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AN AVERAGING TRICK FOR SMOOTH ACTIONS OF COMPACT QUANTUM GROUPS ON MANIFOLDS

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Dedicated to Prof. Kalyan B. Sinha on his 70th birthday.

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We prove that, given any smooth action of a compact quantum group (in the sense of [9]) on a compact smooth manifold satisfying some more natural conditions, one can get a Riemannian structure on the manifold for which the corresponding $C^\infty(M)$-valued inner product on the space of one-forms is preserved by the action.

Key words : Quantum isometry groups; Riemannian metric; averaging trick.

1. Introduction

It is both interesting as well as important to study quantum group actions on classical (commutative) and noncommutative spaces. Indeed, quantum group actions can be viewed as generalised symmetries of a classical or quantum system modelled by commutative or noncommutative manifolds. In this context, it is natural ask the question whether one can have genuine (i.e. which are not groups) compact quantum group actions on (compact) classical spaces. Indeed, this has an affirmative answer in general. First examples of this kind were produced by Wang ([13], see also later of other mathematicians in this direction, e.g. [1], [3] etc.) who defined and studied a quantum-group generalisation of the group of permutations of n objects, called the quantum permutation group, and gave its action on the algebra of functions on finite set of cardinality n. For $n \geq 4$ this quantum group is a genuine one. However, in all such cases the underlying set is disconnected. It took quite a long time since the

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work of Wang before Huang [11] came up with several example of genuine compact quantum groups acting faithfully on compact connected topological spaces. On the other hand, there were indications (e.g. [2]) that such a construction would not be possible if the space is a connected smooth manifold. One of the author of the present paper (Goswami) made this conjecture and finally both the authors, together with Das could prove [9] the non-existence of any faithful action of a genuine compact quantum group on a compact connected manifold if the action is assumed to be smooth in a natural sense. This article is rather long and involved and one of the key step was to convert any smooth action on a compact manifold to an 'isometric' action. The arguments employed there used some sort of averaging of the Laplacian for any given Riemannian structure. For this, one has to be very careful about various domain issues involved in handling the unbounded Laplacian operators. In this note, our aim is to give an alternative and somewhat simpler proof of this fact by averaging the Riemannian inner product instead of the unbounded Laplacian operator, under some additional algebraic conditions on the action of the compact quantum group. In some sense, this is closer to the classical arguments for obtaining an invariant Riemannian structure for a compact group action on a compact manifold, and it is interesting in its own right, with the potential of generalisation to the context of noncommutative manifold a la Connes [7].

2. Notation and Preliminaries

We denote by $\hat{\otimes}$ spatial (minimal) C^* tensor product of C^* algebras.

Definition 2.1 — A compact quantum group (CQG for short) is a unital C^* algebra $\mathcal{Q}$ with a coassociative coproduct Δ from $\mathcal{Q}$ to $\mathcal{Q}\hat{\otimes}\mathcal{Q}$ such that each of the linear spans of $\Delta(\mathcal{Q})(\mathcal{Q}\otimes 1)$ and $\Delta(\mathcal{Q})(1\otimes\mathcal{Q})$ is norm-dense in $\mathcal{Q}\hat{\otimes}\mathcal{Q}$.

An action of $\mathcal{Q}$ on a unital C^* algebra $\mathcal{C}$ is a unital $*$-homomorphism $\alpha : \mathcal{C} \to \mathcal{C}\hat{\otimes}\mathcal{Q}$ such that $(\alpha\otimes id)\alpha = (id\otimes\Delta)\alpha$ and $\overline{Sp}\{\alpha(\mathcal{C})(1\otimes\mathcal{Q})\} = \mathcal{C}\hat{\otimes}\mathcal{Q}$.

We denote by $\mathcal{Q}_0$ the dense unital Hopf $*$ algebra in $\mathcal{Q}$ spanned by the matrix coefficients of irreducible unitary representations of $\mathcal{Q}$ (see, e.g. [14]). Given an action α of $\mathcal{Q}$ on $\mathcal{C}$ we always get a dense unital $*$-subalgebra $\mathcal{C}_0$ of $\mathcal{C}$ on which α is algebraic, i.e. maps $\mathcal{C}_0$ to the algebraic tensor product $\mathcal{C}_0\otimes\mathcal{Q}_0$.

