

THE t -NAGATA RING OF t -SCHREIER DOMAINS

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(Received 7 December 2013; accepted 25 December 2014)

Let D be an integral domain, X be an indeterminate over D , v (respectively, t) be the so-called v -operation (respectively, t -operation) on D and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. In this paper, we completely characterize when the t -Nagata ring $D[X]_{N_v}$ is a Schreier domain.

Key words : t -Schreier domain, t -Nagata ring.

1. INTRODUCTION

Throughout this paper, D denotes an integral domain with quotient field K , X is an indeterminate over D , $D[X]$ means the polynomial ring over D , and for an $f \in K[X]$, $c(f)$ stands for the D -submodule of K generated by the coefficients of f .

In [5], Cohn called an element $0 \neq a \in D$ *primal* if for all $0 \neq b_1, b_2 \in D$ with $a \mid b_1 b_2$, a can be written as $a = a_1 a_2$ for some $a_1, a_2 \in D$ with $a_i \mid b_i$, $i = 1, 2$. He also defined D to be a *Schreier domain* if D is an integrally closed domain in which every nonzero element is primal. He noted that the condition for D to be a Schreier domain, when expressed in terms of the ordered group of fractional principal ideals of D (ordered by reverse inclusion), becomes a Riesz group [5, page 255]. (Recall that a directed partially ordered abelian group G is a *Riesz group* if whenever $x \leq a_1 + a_2$ with $x, a_1, a_2 \geq 0$ elements of G , there exist $x_1, x_2 \in G$ such that $x = x_1 + x_2$ and $0 \leq x_i \leq a_i$, $i = 1, 2$). Later, Zafrullah weakened the condition and introduced the notion of pre-Schreier domains [15]. He called D a *pre-Schreier domain* if every nonzero element of D is primal. Hence an integrally closed pre-Schreier domain is a Schreier domain. As a generalization of the class of pre-Schreier domains, Dumitrescu and Moldovan introduced the notion of quasi-Schreier domains. They defined D to be a *quasi-Schreier domain* if whenever $A_1 A_2 \subseteq B$ with B, A_1, A_2 invertible ideals of D , it follows that $B = B_1 B_2$ for some (invertible) ideals B_i of D such that $A_i \subseteq B_i$, $i = 1, 2$. It is known that D

is a quasi-Schreier domain if and only if the group of invertible fractional ideals of D (ordered by reverse inclusion) is a Riesz group. In order to study integral domains whose groups of t -invertible fractional t -ideals are Riesz groups, Dumitrescu and Zafrullah introduced the concept of t -Schreier domains, and investigate several properties [8]. They defined D to be a t -Schreier domain if whenever B, A_1, A_2 are t -invertible t -ideals of D and $A_1 A_2 \subseteq B, B = (B_1 B_2)_t$ for some (t -invertible) ideals B_i of D such that $A_i \subseteq (B_i)_t, i = 1, 2$.

Let $N = \{f \in D[X] \mid c(f) = D\}$ and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then N and N_v are saturated multiplicative subsets of $D[X]$ with $N \subseteq N_v$. The quotient ring $D[X]_N$ (respectively, $D[X]_{N_v}$) is called the *Nagata ring* (respectively, *t -Nagata ring*) of D . It was shown in [2, Theorem 13] that D is an integrally closed quasi-Schreier domain if and only if $D[X]_N$ is a Schreier domain. In this paper, we study the t -Nagata ring of t -Schreier domains. In fact, we show that D is an integrally closed t -Schreier domain if and only if $D[X]_{N_v}$ is a Schreier domain. As a corollary, we regain the t -Schreier domain version of Bazzoni's conjecture, which states that if D is a t -Schreier domain, then D has finite t -character if and only if D is a w -LPI domain.

