Indian J. Pure Appl. Math., **46**(5): 633-645, October 2015

DOI: 10.1007/s13226-015-0130-y

COMPLETE JOIN HYPERLATTICES

A. Soltani Lashkenari and B. Davvaz

Department of Mathematics, Yazd University, Yazd, Iran,
e-mails: davvaz@yazd.ac.ir, bdavvaz@yahoo.com

(*Received* 19 *April* 2014; *accepted* 30 *October* 2014)

Hyperlattices are a suitable generalization of ordinary lattices. In this paper, we consider join hyperlattices and define complete join hyperlattices. We investigate some properties of them. Also, we consider some well-known hyperlattices such as Nakano superlattices and P-hyperlattices, and we show that under certain conditions, they are complete. Then, we define completion of a join hyperlattice.

Key words : Algebraic hyperstructures; hyperlattice; ideal; filter; prime element; complete hyperlattice.

1. Introduction

Lattices, especially Boolean algebras, arise naturally in logic, and thus some of the elementary theory of lattices had been worked out earlier. Nonetheless, there is the connection between modern algebra and lattice theory. Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. The hyperstructure theory was introduced in 1934 by the French mathematician, Marty, at the 8th congress of scandinavian mathematicians [24]. Since then, many papers and several books have been written on this topic (see for instance [2, 7, 8, 9, 10, 30]). Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In [23], Leoreanu-Fotea and Davvaz characterized distributive lattices and modular lattices using n-hyperstructures. The study of ordered hypergroupoids and hypergroups appears in the work of Konstantinidou [19], Chajda and Hoskova [4, 5, 15], Chvalina [6], Heidari and Davvaz [14]. The concept of hyperlattice is a generalization of the concept of lattice [3]. Hyperlattices were for the first time introduced by Konstantinidou and Mittas in [20]. The hyperlattice and homomorphism were studied in [12] too. The distributive hyperlattice and their properties were discussed in [13]. In [22], a strong Boolean hyperalgebra is constructed, which has carrier the set of Boolean functions over such hyperalgebra. In [18], starting with a lattice a class of hyperlattices, the P-hyperlattices, are introduced and classified with respect to different types of distributivity. In [16], Jakubik studied several aspects of the theory of superlattices and defined congruences on superlattices. Also, the congruence on hyperlattices and their properties are studied in [17]. In [11], Dehghan Nezhad and Davvaz introduced the concept of H_v-semilattices and obtained some characterizations of them. In [27], Rasouli and Davvaz by considering the notion of hyperlattice, introduced good and s-good hyperlattices, homomorphism of hyperlattices and s-reflexives. They gave some examples of them and studied their structures. In [26], Rasouli and Davvaz studied hyperlattice structure and its quotient structure with a regular relation. Other contributor to the development of hyperlattice theory are Konstantinidou [19, 21], Ashrafi [1], Rahnamai-Barghi [28, 29], Guo and Xin [12]. In this paper, we introduce the concept of complete join hyperlattices, weak homomorphisms in join hyperlattice and we investigate their properties.

2. Basic Concepts

In this section we provide background information needed in the paper. First, we present some basic definitions and well-known facts about hyperlattices.

Let H be a non-empty set. A hyperoperation on H is a map $\circ$ from $H \times H$ to $\wp^*(H)$, the family of non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$; $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$. Moreover, if for any element $a \in H$ equalities $a \circ H = H \circ a = H$ hold, then the pair $(H, \circ)$ is called a hypergroup.

A lattice is a partially ordered set L such that for any two elements x, y of L, $glb\{x, y\}$ and $lub\{x, y\}$ exist. If L is a lattice, then we define $x \vee y = glb\{x, y\}$ and $x \wedge y = lub\{x, y\}$. This definition is equivalent to the following definition [3]. Let L be a non-empty set with two binary operations $\wedge$ and $\vee$. Let for all $a, b, c \in L$, the following conditions satisfied:

(1) $a \wedge a = a$ and $a \vee a = a$;

(2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;

(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;

(4) $(a \wedge b) \vee a = a$ and $(a \vee b) \wedge a = a$;

Then, $(L, \vee, \wedge)$ is a lattice.

Now, we recall the notions of four types of hyperlattices.

Join hyperlattice : Let L be a non-empty set, $\bigvee : L \times L \longrightarrow \wp^*(L)$ be a hyperoperation, and $\wedge : L \times L \longrightarrow L$ be an operation. Then, $(L, \bigvee, \wedge)$ is a join hyperlattice if for all $x, y, z \in L$ the following conditions hold:

(1) $x \in x \bigvee x$ and $x = x \wedge x$;

(2) $x \bigvee (y \bigvee z) = (x \bigvee y) \bigvee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

(3) $x \bigvee y = y \bigvee x$ and $x \wedge y = y \wedge x$;

4) $x \in x \wedge (x \bigvee y) \cap x \bigvee (x \wedge y)$.

