \ge R\}$$

from (4.5),

$$\begin{aligned} C(\tau)\|J\|^2_{\mathcal{N}} &\ge \sum_{\lambda\in\sigma_{disc}(Z)} \text{dist}(\lambda,\sigma_{ess}(Z))^{1+\tau} \\ &\ge \sum_{\lambda\in\sigma_{disc}(Z)\cap M_R} \text{dist}(\lambda,\sigma_{ess}(Z))^{1+\tau} \\ &\ge \sum_{\lambda\in\sigma_{disc}(Z)\cap M_R} R^{1+\tau} = \#\big(\sigma_{disc}(Z)\cap M_R\big)R^{1+\tau}, \end{aligned}$$

which holds for every $\tau > 0$.

Hence, as an upper bound we obtain

$$\#\big(\sigma_{disc}(Z)\cap M_R\big) \le \frac{C(\tau)\|J\|^2_{\mathcal{N}}}{R^{1+\tau}}.$$

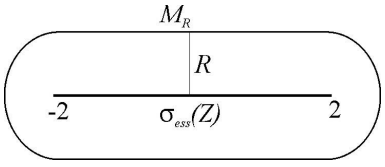

Figure 3: Region M_R

5. Nuclear Perturbations of the Multiplication Operator on $C[\alpha,\beta]$

As another easy application, consider $X = C[\alpha,\beta]$ (the space of continuous functions). We define

$$Z_0 : X \to X \text{ with } (Z_0 f)(t) := M(t)f(t),$$

where M is a real-valued contiuous function on $[\alpha,\beta]$.

We know $Z_0 \in \mathcal{B}(X), \sigma(Z_0) = \sigma_{ess}(Z_0) = [\min(M), \max(M)] =: [a, b]$ and

$$\big(R_{Z_0}(\lambda)f\big)(x) = \frac{f(x)}{M(x) - \lambda},\ \lambda \in \rho(Z_0).$$

In this example it is possible to compute the operator norm of the resolvent exactly and we receive

$$\|R_{Z_0}(\lambda)\| = \|\frac{1}{M - \lambda}\|_\infty = \frac{1}{\mathrm{dist}(\lambda, [a, b])}.$$

Defining an integral operator

$$K : X \to X \text{ with } Kf(t) := \int_\alpha^\beta k(t, s)f(s)ds$$

with $k \in C[\alpha, \beta]^2$ we know by Example 2.2 that this operator is nuclear.

Then for

$$Z := Z_0 + K$$

the function

$$d(\lambda) := \det\big(\mathbb{1} - KR_{Z_0}(\lambda)\big),\ \lambda \in \rho(Z_0)$$

defines a holomorphic function with zero-set equal to $\sigma_{disc}(Z)$ and

$$|d(\lambda)| \leq \frac{1}{2}\|K\|_{\mathcal{N}}^2 \frac{1}{\mathrm{dist}(\lambda, [a, b])^2}.$$

Setting $\phi(w) := \frac{b-a}{4}(w + w^{-1} + 2),\ w \in \mathbb{D} \setminus \{0\}$ (ϕ maps $\mathbb{D} \setminus \{0\}$ to $\mathbb{C} \setminus [a, b]$) we receive (for the holomorphic function $d \circ \phi$)

$$|(d \circ \phi)(w)| \leq \frac{1}{2}\|K\|_{\mathcal{N}}^2 \frac{1}{\mathrm{dist}(\phi(w), [a, b])^2}.$$

Using the estimate

$$\frac{b-a}{8}\frac{|w^2 - 1|(1 - |w|)}{|w|} \leq \mathrm{dist}(\phi(w), [a, b]) \leq \frac{(b-a)(1+\sqrt{2})}{8}\frac{|w^2 - 1|(1 - |w|)}{|w|}$$

with $w \in \mathbb{D} \setminus \{0\}$, which is a generalization of the estimate in Lemma 4.2 (see [6], Lemma 4.2.1) we obtain

$$|d \circ \phi(w)| \leq \frac{1}{2}C(a, b)\|K\|_{\mathcal{N}}^2 \frac{|w|^2}{|w^2 - 1|^2(1 - |w|)^2} \tag{5.1}$$

and so by Theorem 3.3

$$\sum_{w\in\mathcal{Z}(d\circ\phi)} \frac{(1-|w|)^{3+\tau}}{|w|^{1+\tau}}|w^2-1|^{1+\tau} \leq C(\tau,a,b)\|K\|_{\mathcal{N}}^2.$$

In analogy to the proof of Theorem 4.3 we can deduce:

***Theorem* 5.1** — *Let $Z = Z_0 + K$ defined as described above, then*

$$\sum_{\lambda\in\sigma_{disc}(Z)} \frac{\operatorname{dist}(\lambda,[a,b])^{3+\tau}}{|\lambda-a||\lambda-b|} \leq C(\tau,a,b)\|K\|_{\mathcal{N}}^2,\ \tau>0.$$

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