Definition 2.2 — An action α is said to be faithful if the $*$-subalgebra of $\mathcal{Q}$ generated by $\{(\omega\otimes id)\alpha(a),\ \omega\in\mathcal{C}^*,\ a\in\mathcal{C}\}$, where $\mathcal{C}^*$ is the set of bounded linear functionals on $\mathcal{C}$, is dense in $\mathcal{Q}$.

We refer [15], [14] for the theory of unitary representation of CQG's and to [7] for the framework of noncommutative geometry given by spectral triples.

Definition 2.3 — For a compact Riemannian manifold M, we say that action α of a CQG $\mathcal{Q}$ on $C(M)$ to be isometric if it maps $C^\infty(M)$ to $C^\infty(M, \mathcal{Q})$ and for every bounded linear functional ϕ on $\mathcal{Q}$, one has $\mathcal{L} \circ \alpha_\phi = \alpha_\phi \circ \mathcal{L}$, where $\alpha_\phi = (\mathrm{id} \otimes \phi) \circ \alpha$ and $\mathcal{L}$ is the restriction of the Hodge Laplacian $-d^* d$ on the space of smooth functions.

The following result is proved in [8].

Proposition 2.4 — For a compact Riemannian manifold M, there is a universal object in the category of CQG's having isometric actions on M. We call this CQG the quantum isometry group of M.

An action α of a CQG $\mathcal{Q}$ on $C(M)$ is called smooth if it maps $C^\infty(M)$ to $C^\infty(M, \mathcal{Q})$ and the span of $\alpha(C^\infty(M))(1 \otimes \mathcal{Q})$ is dense in $C^\infty(M, \mathcal{Q})$ in the natural Frechet topology. It has been proved in [9] that smooth actions are automatically continuous as a map from $C^\infty(M)$ to $C^\infty(M, \mathcal{Q})$ in the respective Frechet topologies coming from that of $C^\infty(M)$. For any C^*-algebra $\mathcal{C}$, We consider the set of smooth $\mathcal{C}$-valued one-forms $\Omega^1(M, \mathcal{C})$ with the natural Frechet topology coming from M ($\Omega^1(M) := \Omega^1(M, \mathbb{C})$) and the obvious $C^\infty(M, \mathcal{C})$-bimodule structure. Given a smooth action α of $\mathcal{Q}$ we call a continuous $\mathbb{C}$-linear map $\Gamma : \Omega^1(M) \to \Omega^1(M, \mathcal{Q})$ to be a representation if Γ is co-associative in the obvious sense and $\Gamma(\xi f) = \Gamma(\xi)\alpha(f) = \alpha(f)\Gamma(\xi)$ for $\xi \in \Omega^1(M), f \in C^\infty(M)$.

We often say that α is a smooth action on M to mean that it is a smooth action on $C(M)$ in the sense discussed above. For such an action we denote $(d \otimes \mathrm{id})(df)$ by $d\alpha(df)$. The $C^\infty(M)$-valued inner product on $\Omega^1(M)$ coming from the Riemannian structure is denoted by $<< \cdot, \cdot >>$ and we say that a smooth action α preserves the Riemannian structure (or the Riemannian inner product) if $<< d\alpha(df), d\alpha(dg) >>= \alpha(<< df, dg >>)$ for all $f, g \in C^\infty(M)$. It is proved in [9] that a smooth action on a compact Riemannian manifold M (without boundary) is isometric if and only if it preserves the inner product.

Before we state and prove the main result in the next section, let us collect a few facts about a smooth faithful action of compact quantum groups on compact manifolds, for the details of which the reader may be referred to [9] and references therein.

Proposition 2.5 — If a CQG $\mathcal{Q}$ acts faithfully and smoothly on a smooth compact manifold M then we have:

(i) $\mathcal{Q}$ has a tracial Haar state, i.e. it is Kac type CQG.

(ii) The action is injective.

(iii) The antipode κ satisfies $\kappa(a^*) = \kappa(a)^*$.