Meet hyperlattice : Let L be a non-empty set, $\bigwedge : L \times L \longrightarrow \wp^*(L)$ be a hyperoperation, and $\vee : L \times L \longrightarrow L$ be an operation. Then, $(L, \vee, \bigwedge)$ is a meet hyperlattice if for all $x, y, z \in L$ the following conditions hold:

(1) $x \in x \bigwedge x$ and $x = x \vee x$;

(2) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \bigwedge (y \bigwedge z) = (x \bigwedge y) \bigwedge z$;

(3) $x \vee y = y \vee x$ and $x \bigwedge y = y \bigwedge x$;

(4) $x \in x \bigwedge (x \vee y) \cap x \vee (x \bigwedge y)$.

Total hyperlattice or superlattice : Let L be both join and meet hyperlattice (which means that $\bigvee$ and $\bigwedge$ are both hyperoperations). Then, we call L is total hyperlattice.

Example 1 : [26] Let $(L, \vee, \wedge)$ be a lattice. We define $\bigvee$ on L as follows:

$$x \bigvee y = \{z \mid z \geq x \vee y\}.$$

Then, $(L, \bigvee, \wedge)$ is a join hyperlattice.

Example 2 : [25] Let $Sub(V)$ be the set of all subspaces of n-dimensional vectors space V. Define hyperoperations $\oplus$ and $\otimes$ on $Sub(V)$ as follows: for all $V_1, V_2 \in Sub(V)$, $V_1 \otimes V_2 = Sub(V_1 \cap V_2)$, $V_1 \oplus V_2 = Sub(V_1 + V_2)$, where $V_1 + V_2$ represents the sum space of V_1 and V_2. Then, we can check that $(Sub(V), \otimes, \oplus)$ is a hyperlattice.

Example 3 : [25] Let $(L, \leq)$ be a partial order set. we define hyperoperations as follows: $a \bigvee b = \{x \in L : x \leq a, x \leq b\}$ and $a \bigwedge b = \{x \in L : a \leq x, b \leq x\}$. Then, $(L, \bigvee, \bigwedge)$ is a total hyperlattice.

3. Complete Hyperlattices

In this section, we consider join hyperlattices. We define complete join hyperlattice, and we study some properties of them. Also, we consider some well-known hyperlattices such as Nakano superlattice and P-hyperlattice, and we show that under certain conditions, they are complete. Then, we define completion of a join hyperlattice.

Let $(L, \bigvee, \wedge)$ be a join hyperlattice. According to [26], we say L is a strong join hyperlattice if for all $x, y \in L$, $y \in x \bigvee y$ implies that $x = x \wedge y$. Notice that by [26], there exists an order relation on a join hyperlattice $(L, \bigvee, \wedge)$ such that $x \leq y$ if and only if $x = x \wedge y$. We say that 0 is a zero element of L, if for all $x \in L$ we have $0 \leq x$ and 1 is a unit of L if for all $x \in L$, $x \leq 1$. We say L is bounded if L has 0 and 1. And y is a complement of x if $1 \in x \bigvee y$ and $0 = x \wedge y$. A complemented hyperlattice is a bounded hyperlattice which every element has a complement. We say L is distributive if for all $x, y, z \in L$, $x \wedge (y \bigvee z) = (x \wedge y) \bigvee (x \wedge z)$. And L is s-distributive if $x \bigvee (y \wedge z) = (x \bigvee y) \wedge (x \bigvee z)$. Notice that in lattices, the concepts of distributivity and s-distributivity are equivalent but in hyperlattice this is not true. Every complemented distributive join hyperlattice is called a Boolean hyperlattice. Let $\leq$ be an order relation on L such that if $x \leq y$ and for every $a \in L$ implies that $a \bigvee x \leq a \bigvee y$, then $(L, \bigvee, \wedge)$ is an order hyperlattice and $\leq$ is a hyperorder. Let I and F be non-empty subsets of L. Then, I is called an ideal of L if: (1) for every $x, y \in I$, $x \bigvee y \subseteq I$; (2) $x \leq I$ implies $x \in I$. Also, F is called a filter of L if: (1) for every $x, y \in F$, $x \wedge y \in F$; (2) $F \leq x$ implies $x \in F$. An ideal I of L is prime if for all $a, b \in L$, $a \wedge b \in I$ implies that $a \in I$ or $b \in I$. Similarly, a filter F is prime if $(a \bigvee b) \cap F \neq \emptyset$ implies that $a \in F$ or $b \in F$. Let $(L, \bigvee, \wedge)$ be a bounded strong hyperlattice. Then, $p \in L$ is called a prime element if $p \neq 1$ and when $p = x \wedge y$ implies $p = x$ or $p = y$. Let L_1 and L_2 be two join hyperlattices. The map $f : L_1 \longrightarrow L_2$ is called a homomorphism if for all $x, y \in L_1$ we have $f(x \bigvee y) = f(x) \bigvee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$. Moreover, f is an isomorphism if it is bijection too.

