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THE DISTANCE FORMULA FOR THE DERIVATION PROBLEM[1]

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Dedicated to Professor Kalyan B. Sinha on his 70*th birthday*.

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In this paper we discuss questions related to the perturbation of von Neumann algebras. We will show that the distance between an arbitrary operator of $\mathcal{M}$ and the commutant $\mathcal{M}'$ is the same as the distance from the same operator to the trivial center $\mathbb{C}I$ of $\mathcal{M}$ where $\mathcal{M}$ is a type II-factor in a separable Hilbert space. As a consequence, the derivation implemented by some element $x \in \mathcal{M}$ has the same norm as the distance from x to the commutant $\mathcal{M}'$.

Key words : Type II factor; distance of two C^*-algebras; derivation.

1. INTRODUCTION

Kadison and Kastler [5] initiated the study of perturbations of von Neumann algebras. They conjectured that two von Neumann algebras might be unitarily equivalent if they are sufficiently close. Here the closeness of two von Neumann algebras implies that their unit balls are close in the Hausdorff metric derived from the norm. To quantify the statement, they defined the distance between two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ on a Hilbert space $\mathcal{H}$. This conjecture was affirmatively answered for injective von Neumann algebras [3, 8].

If $\mathcal{F}$ is a family of bounded linear operators on a Hilbert space $\mathcal{H}$ and T is any bounded linear operator on $\mathcal{H}$, then the distance between the family $\mathcal{F}$ and an operator T is given by the formula

$$d(T, \mathcal{F}) = \inf\{\|T - S\| : S \in \mathcal{F}\}.$$

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Let $\mathfrak{A}$ be a (possibly non-selfadjoint) algebra of bounded operators on a Hilbert space $\mathcal{H}$ and let T be an arbitrary bounded operator on $\mathcal{H}$. If P is a projection whose range E is invariant under $\mathfrak{A}$ that is, $A(E) \subset E$ for all $A \in \mathfrak{A}$, then for each $A \in \mathfrak{A}$ one has $(I-P)AP = 0$, so that

$$\|T - A\| \geq \|(I-P)(T-A)P\| = \|(I-P)TP\|.$$

It follows that

$$d(T, \mathfrak{A}) \geq \sup\{\|(I-P)TP\| : P \in \text{Lat}(\mathfrak{A})\} \tag{1}$$

where Lat$(\mathfrak{A})$ is the lattice of all $\mathfrak{A}$-invariant projections.

A *nest* on a Hilbert space $\mathcal{H}$ is a family $\mathcal{C}$ of subspaces of $\mathcal{H}$, totally ordered under inclusion and such that for any subfamily $(E_i)_{i\in I}$ in $\mathcal{C}$ we have

$$\bigcap_{i\in I} E_i \in \mathcal{C} \text{ and } \overline{\bigcup_{i\in I} E_i} \in \mathcal{C}$$

where $\overline{\cup_{i\in I} E_i}$ is the closed subspace generated by $\cup_{i\in I} E_i$. A *nest algebra* associated with a nest $\mathcal{C}$ is the subalgebra $\mathfrak{A}(\mathcal{C})$ consisting of all operators T in $\mathcal{B}(\mathcal{H})$ such that for all $E \in \mathcal{C}$, $T(E) \subset E$. Let $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ be a subalgebra and let Lat$(\mathfrak{A})$ be the set of all closed subspaces $E \subset \mathcal{H}$ which are $\mathfrak{A}$-invariant. It is not hard to see for any $\mathfrak{A}$-invariant subspace E, we have

$$\|(I - P_E)TP_E\| \leq d(T, \mathfrak{A}),$$

where P_E is the projection onto E, so that for any T in $\mathcal{B}(\mathcal{H})$

$$\sup\{\|(I-P_E)TP_E\| : E \in \text{Lat}(\mathfrak{A})\} \leq d(T, \mathfrak{A}). \tag{2}$$

Arveson's distance formula [1] says that if $\mathfrak{A}$ is a nest algebra then (2) becomes an equality, that is, for any T in $\mathcal{B}(\mathcal{H})$ we have

$$\sup\{\|(I-P_E)TP_E\| : E \in \text{Lat}(\mathfrak{A})\} = d(T, \mathfrak{A}). \tag{3}$$

This formula has proved to be useful in studying problems involving compact perturbations and similarity theory for nests [2]. Solel also proved a distance formula for analytic operator algebras [10]. Furthermore, this formula led him to study this distance formula for more general classes of algebras: an algebra is said to be *reflexive* if it satisfies the implication

$$T(E) \subset E \text{ for all } E \in \text{Lat}(\mathfrak{A}) \Rightarrow T \in \mathfrak{A};$$