For the reader's convenience, we review some preliminaries. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D . For an $I \in \mathbf{F}(D)$, $I^{-1} = \{x \in K \mid xI \subseteq D\}$. The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the v -operation on D ; the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t = \bigcup \{J_v \mid J \text{ is a nonzero finitely generated fractional subideal of } I\}$ is called the t -operation on D ; and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w = \{x \in K \mid xJ \subseteq I \text{ for some finitely generated ideal } J \text{ of } D \text{ such that } J^{-1} = D\}$ is called the w -operation on D . An $I \in \mathbf{F}(D)$ is a v -ideal (respectively, t -ideal, w -ideal) if $I_v = I$ (respectively, $I_t = I, I_w = I$). Clearly, if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. A t -ideal $I \in \mathbf{F}(D)$ is of *finite type* if $I = J_v$ for some finitely generated ideal J of D . An $I \in \mathbf{F}(D)$ is said to be *invertible* (respectively, *t -invertible*) if $II^{-1} = D$ (respectively, $(II^{-1})_t = D$). It is well known that if an $I \in \mathbf{F}(D)$ is t -invertible, then I_t is of finite type. The t -class group is the abelian group $\text{Cl}(D) = \text{T}(D)/\text{Prin}(D)$, where $\text{T}(D)$ is the group of t -invertible fractional t -ideals of D under the t -multiplication $I * J = (IJ)_t$ and $\text{Prin}(D)$ is the subgroup of $\text{T}(D)$ of nonzero principal fractional ideals of D . The *Picard group* of D is the subgroup $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$ of $\text{Cl}(D)$, where $\text{Inv}(D)$ is the group of invertible fractional ideals of D .

2. MAIN RESULTS

We start this section with the following two known facts which help us prove the main result. The readers may consult [8] and [6].

Lemma 1 — The following assertions hold for a *t*-Schreier domain *D*.

- (1) If *S* is a multiplicative subset of *D*, then *D_S* is a *t*-Schreier domain.
- (2) If *D* is integrally closed, then *D*[*X*] is a *t*-Schreier domain.
- (3) *D* is a pre-Schreier domain if and only if $\text{Cl}(D) = 0$.

PROOF : These statements appear in [8, Proposition 2(a) and Theorems 7 and 9(b)]. □

Lemma 2 — *D* is a pre-Schreier domain if and only if *D* is a quasi-Schreier domain and $\text{Pic}(D) = 0$.

PROOF : This was shown in [6, Proposition 2.2]. □

It is well known that *D* is integrally closed if and only if $c(fg)_v = (c(f)c(g))_v$ for all $0 \neq f, g \in D[X]$ [10, Theorem 34.8]. We give another criteria for *D* to be integrally closed in terms of the *t*-invertibility.

Lemma 3 — The following conditions are equivalent.

- (1) *D* is integrally closed.
- (2) If $c(fg)$ is *t*-invertible for some $0 \neq f, g \in D[X]$, then $c(f)$ is *t*-invertible.
- (3) If $c(fg)_v$ is principal for some $0 \neq f, g \in D[X]$, then $c(f)$ is *t*-invertible.
- (4) If $c(fg)$ is *t*-invertible for some $0 \neq f, g \in D[X]$, then $c(fg)_v = (c(f)c(g))_v$.
- (5) If $c(fg)_v$ is principal for some $0 \neq f, g \in D[X]$, then $c(fg)_v = (c(f)c(g))_v$.

PROOF : (1) \Rightarrow (2) If *D* is integrally closed, then $c(fg)_v = (c(f)c(g))_v$ for all $0 \neq f, g \in D[X]$ [10, Theorem 34.8]. Thus this implication follows directly from the fact that for $I, J \in \mathbf{F}(D)$, if *IJ* is *t*-invertible, then both *I* and *J* are *t*-invertible.

(2) \Rightarrow (3) and (4) \Rightarrow (5) It suffices to note that every nonzero principal fractional ideal is *t*-invertible.

(2) \Rightarrow (4) Recall that $c(f)^{m+1}c(g) = c(f)^m c(fg)$ for some positive integer *m* [10, Theorem 28.1]. Since $c(fg)$ is *t*-invertible, $c(f)$ is *t*-invertible by (2). Thus we have

$$(c(f)c(g))_v = (c(f)^{m+1}(c(f)^{-1})^m c(g))_v = (c(f)^m (c(f)^{-1})^m c(fg))_v = c(fg)_v,$$

which is the desired equality.

(3) \Rightarrow (5) Note again that a nonzero principal fractional ideal is *t*-invertible. Thus the same argument as in the proof of (2) \Rightarrow (4) shows this implication.