Let $(L, \bigvee, \wedge)$ be a join strong hyperlattice and $A \subseteq L$. The intersection of all ideals of L containing A is the ideal generated by A and it is denoted by $\langle A \rangle$. By [14], in order hyperlattices, we have $(A] = \{t \in L \mid \exists a \in A, t \leq a\}$.

***Theorem* 3.1** — *Let $(L, \bigvee, \wedge)$ be a join order hyperlattice and $X \subseteq L$. Then, we have*

$$\langle X \rangle = (X] \cup (L \bigvee X] = \{t \in L \mid t \leq x_1 \bigvee x_2 \bigvee \ldots \bigvee x_n, x_i \in X\}.$$

PROOF : First, we show that $\langle X \rangle$ is an ideal of $(L, \bigvee, \wedge, \leq)$. Let $t_1, t_2 \in \langle X \rangle$, there exist $x_i, y_i \in X$ such that $t_1 \leq x_1 \bigvee x_2 \bigvee \ldots \bigvee x_n$ and $t_2 \leq y_1 \bigvee y_2 \bigvee \ldots \bigvee y_n$. Since $\leq$ is a hyperorder $t_1 \bigvee t_2 \leq x_1 \bigvee x_2 \bigvee \ldots \bigvee x_n \bigvee y_1 \bigvee y_2 \bigvee \ldots \bigvee y_n$. Thus, $t_1 \bigvee t_2 \in \langle X \rangle$. If $x' \leq y$ and $y \in \langle X \rangle$, by transitivity of $\leq$ we have $x' \in X$. Thus, $\langle X \rangle$ is an ideal. We can easily prove that $\langle X \rangle$ is the least ideal containing X and the proof is completed. □

Proposition 3.2 — Let $(L, \bigvee, \wedge)$ be a join strong hyperlattice. For every $(a, b) \in L \times L$ there exist $a_1, a_2 \in a \bigvee b$ such that $a \leq a_1$ and $b \leq b_1$.

PROOF : Since $a \in a \wedge (a \bigvee b)$, there exists $a_1 \in a \bigvee b$ such that $a \leq a_1$. □

Proposition 3.3 — Let $(L, \bigvee, \wedge)$ be a join strong hyperlattice. If for every $x, y \in L$, $x \bigvee y$ is an ideal of L, then $x = y$.

PROOF : Suppose that $x, y \in L$ such that $x \bigvee y$ is an ideal of L. By the previous proposition, there exist $x_1, y_1 \in x \bigvee y$ such that $x \leq x_1$ and $y \leq y_1$. Since $x \bigvee y$ is an ideal of L and L is strong, we have $x = y$. □

Let L be a join hyperlattice. We can define relation $\preceq$ as follows: $a \preceq b$ if and only if $b \in a \bigvee b$. If L is strong, then this relation coincide with $\leq$ which is defined already.

Proposition 3.4 — Let $(L, \bigvee, \wedge)$ be a distributive join strong hyperlattice and $a \in L$. Then, $I = (a] = \{x \in L \mid x \leq a\}$ is an ideal of L.

PROOF : Suppose that $x, y \in L$ such that $x \leq y$ and $y \in I = (a]$. Since $x \leq y \leq a$, we have $x \in I$. Moreover, if $p, q \in I$ by distributivity of L, we have $a \wedge (p \bigvee q) = (a \wedge p) \bigvee (a \wedge q)$. Thus, for every $x \in p \bigvee q$ there exists $y \in p \bigvee q$ such that $x = a \wedge y \leq a$. Therefore, $p \bigvee q \subseteq I$. □

The ideal $(a]$ in Proposition 3.4 is called a principal ideal and we denote it by $\downarrow a$.

Definition 3.5 — Let $(L, \bigvee, \wedge)$ be a join hyperlattice. Then, L is a complete join hyperlattice if for every $S \subseteq L$ and subsets $S^u = \{x \in L \mid (\forall s \in S) s = s \wedge x\}$, $S^l = \{x \in L \mid (\forall s \in S) s \in s \vee x\}$, S^u has a least element and S^l has a greatest element with the order relation $\leq$ on L.