It is called *hyper-reflexive* if moreover there is a constant K such that for any T in $\mathcal{B}(\mathcal{H})$

$$d(T,\mathfrak{A}) \le K \sup\{\|(I-P_E)TP_E\| : E \in \text{Lat}(\mathfrak{A})\}.$$

As we just saw, this inequality holds with $K = 1$ for nest algebras. The von Neumann's double commutant theorem implies that every von Neumann algebra is reflexive. The Kadison-Kastler's question concerning the unitary equivalence of two close von Neumann algebras was studied by many people. See [2] and its references for detailed information. Moreover, when the von Neumann algebras in the question are injective, positive answers give a precise description of a relationship between the commutants. In order to generalize these results, Christensen [4] considered the following question: *"If a bounded linear operator x on a Hilbert space $\mathcal{H}$ nearly commutes with elements in a unit ball of a C^*-algebra $\mathcal{A}$ acting on $\mathcal{H}$, must x be close to the commutant of $\mathcal{A}$?"* To study this question, he estimated the norms of the inner derivations of a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ into $\mathcal{B}(\mathcal{H})$.

In this paper we discuss the question related to the question considered by Christensen. Let $\mathcal{M}$ be a type II-factor with separable predual. To do this, we will prove that the distance between an arbitrary operator x in $\mathcal{M}$ and the commutant $\mathcal{M}'$ is same as the distance between the same operator x and the trivial center $\mathbb{C}I$. As a consequence, we see that the norm of the derivation D implemented by some element $x \in \mathcal{M}$ is the same as the distance $d(x,\mathcal{M}')$ from x to the commutant $\mathcal{M}'$.

2. A Distance Formula

The distance between two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is defined as

$$d(\mathcal{M},\mathcal{N}) = \sup\big\{d(x,\mathcal{N}_1), d(y,\mathcal{M}_1) : x \in \mathcal{M}_1, y \in \mathcal{N}_1\big\}$$

where $\mathcal{M}_1$ and $\mathcal{N}_1$ are closed unit balls of $\mathcal{M}$ and $\mathcal{N}$, respectively [5]. Let $\mathcal{A}$ be an abstract C^*-algebra and let π be a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Then it is not hard to see that for all $x \in \mathcal{B}(\mathcal{H})$ one has the inequality

$$2d(x,\pi(\mathcal{A})') \ge \sup\{\|x\pi(a)-\pi(a)x\| : a \in \mathcal{A} \text{ and } \|a\| \le 1\}.$$

Conversely, if $\mathcal{A}$ is a nuclear C^*-algebra or a properly infinite von Neumann algebra, it was proved in [4] that

$$d(x,\pi(\mathcal{A})') \le K \sup\{\|x\pi(a)-\pi(a)x\| : a \in \mathcal{A} \text{ and } \|a\| \le 1\}$$

for some constant $K \ge 1$. Such two inequalities mean that two induced norms on the quotient space $\mathcal{B}(\mathcal{H})/\pi(\mathcal{A})'$ are equivalent for a nuclear C^*-algebra $\mathcal{A}$ or a properly infinite von Neumann algebra $\mathcal{A}$.

To investigate the distance between an arbitrary operator of a type II_1-factor $\mathcal{M}$ and the commutant $\mathcal{M}'$, we first recall the well known result proved by Popa [7].

***Theorem* 2.1** — *Let $\mathcal{M}$ be a type II-factor with separable predual. There is an irreducible hyperfinite subfactor $\mathcal{R}$ of $\mathcal{M}$, that is, $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$.*

When the II_1-factor $\mathcal{M}$ in the following theorem has property P or is a McDuff factor, the distance formula in Theorem 2.2 follows from results in [4]. Hence we can regard the following theorem as a generalization of the results.

***Theorem* 2.2** — *If $\mathcal{M}$ is a type II_1-factor acting on a separable Hilbert space $\mathcal{H}$, then we have that*

$$d(x, \mathbb{C}I) = d(x, \mathcal{M}') \quad \text{for each } x \in \mathcal{M}$$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$.

PROOF : It is clear that the inequality $d(x, \mathcal{M}') \leq d(x, \mathbb{C}I)$ always holds for each $x \in \mathcal{M}$. We only have to show that $d(x, \mathcal{M}') \geq d(x, \mathbb{C}I)$ for each $x \in \mathcal{M}$.