We usually denote by $\otimes$ algebraic tensor product of vector spaces or algebras. We also use Sweedler convention for Hopf algebra coproduct as well as its analogue for (co)-actions of Hopf algebras. That is, we simply write $\Delta(q) = q_{(1)} \otimes q_{(2)}$ suppressing finite summation, where Δ denote the co-product map of a Hopf algebra and q is an element of the Hopf algebra. Similarly, for an algebraic (co)action α of a Hopf algebra on some algebra $\mathcal{C}$, we write $\alpha(a) = a_{(0)} \otimes a_{(1)}$.

3. The Main Result

Fix a compact Riemannian manifold M (not necessarily orientable) and a smooth action α of a CQG $\mathcal{Q}$. We make the following assumptions for the rest of the paper.

Assumption I : There is a Fréchet dense unital $*$-subalgebra $\mathcal{A}$ of $C^\infty(M)$ such that $<< d\alpha(df), d\alpha(dg) >> \in \mathcal{A}$ for all $f, g \in \mathcal{A}$.

Assumption II : There is a well-defined representation Γ on $\Omega^1(M)$ in the sense discussed earlier, such that $\Gamma(df) = (d \otimes \mathrm{id})(\alpha(f))$ for all $f \in C^\infty(M)$. We'll denote this Γ by $d\alpha$.

We now state and prove the main result that we can equip M with a new Riemannian structure with respect to which the action becomes inner product preserving using an analogue of the averaging technique of classical differential geometry.

***Theorem* 3.1** — *M has a Riemannian structure such that α is inner product preserving.*

Note that the first assumption holds for a large class of examples, such as algebraic actions of CQG's compact, smooth, real varieties where the complexifed coordinate algebra of the variety can be chosen as $\mathcal{A}$. On the other hand, the second assumption means that the action on M in some sense lifts to the space of one-forms. This is always automatic for a smooth action by (not necessarily compact) groups, and in fact is nothing but the differential of the map giving the action. Moreover, it is easy to see that any CQG action which preserves the Riemannian inner product does admit such a lift on the bimodule of one-forms, i.e. satisfies the assumption II. Therefore, it is a reasonable assumption too.

Remark 3.2 : We have already mentioned in the introduction that the existence of an invariant Riemannian inner product, i.e. the conclusion of Theorem 3.1 has been proved in [9] as a part of much more general scheme; in fact, without even Assumption I, i.e. only under the smoothness assumption. It also follows from the results of [9] that for a smooth action, Assumption II is equivalent to the existence of invariant Riemannian structure. However, our aim in this paper is to give a direct and easier construction of the invariant Riemannian inner product bypassing the longer and more involved arguments of [9]. But in doing this, we had to pay a price: the scope of our methods are slightly restrictive as we had to impose Assumption I. Another thing: we refrain from using the result of [9] that smoothness implies Assumption II because the proof of that result in [9] made use of an invariant Riemannian structure obtained by a different technique. Thus, to avoid logical circularity and to make our proof self-contained, we assumed the condition II instead of treating it as a consequence of smoothness.

Nevertheless, we believe that the alternative construction of an invariant Riemannian structure presented here should be valuable beyond classical manifolds, i.e. in the wider context of noncommutative geometry.

PROOF OF THEOREM 3.1: We break the proof of into a number of lemmas.

Lemma 3.3 — Define the following map Ψ from $\mathcal{A} \otimes \mathcal{Q}_0$ to $\mathcal{A}$:

$$\Psi(F) := (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id})(F).$$

Here $m : \mathcal{Q}_0 \otimes \mathcal{Q}_0 \to \mathcal{Q}_0$ is the multiplication map. Then Ψ is a completely positive map.