Example 4 — [27] Let $L = \{0, x_1, x_2, 1\}$. Consider the following tables:

$\bigvee$	0	x_1	x_2	1
0	0	$\{x_1, x_2, 1\}$	$\{x_2, 1\}$	1
x_1	$\{x_1, x_2, 1\}$	x_1	$\{x_2, 1\}$	1
x_2	$\{x_2, 1\}$	$\{x_2, 1\}$	x_2	1
1	1	1	1	1

$\wedge$	0	x_1	x_2	1
0	0	0	0	0
x_1	0	x_1	x_1	x_1
x_2	0	x_1	x_2	x_2
1	0	x_1	x_2	1

Then, $(L, \bigvee, \wedge)$ is a complete join hyperlattice.

Example 5 — [21] Let $L = \{a, b, c, d\}$. Consider the following table and operations:

$\bigvee$	a	b	c	d
a	a	b	c	d
b	b	$\{a, b\}$	d	d
c	c	d	$\{a, c\}$	d
d	d	d	d	L

$$a \wedge a = a \wedge b = b \wedge a = a \wedge c = c \wedge a = b \wedge c = c \wedge b = a \wedge d = d \wedge a = a$$

$$b \wedge d = d \wedge b = b, c \wedge d = d \wedge c = c, d \wedge d = d$$

Then, $(L, \bigvee, \wedge)$ is a complete join hyperlattice.

Example 6 — Let $H = \{0, x_1, x_2, 1\}$. Consider the following tables:

$\bigvee$	0	x_1	x_2	1
0	0	x_1	x_2	1
x_1	x_1	$\{0, x_1\}$	1	$\{x_2, 1\}$
x_2	x_2	1	$\{0, x_2\}$	$\{x_1, 1\}$
1	x_2	1	1	H

$\wedge$	0	x_1	x_2	1
0	0	0	0	0
x_1	0	x_1	0	x_1
x_2	0	0	x_2	x_2
1	0	x_1	x_2	1

Then, $(L, \bigvee, \wedge)$ is a join complete hyperlattice.

Now, we introduce some examples of join hyperlattices which are not complete.

Example 7 : Let $\mathbb{R}$ be the set of real numbers with hyperoperation $x \vee y = [x, y]$ and operation $x \wedge y = \min\{x, y\}$. Then, $\mathbb{R}$ is a join hyperlattice but is not complete since for $S = (-1, 2)$ the subset S^l has not greatest element.

Example 8 : Consider $H = \{a, b, c\}$ and the following tables:

$\bigvee$	a	b	c
a	a	b	c
b	b	b	H
c	c	H	c

$\wedge$	a	b	c
a	a	a	a
b	a	b	a
c	a	b	c

Then, H is not a complete hyperlattice, since for $S = \{b, c\}$, S^u does not exist.

Consider a modular lattice $(L, \vee, \wedge)$. We recall the Nakano hyperoperations $\sqcup$ and $\sqcap$ on L. For all $x, y \in L$ we define

$$x \sqcup y = \{z : z \vee x = z \vee y = x \vee y\}, \;\; x \sqcap y = \{z : z \wedge x = z \wedge y = x \wedge y\}.$$

According to [17] $(L, \sqcup, \wedge)$ is a strong join hyperlattice.

***Theorem* 3.6** — *Let $(L, \sqcup, \wedge)$ be a strong join hyperlattice which is defined as above and $(L, \vee, \wedge)$ be a complete lattice. If the hyperlattice L has greatest and least elements, then $(L, \sqcup, \wedge)$ is a complete hyperlattice.*

PROOF : Let $S \subseteq L$. We have $S^l = \{x \in L \mid \forall s \in S, s \in s \sqcup x\}$. According to the definition of $\sqcup$ we have $S^l = \{x \in L \mid \forall s \in S, s = s \vee s = s \vee x\}$. If we denote partial order on a lattice L by $\leq_1$, then we have $S^l = \{x \in L \mid \forall s \in S, x \leq_1 s\}$. Since $(L, \vee, \wedge)$ is a complete lattice, so S^l with order $\leq_1$ has the greatest element x'. Hence, for every $s \in S$ we have $x' \vee s = s \vee s = s \vee x$. Therefore, $x' \in s \sqcup x'$. Thus, $s \leq x'$ and similarly we show that S^u has the least element with order $\leq$ on L. Now, since lattice L is complete the rest of proof is completed. (Notice that if $S = \emptyset$, then the hyperlattice L should has the greatest and least elements). □

In [18] Konstantinidou investigated P-hyperlattice and defined the hyperoperation $\bigvee^P$ as follows:

$$a \bigvee^P b = a \vee b \vee P = \{a \vee b \vee p \mid p \in P\}.$$

She showed that $(L, \bigvee^p, \wedge)$ is a join strong hyperlattice if and only if for each $x \in L$ there exists $p \in P$ such that $p \leq x$.