(5) \Rightarrow (1) Let $a \in K$ be integral over D . Then there exists a monic polynomial $f \in D[X]$ such that $f(a) = 0$. Factorize f in $K[X]$ as $f = (X - a)g$ for some $g \in K[X]$, and take a nonzero element $d \in D$ such that $ad \in D$ and $dg \in D[X]$. Hence by (5), we obtain

$$d^2D = (d^2c(f))_v = c((d(X - a))(dg))_v = (c(d(X - a))c(dg))_v = d^2(c(X - a)c(g))_v,$$

and therefore $D = (c(X - a)c(g))_v$. Note that g is a monic polynomial in $K[X]$; so $a \in (c(X - a)c(g))_v$. Thus $a \in D$, which indicates that D is integrally closed. \square

We give the main result of this paper.

Theorem 4 — *The following statements are equivalent.*

- (1) D is an integrally closed t -Screier domain.
- (2) $D[X]_{N_v}$ is a Schreier domain.
- (3) $D[X]_{N_v}$ is an integrally closed quasi-Schreier domain.
- (4) $D[X]_{N_v}$ is an integrally closed t -Screier domain.

PROOF : (1) \Rightarrow (2) If D is an integrally closed t -Screier domain, then $D[X]$ is a t -Screier domain by Lemma 1(2), and hence $D[X]_{N_v}$ is also a t -Screier domain by Lemma 1(1). Note that $\text{Cl}(D[X]_{N_v}) = 0$ [12, Theorems 2.4 and 2.14]. Thus $D[X]_{N_v}$ is a Schreier domain by Lemma 1(3).

(2) \Rightarrow (1) Note that $D = D[X]_{N_v} \cap K$ [12, Proposition 2.8(1)]; so D is integrally closed, because $D[X]_{N_v}$ is integrally closed. Let B, A_1, A_2 be t -invertible t -ideals of D with $A_1A_2 \subseteq B$. Then B is of finite type [12, Corollary 2.7]; so there exists an $f \in D[X]$ such that $B = c(f)_v$. Since $c(f)_v$ is t -invertible, $BD[X]_{N_v} = fD[X]_{N_v}$ [12, Lemma 2.11]. Similarly, for each $i = 1, 2$, we can find an element $g_i \in D[X]$ such that $A_i = c(g_i)_v$ and $A_iD[X]_{N_v} = g_iD[X]_{N_v}$. Then we have

$$\begin{aligned} (g_1D[X]_{N_v})(g_2D[X]_{N_v}) &= (A_1D[X]_{N_v})(A_2D[X]_{N_v}) \\ &\subseteq BD[X]_{N_v} \\ &= fD[X]_{N_v}, \end{aligned}$$

and hence f divides g_1g_2 in $D[X]_{N_v}$. Since $D[X]_{N_v}$ is a Schreier domain, f can be written as $f = f_1f_2$ for some $f_1, f_2 \in D[X]_{N_v}$ such that f_i divides g_i in $D[X]_{N_v}$ for each $i = 1, 2$. Write $f_i = \frac{f'_i}{f''_i}$ for some $f'_i, f''_i \in D[X]$ with $c(f''_i)_v = D$. Then $f f'_1 f'_2 = f'_1 f'_2$. Note that

$$c(f'_1 f'_2)_v = c(f f'_1 f'_2)_v = (c(f)c(f'_1)c(f'_2))_v = c(f)_v = B,$$

where the second equality comes from [10, Theorem 34.8]; so $c(f'_1 f'_2)_v$ is t -invertible. By letting $B_i = c(f'_i)_t$, we have $B = (B_1 B_2)_t$ by Lemma 3 (or [10, Theorem 34.8]). Since f_i divides g_i , we have

$$A_i D[X]_{N_v} = g_i D[X]_{N_v} \subseteq f_i D[X]_{N_v} = f'_i D[X]_{N_v} = B_i D[X]_{N_v}.$$

Thus we obtain

$$A_i = A_i D[X]_{N_v} \cap K \subseteq B_i D[X]_{N_v} \cap K = B_i,$$

which indicates that D is a t -Schreier domain.

(2) \Leftrightarrow (3) \Leftrightarrow (4) Note again that $\text{Cl}(D[X]_{N_v}) = 0$. Thus these equivalences are immediate consequences of Lemmas 1(3) and 2. □

Recall that D is a pre-Schreier domain (respectively, quasi-Schreier domain, t -Schreier domain) if and only if the group of nonzero principal fractional ideals (respectively, invertible fractional ideals, t -invertible fractional t -ideals) of D is a Riesz group. Thus by Theorem 4, we have

Corollary 5 — If D is integrally closed, then the following are equivalent.