PROOF : As the range is a subalgebra of a unital commutative C^* algebra, it is enough to prove positivity. Let $F = G^*G$ in $\mathcal{A} \otimes \mathcal{Q}_0$ where $G = \sum_i f_i \otimes q_i$, (finite sum) for some $f_i \in \mathcal{A}, q_i \in \mathcal{Q}_0$. We write $\alpha(f) = f_{(0)} \otimes f_{(1)}$ in Sweedler notation as usual, and observe that

$$\begin{aligned} &\Psi(F) \\ &= \sum_{ij} f^*_{i(0)} f_{j(0)} h(\kappa(f^*_{i(1)} f_{j(1)}) q^*_i q_j) \\ &= \sum_{ij} f_{j(0)} f^*_{i(0)} h(q_j(\kappa(f_{j(1)}))^* \kappa(f_{i(1)}) q^*_i) \\ &= (\mathrm{id} \otimes h)(\xi^* \xi) \geq 0, \end{aligned}$$

where $\xi = \sum_i f^*_{i(0)} \otimes \kappa(f_{i(1)}) q^*_i$, and note also that we have used above the facts that h is tracial and κ is $*$-preserving. □

For $\omega, \eta \in \Omega^1(\mathcal{A})$ We define

$$<< \omega, \eta >>' := \Psi(<< d\alpha(\omega), d\alpha(\eta) >>),$$

which is well defined as we have assumed that $<< d\alpha(ds_1), d\alpha(ds_2)) >> \in \mathcal{A} \otimes \mathcal{Q}_0$ for $s_1, s_2 \in \mathcal{A}$. Moreover, by complete positivity of Ψ this gives a non-negative definite sesquilinear form on $\Omega^1(\mathcal{A})$. As the action is algebraic over $\mathcal{A}$, we shall use Sweedler's notation to prove the following.

Lemma 3.4 — For $\omega, \eta \in \Omega^1(\mathcal{A}),\ f \in \mathcal{A}$, $<< \omega, \eta >>' = (<< \eta, \omega >>')^*$ and $<< \omega, \eta f >>'$ $= << \omega, \eta >>' f$.

PROOF : It is enough to prove the lemma for $\omega = d\phi$ and $\eta = d\psi$ for $\phi, \psi \in \mathcal{A}$. First observe that as we have $\kappa = \kappa^{-1}$, for $z \in \mathcal{Q}_0$ applying κ on $z_{(1)}\kappa(z_{(2)}) = \epsilon(z).1$, we get

$$z_{(2)}\kappa(z_{(1)}) = \epsilon(z).1. \tag{1}$$

We denote $<< d\phi_{(0)}, d\psi_{(0)} >>$ by x and $\phi^*_{(1)}\psi_{(1)}$ by y. Then

$$\begin{aligned}
& << d\phi, d\psi f >>' \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id) << d\alpha(d\phi), d\alpha(d\psi f) >> \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id)(xf_{(0)} \otimes yf_{(1)}) \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(x_{(0)}f_{(0)(0)} \otimes x_{(1)}f_{(0)(1)} \otimes yf_{(1)}) \\
= & (id \otimes h)(x_{(0)}f_{(0)(0)} \otimes \kappa(x_{(1)}f_{(0)(1)})yf_{(1)}) \\
= & x_{(0)}f_{(0)(0)}h(f_{(1)}\kappa(f_{(0)(1)})\kappa(x_{(1)})y)(by\ tracial\ property\ of\ h) \\
= & x_{(0)}f_{(0)}h(f_{(1)(2)}\kappa(f_{(1)(1)})\kappa(x_{(1)})y) \\
= & x_{(0)}f_{(0)}h(\epsilon(f_{(1)}).1.\kappa(x_{(1)})y) \\
= & x_{(0)}(id \otimes \epsilon)\alpha(f)h(\kappa(x_{(1)})y) \\
= & x_{(0)}fh(\kappa(x_{(1)})y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
<< d\phi, d\psi >>' f &= [(id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id) << d\alpha(d\phi), d\alpha(d\psi) >>]f \\
&= [(id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(x_{(0)} \otimes x_{(1)} \otimes y)]f \\
&= x_{(0)}fh(\kappa(x_{(1)})y).
\end{aligned}$$

Also we have

$$\begin{aligned}
& << d\phi, d\psi >>' \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id)(<< d\phi_{(0)}, d\psi_{(0)} >> \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id)(<< d\psi_{(0)}, d\phi_{(0)} >>^* \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & (id \otimes h)(id \otimes m)(id \otimes \kappa \otimes id)(<< d\psi_{(0)}, d\phi_{(0)} >>^*_{(0)} \otimes << d\psi_{(0)}, d\phi_{(0)} >>^*_{(1)} \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & << d\psi_{(0)}, d\phi_{(0)} >>^* h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))^*\phi^*_{(1)}\psi_{(1)})(\ since\ \kappa\ is\ *\ preserving).
\end{aligned}$$

Hence we have

$$<< d\phi, d\psi >>'^* = << d\psi_{(0)}, d\phi_{(0)} >> h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))\psi^*_{(1)}\phi_{(1)})$$

$(\ since \ h \ is \ tracial \ and \ h(a^*) = \overline{h(a)})$.