***Theorem* 3.7** — *Let $(L, \vee, \wedge)$ be a complete lattice. Then, the P-hyperlattice $(L, \bigvee^P, \wedge)$ is a join complete hyperlattice.*

PROOF : If $S \subseteq L$, then $S^l = \{x \in L \mid \forall s \in S, s \in s \bigvee^p x\}$. So, there exists $p \in P$ such that $s = s \vee x \vee p$. Therefore, $x \leq_1 s$ where $\leq_1$ is an order relation on L. Since L is complete, S^l has the greatest element with order relation $\leq_1$. Let x' be such element. Thus, for all $s \in S$, we have $x' \leq_1 s$. Therefore, $s \vee x' = s$. By the condition of L, there exists $p \in P$ such that $p \leq_1 s$. So, $s \in s \bigvee^p x'$. Existence of greatest element of S^l is concluded from the complete lattice L. The same arguments can be applied for S^u . □

Notice that if $(L, \bigvee, \wedge)$ is a join strong hyperlattice, then for every $s \in S$ which $s = s \wedge x$, we have $s \in s \bigvee (s \wedge x)$. Thus, we get $s \in s \bigvee x$. Therefore, for every $x \in S^U$ we have $x \preceq s$. Thus, for every $x \in S^u$ we have $x \in (S^u)^l = S^{ul}$. If $s \in S$ and $x \in S^l$, then $s \in s \vee x$. Since L is strong, we have $x = s \wedge x$. Therefore, $s \in (S^l)^u$. Thus, $S \subseteq (S^l)^u$. If $(L, \bigvee, \wedge)$ is a strong join hyperlattice and $S \subseteq L$, then we can easily show that $S^u = S^{ulu}$ and $S^l = S^{lul}$.

Definition 3.8 — Let L and L' be two join hyperlattices and $\phi : L' \hookrightarrow L$ be an embedding (onto homeomorphism between join hyperlattice). If L is a complete join hyperlattice, then L is called a completion of L'.

***Theorem* 3.9** — *Let L be a join strong hyperlattice and $\bar{L} = \{A \subseteq L \mid A^{ul} = A\}$. Then, $\bar{L}$ is a complete join hyperlattice and $\bar{L}$ is a completion of L.*

PROOF : We define $A_1 \vee A_2 = \cap\{B \subseteq L \mid A_1 \cup A_2 \subseteq B\}$ and $A_1 \wedge A_2 = A_1 \cap A_2$. Now, we show that $\bar{L}$ is a join strong hyperlattice. We can easily show that $A \in A \vee A$ and $A_1 \vee A_2 = A_2 \vee A_1$. Also, we obtain $A \in A \vee (A \cap B) = \cap\{C \mid A \cup (A \cap B) \subseteq C\}$. If $A \in A \vee B$ then $A \subseteq A \cup B \subseteq A$. So, $A \wedge B = B$. Therefore, $(\bar{L}, \vee, \wedge)$ is a join strong hyperlattice. Now, suppose that $\{A_i\}_{i \in I} = S$ is a family of non-empty subsets of L. Hence, we have $S^l = \{A_j \in L \mid \forall A_i \in S,\ \ A_i \in A_i \vee A_j\}$. Therefore, $\cap A_i$ is the greatest element of S^l. Similarly, we can prove that S^u has the least element. Now, we show that $\bar{L}$ is a completion of L. It is enough to define $\phi : L \hookrightarrow \bar{L}$ such that $\phi(x) = x^l$. □

Now, we define a class of complete join hyperlattices and prove some related results.

Definition 3.10 — Let $(L, \bigvee, \wedge)$ be a complete join hyperlattice which is bounded. In addition, the infinite distributive law holds in L, which means that for every $a \in L$ and $S \subseteq L$ we have

$$a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s).$$

Then, we say L is a D-hyperlattice.

Example 9 : The hyperlattice which introduced in Example 6 is a complete distributive join hyperlattice. Since hyperlattice L is finite, infinite distributive law is the same finite distributive law. So, L is a D-hyperlattice.

Example 10 : Every complete Boolean hyperlattice is a D-hyperlattice.

Proposition 3.11 — Let $(L, \bigvee, \wedge)$ be a complete join strong hyperlattice which has the property that every two elements of L are comparable with the order relation $\leq$ on L. Then, L is a D-hyperlattice.

PROOF : Let $\leq$ be an order relation on hyperlattice L. We consider two cases;

(1) $a < \bigvee S$. In this case, there exists $s \in S$ such that $s \nleq a$. Thus, by hypothesis $a \preceq s$, which means that $s \in s \bigvee a$. Since $a \in a \bigvee a$, $a \in s \bigvee a$ and L is a strong hyperlattice. Thus, we have $a = s \wedge a$ and $\bigvee_{s \in S}(a \wedge s) = a$. Also, since $a < \bigvee S$, we have $a = a \wedge s \leq a \wedge \bigvee S \leq a$. Therefore, $a \wedge (\bigvee S) = a$.