- (1) The group of t -invertible fractional t -ideals of D is a Riesz group.
- (2) The group of nonzero principal fractional ideals of $D[X]_{N_v}$ is a Riesz group.
- (3) The group of invertible fractional ideals of $D[X]_{N_v}$ is a Riesz group.
- (4) The group of t -invertible fractional t -ideals of $D[X]_{N_v}$ is a Riesz group.

We say that D is a w -LPI domain if every nonzero t -locally principal ideal of D is t -invertible. In [4, Theorem 2.10], the authors showed that D is a w -LPI domain if and only if $D[X]_{N_v}$ is a w -LPI domain. It is known that if D is a quasi-Schreier domain, then D is a t -sub-Prüfer domain; so D has finite t -character if and only if D is a w -LPI domain [7, Corollary 10]. (Recall that D is a t -sub-Prüfer domain if every proper t -ideal of finite type of D is contained in a proper t -invertible t -ideal of D). Hence by Theorem 4, if D is a t -Schreier domain, then D has t -finite character if and only if D is a w -LPI domain [8, Proposition 17]. Since a PvMD is a t -Schreier domain, we recover the PvMD version of Bazzoni’s conjecture as follows. (Recall that D is a Prüfer v -multiplication domain (PvMD) if each nonzero finitely generated ideal of D is t -invertible; and Bazzoni’s conjecture is “if D is a Prüfer domain in which every nonzero locally principal ideal is invertible, then D is of finite character”).

Corollary 6 ([11, Theorem 6.11] or [16, Proposition 5]) — If D is a PvMD, then D has finite t -character if and only if D is a w -LPI domain.

Recall that D is a w -ideal; D is a *strong Mori domain* (SM-domain) if it satisfies the ascending chain condition on integral w -ideals of D ; and D is a *Krull domain* if (i) $D = \bigcap_{P \in X^1(D)} D_P$, (ii) D_P is a local PID for each $P \in X^1(D)$ and (iii) the intersection $D = \bigcap_{P \in X^1(D)} D_P$ has finite character, where $X^1(D)$ is the set of height-one prime ideals of D . It was shown that D is an SM-domain (respectively, Krull domain) if and only if $D[X]_{N_v}$ is a Noetherian domain (respectively, PID) [3, Theorem 1.4] (respectively, [13, Lemma 3.8]).

Corollary 7 (cf. [8, Corollary 6(b)]) — D is a Krull domain if and only if D is a t -Schreier SM-domain.

PROOF : (\Rightarrow) This implication follows directly from the facts that a Krull domain is both a PvMD and an SM-domain [14, Theorem 2.8]; and a PvMD is a t -Schreier domain [9, Lemma 1.8].

(\Leftarrow) If D is a t -Schreier SM-domain, then by Theorem 4 and [3, Theorem 1.4], $D[X]_{N_v}$ is a Schreier Noetherian domain; so $D[X]_{N_v}$ is a UFD. Hence $D[X]_{N_v}$ is a Krull domain (cf. [13, page 284]), and thus $D = D[X]_{N_v} \cap K$ is a Krull domain [10, Corollary 44.10]. \square

We end this article with an example of a non-integrally closed t -Schreier domain D such that $D[X]_{N_v}$ is a t -Schreier domain.

Example 8 : Let V be a valuation domain which has nonperfect residue field L , and M be the maximal ideal of V such that $M = M^2 \neq (0)$. Let $\pi : V \rightarrow L$ be the canonical epimorphism and consider the pullback $D = \pi^{-1}(L^p)$, where p is the characteristic of L . Then D is a non-integrally closed t -Schreier domain (cf. [1, Proposition 2.6] and [15, Theorem 4.4]). Note that $D[X]_N$ is a t -Schreier domain, where $N = \{f \in D[X] \mid c(f) = D\}$ (cf. [2, Example 14]). Since $N \subseteq N_v$, $D[X]_{N_v}$ is a t -Schreier domain by Lemma 1(1).

ACKNOWLEDGEMENT

The author would like to thank the referee for several valuable suggestions and to thank professor M. Zafrullah for indicating that if D is a quasi-Schreier domain, then D has finite t -character if and only if D is a w -LPI domain.

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