But we can readily see that

$$<< d\psi, d\phi >>' = << d\psi_{(0)}, d\phi_{(0)} >> h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))\psi^*_{(1)}\phi_{(1)}),$$

which completes the proof of the lemma. □

Actually we can extend $<<, >>'$ to a slightly bigger set than $\Omega^1(\mathcal{A})$ namely $\Omega^1(\mathcal{A})C^\infty(M) = \ Sp\,\{\omega f : \omega \in \Omega^1(\mathcal{A}), f \in C^\infty(M)\}$.

For $\omega, \eta \in \Omega^1(\mathcal{A})C^\infty(M)$, $\omega = \sum \omega_i f_i$ and $\eta = \sum \eta_i g_i(\text{ finite sums})$, $\omega_i, \eta_i \in \Omega^1(\mathcal{A})$ and $f_i, g_i \in C^\infty(M)$ (say) we can choose sequences $f_i^{(n)}, g_i^{(n)}$ from $\mathcal{A}$ such that $f_i^{(n)} \to f_i$ and $g_i^{(n)} \to g_i$ in the corresponding Fréchet topology and by Lemma 3.4 observe that

$$\begin{aligned} & << \sum_i \omega_i f_i^{(n)}, \sum_j \eta_j g_j^{(n)} >>' \\ = \ & \sum_{i,j} \overline{f_i^{(n)}} << \omega_i, \eta_j >>' \ g_j^{(n)} \\ \to \ & \sum_{i,j} \overline{f_i} << \omega_i, \eta_j >>' \ g_j := << \omega, \eta >>' \ . \end{aligned} \quad (2)$$

Clearly this definition is independent of the choice of sequences $f_i^{(n)}$ and $g_i^{(n)}$.

We next prove the following

Lemma 3.5 — For $\phi, \psi \in \mathcal{A}$,

$$<< d\alpha(d\phi), d\alpha(d\psi) >>' = \alpha(<< d\phi, d\psi >>'). \quad (3)$$

PROOF : With x, y as before we have

Claim 2 : We can extend the definition of $<<, >>'$ for $\omega, \eta \in \Omega^1(\mathcal{A})C^\infty(M)$ such that

$$\forall \ \ f \in C^\infty(M), << (d\phi), (d\psi)f >>' = << d\phi, d\psi >>' \ f. \quad (4)$$

PROOF : For $f \in C^\infty(M)$, define $<< (d\phi), (d\psi)f >>' := \ lim \ << d\phi, d\psi \ f_n >>'$, where $f_n \in \mathcal{A}$ with $lim \ f_n = f$, where the limits are taken in the Fréchet topology.

Observe that $<< d\phi, d\psi \ f_n >>'$ is Fréchet Cauchy as

$$\begin{aligned} & << d\phi, d\psi \ f_n >>' - << d\phi, d\psi \ f_m >>' \\ = \ & << d\phi, d\psi >>' \ (f_n - f_m). \end{aligned}$$

So $<< d\phi, d\psi f >>' = lim << d\phi, d\psi >>' f_n = << d\phi, d\psi >>' f$, again the limit is taken in the corresponding Fréchet topology.