(2) $\bigvee S \leq a$. In this case, $\bigvee S \wedge a = \bigvee S$. And for every $s \in S$ we have $s \leq a$. Therefore, $s = s \wedge a$ and $\bigvee_{s \in S}(a \wedge s) = \bigvee S$. Thus, L is a D-hyperlattice. □

Definition 3.12 — Let L be a D-hyperlattice and F be a filter of L. We say F is completely prime if for every $S \subseteq L$ we have

$$\bigvee S \in F \Leftrightarrow S \cap F \neq \emptyset.$$

Notice that every completely prime filter in a D-hyperlattice is prime.

***Theorem* 3.13** — *Let L be a complete bounded distributive join strong hyperlattice which is s-distributive. Then, $p \in L$ is a prime element if and only if $A = L - \downarrow p$ is a completely prime filter of L.*

PROOF : Suppose that p is a prime element. We consider $A = L - \downarrow p$. Then, for every $x, y \in A$ we have $x \notin \downarrow p$ and $y \notin \downarrow p$. If $x \wedge y \leq p$, then $x \wedge y \in \downarrow p$. Since L is strong $x \wedge y \preceq p$, hence $p \in (x \wedge y) \bigvee p = (x \bigvee p) \wedge (y \bigvee p)$. Thus, there exist $a \in x \bigvee p$ and $b \in y \bigvee p$ such that $p = a \wedge b$. Since p is a prime element, we have $p = a \in x \bigvee p$ or $p = b \in x \bigvee p$. Thus, $x \preceq p$ or $y \preceq p$. Therefore, $x \leq p$ or $y \leq p$ and this is a contradiction. Thus, $x \wedge y \in A$. If $x \leq y$ and $x \notin p$, then $y \notin p$. Therefore, L is a filter. Now, let $X \subseteq L$ and $\bigvee X \in A$. Hence, $\bigvee X \nleq p$. Thus, there exists $x \in X$ such that $x \nleq p$. Therefore, $A \cap X \neq \emptyset$.

Similarly, we can prove the converse. □

Example 11 : Let L be a complete strong s-distributive Boolean hyperlattice and p be a prime element of it. We derive the property of elements $x \in L$ such that $p \leq x \leq 1$. Since $p \in 0 \bigvee p = (x \wedge x') \bigvee p = (x \bigvee p) \wedge (x' \bigvee p)$, there exist $y \in x \bigvee p$ and $y' \in x' \bigvee p$ such that $p = y \wedge y'$. Since p is a prime element, we have $p \in x \bigvee p$ or $p \in x' \bigvee p$. Since L is strong, $x \leq p$ or $x' \leq p$. If we consider $p \leq x \leq 1$, then $x = p$ or $x' \leq p$. Thus, $1 \in x \bigvee x' \subseteq x \bigvee (x' \wedge p) = (x \bigvee x') \wedge (x \bigvee p)$

and we can say that there exist $y \in x \bigvee x'$ and $y' \in x \bigvee p$ such that $1 = y \wedge y'$. We conclude that $1 = y' \in x \bigvee p$. Hence, if $p \leq x \leq 1$ and p is a prime element, then $x = p$ or $1 \in x \bigvee p$.

Definition 3.14 — Let L and M be two strong bounded join hyperlattices and L be a good hyperlattice ($0 \bigvee 0 = 0$). The map $f : L \longrightarrow M$ is a weak homeomorphism if

(1) $f(a \vee b) = f(a) \vee f(b)$;

(2) $f(1) = 1$;

(3) $a \wedge b = 0 \Leftrightarrow f(a) \wedge f(b) = 0$;

(4) $x \in a \vee b \Leftrightarrow f(x) \in f(a \vee b)$.

Notice that if L is a D-hyperlattice, then the condition (1) can be reformulated as $f(\bigvee S) = \bigvee f(S)$ for every $S \subseteq L$. The map f is a homomorphism if for every $S \subseteq L$, we have $f(\bigvee S) = \bigvee f(S)$ and $f(\bigwedge S) = \bigwedge f(S)$ for every finite subset $S \subseteq L$.