First, we will prove the inequality for the case when the commutant $\mathcal{M}'$ in $\mathcal{B}(\mathcal{H})$ is a type II_1-factor. Suppose that $\mathcal{M}'$ is of type II_1. By Popa's result (Theorem 2.1), there exists an irreducible hyperfinite subfactor $\mathcal{R}$ of $\mathcal{M}'$, so that $\mathcal{R}' \cap \mathcal{M}' = \mathbb{C}I$. Let E be the conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$ (see Theorem 10.22 in [11] for the existence). Since $\mathcal{R} \subseteq \mathcal{M}'$, we have the inclusion $\mathcal{M} \subseteq \mathcal{R}'$ so that $E(x) = x$ for all $x \in \mathcal{M}$. Let τ' be the unique faithful tracial state on $\mathcal{M}'$. For any $x \in \mathcal{M}$ and $y \in \mathcal{M}'$, we have that

$$\|x - y\| \geq \|E(x - y)\| = \|x - E(y)\| = \|x - \tau'(y)I\|$$

where the second equality follows from the Dixmier approximation theorem ([6, Theorem 8.3.5]) and 8.7.24 in [6]. This implies that $d(x, \mathcal{M}') \geq d(x, \mathbb{C}I)$.

Assume that $\mathcal{M}'$ is a type II_∞-factor. Then there exist a type II_1-factor $\mathcal{N}$ and a Hilbert space $\mathcal{K}$ such that $\mathcal{M}' \cong \mathcal{N} \otimes \mathcal{B}(\mathcal{K})$. If $\mathcal{E}$ is the conditional expectation from $\mathcal{B}(\mathcal{H})$ onto the commutant of $\mathbb{C}I \otimes \mathcal{B}(\mathcal{K})$, then $\mathcal{E}(\mathcal{M}') = \mathcal{N} \otimes \mathbb{C}I$. Since the conditional expectation $\mathcal{E}$ is norm decreasing and $\mathcal{E}(x) = x$ for all $x \in \mathcal{M}$, we have the inequality

$$\|x - y\| \geq \|\mathcal{E}(x - y)\| = \|x - \mathcal{E}(y)\|$$

for any $x \in \mathcal{M}$ and $y \in \mathcal{M}'$. Moreover, the element $\mathcal{E}(y)$ is in the II_1-factor $\mathcal{N} \otimes \mathbb{C}I$.

If we apply the above argument, we get the inequality

$$\|x - \mathcal{E}(y)\| \geq \|x - \sigma(\mathcal{E}(y))I\|$$

where σ is the normalized trace on $\mathcal{N} \otimes \mathbb{C}I$. This completes the proof. □

When $\mathcal{M}$ is a factor of type II_∞, we can also get the distance formula as in Theorem 2.2 by using the same method. The following corollary is true for a type II_∞-factor.

Corollary 2.3 — Let $\mathcal{M}$ be a type II_1-factor with separable predual. Suppose that $x \in \mathcal{M}$ implements a derivation δ on $\mathcal{M}$, that is, $\delta(\cdot) = [x, \cdot]$.

(1) We have that $\|\delta\| = 2d(x, \mathcal{M}')$.

(2) There exists an operator $y \in \mathcal{M}$ of norm $\frac{\|\delta\|}{2}$ such that $y = x - \lambda I$ where $\lambda \in \mathbb{C}$ is the complex number with $\|x - \lambda I\| = d(x, \mathbb{C}I)$. In particular, δ is also implemented by y.

PROOF : (1) If $\mathcal{N}$ is a von Neumann algebra with the center $Z(\mathcal{N})$, by Theorem 2.5.4 in [9] we have that $\|\delta\| = 2d(x, Z(\mathcal{N}))$ where $x \in \mathcal{N}$ implements a derivation δ. However, the center $Z(\mathcal{M})$ is the trivial algebra $\mathbb{C}I$ since $\mathcal{M}$ is a factor. By Theorem 2.2, we have the equality $\|\delta\| = 2d(x, \mathcal{M}')$.

(2) It is known that there exist an operator $y \in \mathcal{M}$ of norm $\frac{\|\delta\|}{2}$ such that δ is implemented by y (see Corollary 2.5.5 in [9]). Since the difference $x - y$ is in the center $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$, there exists some constant $\lambda \in \mathbb{C}$ such that $x - y = \lambda I$. However, Theorem 2.2 and the equality $\|\delta\| = 2d(x, \mathbb{C}I)$ in (1) give the equalities

$$\|x - \lambda I\| = \|y\| = \frac{1}{2}\|\delta\| = d(x, \mathcal{M}') = d(x, \mathbb{C}I).$$

This completes the proof. □

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