That proves the claim.

$$\begin{aligned} & << d\alpha(d\phi), d\alpha(d\psi) >>' \\ = & (id \otimes h \otimes id)(id \otimes m \otimes id)(id \otimes \kappa \otimes id \otimes id)(\alpha \otimes id \otimes id)(x \otimes \Delta(y)) \\ = & (id \otimes h \otimes id)(id \otimes m \otimes id)(id \otimes \kappa \otimes id \otimes id)(x_{(0)} \otimes x_{(1)} \otimes y_{(1)} \otimes y_{(2)}) \\ = & (id \otimes h \otimes id)(x_{(1)} \otimes \kappa(x_{(2)})y_{(1)} \otimes y_{(2)}) \\ = & x_{(0)} \otimes h(\kappa(x_{(1)})y_{(1)})y_{(2)}. \end{aligned}$$

On the other hand

$$\begin{aligned} \alpha(<< d\phi, d\psi >>') &= x_{(0)(0)} h(\kappa(x_{(1)})y) \otimes x_{(0)(1)} \\ &= x_{(0)} \otimes x_{(1)(1)} h(\kappa(x_{(1)(2)})y) \\ &= x_{(0)} \otimes x_{(1)(1)} h(\kappa(y)(x_{(1)(2)}))(\text{since } h(\kappa(a)) = h(a)). \end{aligned}$$

Hence it is enough to show that $h(\kappa(c)b_{(2)})b_{(1)} = h(\kappa(b)c_{(1)})c_{(2)}$ where $b, c \in \mathcal{Q}_0$, for then taking $x_{(1)} = b$ and $y = c$ we can complete the proof.

We make the transformation $T(a \otimes b) = \Delta(\kappa(a))(1 \otimes b)$.

Then

$$\begin{aligned} & (h \otimes id)T(a \otimes b) \\ = & (h \otimes id)\Delta(\kappa(a))(1 \otimes b) \\ = & ((h \otimes id)\Delta(\kappa(a)))b \\ = & h(\kappa(a))b \\ = & (h \otimes id)(a \otimes b). \end{aligned}$$

Hence $h(b_{(2)}\kappa(c))b_{(1)} = (h \otimes id)T(b_{(2)}\kappa(c) \otimes b_{(1)})$.

So, by using traciality of h it is enough to show that $T(b_{(2)}\kappa(c) \otimes b_{(1)}) = c_{(1)}\kappa(b) \otimes c_{(2)}$.

$$\begin{aligned} & T(b_{(2)}\kappa(c) \otimes b_{(1)}) \\ = & \Delta(\kappa(b_{(2)}\kappa(c)))(1 \otimes b_{(1)}) \\ = & \Delta(c\kappa(b_{(2)}))(1 \otimes b_{(1)}) \\ = & (c_{(1)} \otimes c_{(2)})[\kappa(b_{(2)(2)}) \otimes \kappa(b_{(2)(1)})](1 \otimes b_{(1)}) \end{aligned}$$

$$
\begin{aligned}
&= (c_{(1)} \otimes c_{(2)}) m_{23}(\kappa(b_{(2)(2)}) \otimes \kappa(b_{(2)(1)}) \otimes b_{(1)}) \\
&= (c_{(1)} \otimes c_{(2)}) m_{23}(\kappa \otimes \kappa \otimes id)\sigma_{13}(b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)}) \\
&= (c_{(1)} \otimes c_{(2)}) m_{23}(\kappa \otimes \kappa \otimes id)\sigma_{13}(b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}) \\
&= (c_{(1)} \otimes c_{(2)}) m_{23}(\kappa(b_{(2)} \otimes \kappa(b_{(1)(2)}) \otimes b_{(1)(1)}) \\
&= (c_{(1)} \otimes c_{(2)})(\kappa(b_{(2)}) \otimes \epsilon(b_{(1)}).1_{\mathcal{Q}})(by\ (10)) \\
&= (c_{(1)} \otimes c_{(2)})(\kappa \otimes \kappa)((b_{(2)}) \otimes \epsilon(b_{(1)}).1_{\mathcal{Q}}) \\
&= (c_{(1)} \otimes c_{(2)})(\kappa \otimes \kappa)(\epsilon(b_{(1)})b_{(2)} \otimes 1_{\mathcal{Q}}) \\
&= c_{(1)}\kappa(b) \otimes c_{(2)}.
\end{aligned}
$$

Which proves the claim.

Now we proceed to define a new Riemannian structure on the manifold so that the action α will be inner product preserving. For that we are going to need the following

Lemma 3.6 — (i) For $m \in M$, Sp $\{ds(m) : s \in \mathcal{A}\}$ coincides with $T^*_m(M)$.