***Theorem* 3.15** — *Let $(L, \vee, \wedge)$ be a strong D-hyperlattice which is s-distributive and for every $a, b \in L$ there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \leq u \vee b$, $b \leq a \vee v$. Then, every weak homomorphism is a homomorphism.*

PROOF : By hypothesis, there exist u and v such that $a \leq u \vee b$ and $b \leq a \vee v$. Since $a \leq (u \vee a) \wedge (u \vee b) = u \vee (a \wedge b)$ and $b \leq v \vee (a \wedge b)$. Thus, we have $f(a) \leq f(u) \vee f(a \wedge b)$ and $f(b) \leq f(v) \vee f(a \wedge b)$. Since $f(u) \wedge f(v) = 0$, we have $f(a) \wedge f(b) \leq f(a \wedge b) \vee (f(u) \wedge f(v)) = 0 \vee f(a \wedge b) = f(0) \vee f(a \wedge b)$. As before, we define $a \leq X$ if for every $x \in X$, $a \leq x$. Thus, since $0 \leq f(a \wedge b)$ and L is strong, $0 \preceq f(a \wedge b)$. Therefore, $f(a \wedge b) \in 0 = f(0) \vee f(a \wedge b)$. Consequently, $f(a) \wedge f(b) \leq f(a \wedge b)$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $f(a \wedge b) \leq f(a) \wedge f(b)$. Thus, the proof is completed and f is a homomorphism. □

***Theorem* 3.16** — *Let $(L, \bigvee, \wedge)$ be a strong s-distributive Boolean hyperlattice which has the property that for every $x \in L$, $x \bigvee 0 = x$. Then, every onto weak homomorphism is a homomorphism. (A homomorphism in Boolean hyperlattices is a hyperlattice homomorphism which preserves 0, 1 and complements).*

PROOF : Let $f : L \longrightarrow B$ be an onto weak homomorphism in the category of Boolean hyperlattice. First, we show that for every $a, b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$. Since L is complemented, there exists z, z' such that $1 \in a \bigvee z$, $a \wedge z = 0$ and $1 \in b \bigvee z'$, $b \wedge z' = 0$. Thus, we obtain

$$(z \bigvee a) \wedge (z' \bigvee b) \leq z \bigvee z' \bigvee (a \wedge b) = (z \bigvee z' \bigvee a) \wedge (z \bigvee z' \bigvee b).$$

Since $1 \in (z \bigvee a) \wedge (z' \bigvee b)$, there exists $y \in z \bigvee z' \bigvee a$ and $y' \in z \bigvee z' \bigvee b$ such that $1 \leq y \wedge y'$. Since $1 \in z \bigvee z' \bigvee (a \wedge b)$ and f is a weak homomorphism, we have $1 = f(1) \in f(z) \bigvee f(z') \bigvee f(a \wedge$

$b)$. Thus, we have

$$\begin{aligned} f(a) \wedge f(b) &= f(a) \wedge f(b) \wedge 1 \\ &\subseteq (f(a) \wedge f(b)) \wedge (f(z) \bigvee f(z') \bigvee f(a \wedge b)) \\ &= f(a) \wedge f(b) \wedge f(a \wedge b) \leq f(a \wedge b). \end{aligned}$$

The converse is trivial. So, for every $a, b \in L$ we have $f(a \wedge b) = f(a) \wedge f(b)$. Moreover, since f is onto, there exists $a \in L$ such that $f(a) = 0_B$. Since L and B are bounded, we have $f(0_L) = 0_B$. For every $a \in L$ we have $1 \in a \bigvee a'$, $a \wedge a' = 0$. Thus, $f(a) \wedge f(a') = 0$, $1 = f(1) \in f(a \bigvee a') = f(a) \bigvee f(a')$. Therefore, $f(a') \subseteq f(a)'$. Similarly, we can prove that $f(a)' \subseteq f(a')$. This completes the proof. □

Definition 3.17 — Let $(L, \bigvee, \wedge)$ be a distributive join hyperlattice. We say that L is a regular hyperlattice if for every $a, b \in L$ with the condition $a \nleq b$, there exists $y \in L$ such that $1 \in a \bigvee y$ and $a \wedge y = 0$.

Example 12 : Every complemented distributive hyperlattice is a regular hyperlattice.

***Theorem* 3.18** — *Let $(L, \bigvee, \wedge)$ be a regular join hyperlattice such that for every $x \in L$, $x \bigvee 0 = x$. Then, every weak homomorphism in the category of join hyperlattices is a homomorphism.*

PROOF : Suppose that $a, b \in L$. If $a \leq b$ or $b \leq a$, then the proof is trivial. So, we consider $a \nleq b$ and $b \nleq a$. Therefore, by the regularity of L, there exists $y, y' \in L$ such that $1 \in a \bigvee y$, $a \wedge y = 0$ and $1 \in b \bigvee y'$, $y' \wedge b = 0$. Thus, $1 \in (y \bigvee y') \bigvee (a \wedge b)$ and we have

$$\begin{aligned} f(a) \wedge f(b) &= (f(a) \wedge f(b)) \wedge 1 \\ &\subseteq (f(a) \wedge f(b)) \wedge (f(y) \bigvee f(y') \bigvee f(a \wedge b)) \\ &= 0 \bigvee 0 \bigvee (f(a) \wedge f(b) \wedge f(a \wedge b)) \\ &\leq f(a \wedge b). \end{aligned}$$