(ii) If $\{s_1, ..., s_n\}$ and $\{s_1^{'}, ..., s_n^{'}\}$ are two sets of functions in $\mathcal{A}$ such that each of $\{ds_i(m) : i = 1, ..., n\}$ and $\{ds_i^{'}(m) : i = 1, ..., n\}$ are bases for $T^*_m(M)$ and for $v, w \in T^*_m(M)$ with $v = \sum_i c_i ds_i(m) = \sum_i c_i^{'} ds_i^{'}(m)$ and $w = \sum_i d_i ds_i(m) = \sum_i d_i^{'} ds_i^{'}(m)$, then

$$\sum_{i,j} \bar{c}_i d_j << ds_i, ds_j >>^{'} (m) = \sum_{i,j} \bar{c_i^{'}} d_j^{'} << ds_i^{'}, ds_j^{'} >>^{'} (m),$$

where $<<, >>^{'}$ is the new $C^\infty(M)$ valued inner product introduced earlier.

PROOF : Choosing a coordinate neighbourhood U around m and a set of coordinates $x_1, ..., x_n$ we have $ds(m) = \sum_{i=1}^n \frac{\partial s}{\partial x_i}(m) dx_i(m)$.

Pick any $\eta \in T^*_m(M)$ i.e. we have $\eta = \sum_{i=1}^n c_i dx_i(m)$ for some c_i's in $\mathbb{R}$.

Choose any $f \in C^\infty(M)$ with $\frac{\partial f}{\partial x_i}(m) = c_i$.

For $f \in C^\infty(M)$, by Fréchet density of $\mathcal{A}$ we have a sequence $s_n \in \mathcal{A}$ and an $n_0 \in \mathbb{N}$ such that

$$|\frac{\partial s}{\partial x_i}(m) - \frac{\partial f}{\partial x_i}(m)| < \epsilon \,\forall\, n \geq n_0.$$

So Sp $\{ds(m); s \in \mathcal{A}\}$ is dense in $T^*_m(M)$. $T^*_m(M)$ being finite dimensional Sp $\{ds(m) : s \in \mathcal{A}\}$ coincides with $T^*_m(M)$. Which proves (i).

For proving (ii) first we prove the following fact:

Let $m \in M$ and $\omega \in \Omega^1(\mathcal{A})$ such that $\omega = 0$ in a neighbourhood U of m. Then $<< \omega, \eta >>^{'} = 0$ for all $\eta \in \Omega^1(\mathcal{A})$.

For the proof of the above fact Let $V \subset U$ such that $V \subset \bar{V} \subset U$.

Choose $f \in C^\infty(M)_\mathbb{R}$ such that $supp(f) \subset \bar{V}$, $f \equiv 1$ on V and $f \equiv 0$ outside U.

So we can write $\omega = (1-f)\omega$. Then

$$\begin{aligned} & << \omega, \eta >>^{'} (m) \\ = & << (1-f)\omega, \eta >>^{'} (m) \\ = & << \omega, \eta >>^{'} (m)(1-f)(m) \ (by\ (4)) \\ = & \ 0. \end{aligned}$$

Applying the above fact we can show:

Let $m \in M$ and $\omega = \omega^{'}$, $\eta = \eta^{'}$ in a neighbourhood U of m. Then $<< \omega, \eta >>^{'} = << \omega^{'}, \eta^{'} >>^{'}$, $\forall \omega, \omega^{'}, \eta, \eta^{'} \in \Omega^1(\mathcal{A})$.

For the proof it is enough to observe that $<< \omega, \eta >>^{'} (m) - << \omega^{'}, \eta^{'} >>^{'} (m) = << \omega - \omega^{'}, \eta >>^{'} (m) + << \omega^{'}, \eta - \eta^{'} >> (m)$.

As $\{ds_1(m), ..., ds_n(m)\}$ and $\{ds^{'}_1(m), ..., ds^{'}_n(m)\}$ are two bases for $T^*_m(M)$. Then they are actually bases for $T^*_x(M)$ for x in a neighbourhood U of m. So there are $\{f_{ij} : i, j = 1(1)n\}$ in $C^\infty(M)$ such that

$$ds_i = \sum_{j=1}^{n} f_{ij} ds^{'}_j$$

on U for all $i = 1, \ldots, n$. Hence by the previous discussion

$$<< ds_i, ds_j >>^{'} (m) = << \sum_k f_{ik} ds^{'}_k, \sum_l f_{jl} ds^{'}_l >>^{'} (m). \tag{5}$$

Let $v = \sum_{i=1}^n c_i ds_i(m) = \sum_{i=1}^n c^{'}_i ds^{'}_i(m)$ and $w = \sum_{i=1}^n d_i ds_i(m) = \sum_{i=1}^n d^{'}_i ds^{'}_i(m)$. So by definition

$$\begin{aligned} < v, w >^{'} &= \sum_{ij} \bar{c_i} d_j << ds_i, ds_j >>^{'} (m) \\ &= \sum_{ijkl} \bar{c_i} d_j \bar{f_{ik}}(m) f_{jl}(m) << ds^{'}_k, ds^{'}_l >>^{'} (m) \ (\ by\ (4)) \\ &= \sum_{kl} \bar{c_k}^{'} d^{'}_l << ds^{'}_k, ds^{'}_l >>^{'} (m). \end{aligned}$$

□

PROOF OF THEOREM 3.1 : Now we can define a new inner product on the manifold M. For that let $v, w \in T^*_m(M)$ by (i) of Lemma 3.6 we choose $s_1, ..., s_n \in \mathcal{A}$ such that $ds_1(m), ..., ds_n(m)$ is a basis for $T^*_m(M)$. Let $\{c_i, d_i : i = 1, ..., n\}$ be such that $v = \sum_i c_i ds_i(m)$ and $w = \sum_i d_i ds_i$. Then we define

$$< v, w >' := \sum_{i,j} \bar{c}_i d_j << ds_i, ds_j >>' (m).$$

It is evident that this is a semi definite inner product. We have to show that this is a positive definite inner product. To that end let $< v, v >' = 0$ i.e.

$$\sum_{i,j} \bar{c}_i c_j << ds_i, ds_j >>' (x) = 0,$$

where $v = \sum_i c_i ds_i(x) \in T^*_x(M)$. Since the Haar state h is faithful on $\mathcal{Q}_0$ and by assumption $<< d\alpha(ds_i), d\alpha(ds_j) >> \in \Omega^1(\mathcal{A}) \otimes \mathcal{Q}_0$, we can deduce that

$$\sum_{i,j} \bar{c}_i c_j ((id \otimes m)(id \otimes \kappa \otimes id)(\alpha \otimes id) << d\alpha(ds_i), d\alpha(ds_j) >>)(x) = 0.$$

Since $\epsilon \circ \kappa = \epsilon$ on $\mathcal{Q}_0$, applying $(\epsilon \otimes \epsilon)$ to the above equation, we get

$$\sum_{i,j} \bar{c}_i c_j ((id \otimes m)(id \otimes \epsilon \otimes \epsilon)(\alpha \otimes id) << d\alpha(ds_i), d\alpha(ds_j) >>)(x) = 0.$$

Using the fact that ϵ is $*$-homomorphism we get

$$\sum_{i,j} \bar{c}_i c_j < \epsilon(d\alpha(ds_i)(x)), \epsilon(d\alpha(ds_j)(x)) >= 0.$$

It is easy to see that $\epsilon(d\alpha(ds_i)(x)) = ds_i(x)$ for all i. Hence we conclude that

$$< \sum_i c_i ds_i(x), \sum_i c_i ds_i(x) >= 0,$$

i.e $< v, v >= 0$ and hence $v = 0$ (as $< \cdot, \cdot >$ is strictly positive definite, being an inner product on $T^*_x M$) so that $< \cdot, \cdot >'$ is indeed strictly positive definite, i.e. inner product. We have already noted ((ii) of Lemma 3.6) that our definition is independent of choice of s_i's, and also that with respect to this new Riemannian structure on the manifold, α is inner product preserving. This completes the proof of the Theorem 3.1 on $\Omega^1(\mathcal{A})$ and hence on $\Omega^1(C^\infty(M))$. □

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