The reverse inclusion is trivial. Hence, the proof is completed. □

REFERENCES

1. A. R. Ashrafi, About some join spaces and hyperlattices, *Ital. J. Pure Appl. Math.*, **10** (2001), 199-205.

2. A. Asokkumar, Hyperlattice formed by the idempotents of a hyperring, *Tamkang J. Math.*, **38**(3) (2007), 209-215.

3. G. Birkhoff, Lattice Theory, *Amer. Math. Soc*., New York, 1940.

4. I. Chajda and S. Hoskova, A factorization of quasiorder hypergroups, *Comment. Math. Univ. Carolin.*, **45**(4) (2004), 573-581.

5. I. Chajda and S. Hoskova, A characterization of cone preserving mappings of quasiordered sets, *Miscolc Mathematical Notes*, **6**(2) (2005), 147-152.

6. J. Chvalina, *Commutative hypergroups in the sense of Marty and ordered Sets*, Proceedings of International Conference on General Algebra and Ordered Sets, 1994, Olomouc.

7. P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani editore, Second edition, 1993.

8. P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

9. B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, 2013.

10. B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

11. A. Dehghan Nezhad and B. Davvaz, An introduction to the theory of H_v-semilattices, *Bull. Malays. Math. Sci. Soc.*, **32**(3) (2009), 375-390.

12. X. Z. Guo and X. L. Xin, Hyperlattice, *Pure Appl. Math.*, **20** (2004), 40-43.

13. S. W. Han and B. Zhao, Distributive hyperlattice, *J. Northwest Univ.*, **35**(2) (2005), 125-129.

14. D. Heidari and B. Davvaz, On ordered hyperstructures, *University Politehnica of Bucharest Scientific Bulletin, Series A: Applied Mathematics and Physics*, **73**(2) (2011), 85-96.

15. S. Hoskova, *Discrete transformation hypergroups*, Proceedings of 4th International Conference Aplimat, 2005, Bratislava, Slovakia.

16. J. Jakubik, On strong superlattices, *Math. Slovaca*, **44**(2) (1994), 131-138.

17. A. Kehagias, K. Serafimidis and M. Konstantinidou, A note on the congruences of the Nakano Superlattice and some properties of the associated quotients, *Matematica Palermo*, **51**(2) (2002), 333-354.

18. M. Konstantinidou, On P-hyperlattices and their distributivity, *Rendiconti del Circolo Matematico di Palermo*, **42**(3) (1993), 391-403.

19. M. Konstantinidou, On the hyperlattices-ordered groupoids, *Boll. Un. Mat. Ital. A (6)*, **2**(3) (1983), 343-350.

20. M. Konstantinidou and J. Mittas, An introduction to the theory of hyperlattices, *Mathematica Balkanica*, **7** (1977), 187-193.

21. M. Konstantinidou and K. Serafimidis, Modular hyperlattices, *Prakt. Akad. Athenon*, **53** (1978), 202-218.

22. M. Konstantinidou and A.I. Synefaki, A strong Boolean hyperalgebra of Boolean functions, *Ital. J. Pure Appl. Math.*, **2** (1997), 9-18.

23. V. Leoreanu-Fotea and B. Davvaz, Join n-spaces and lattices, *Journal of Multiple-Valued Logic and Soft Computing*, **15**(5-6) (2009), 421-432.

24. F. Marty, *Sur une generalisation de la notion de groupe*, in 8th congres des Mathematiciens Scandinaves, (1934), 45-49.

25. H. Pengfei, X. Xin and J. Zhan, On rough hyperideals in hyperlattice, *J. Appl. Math.*, (2013), Art. ID 915217, 10 pp.

26. S. Rasouli and B. Davvaz, Lattices derived from hyperlattices, *Comm. Algebra*, **38**(8) (2010), 2720-2737.

27. S. Rasouli and B. Davvaz, Construction and spectral topology on hyperlattices, *Mediterranean Journal of Mathematics*, **7**(2) (2010), 249-262.

28. A. Rahnamai-Barghi, The prime ideal theorem and semiprime ideals in meethyperlattices, *Ital. J. Pure Appl. Math.*, **5** (1999), 53-60.

29. A. Rahnamai-Barghi, The prime ideal theorem for distributive hyperlattices, *Ital. J. Pure Appl. Math.*, **10** (2001), 75-78.

30. T. Vougiouklis, *Hyperstructures and Their Representations*, Aviani editor, Hardonic Press, Palm Harbor, USA, 1994.