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ALGEBRAIC SOLUTIONS OF THE HIROTA BILINEAR FORM FOR THE KORTEWEG-DE VRIES AND BOUSSINESQ EQUATIONS

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We reduce the order of the fourth-order Hirota-bilinear equations of the Korteweg-de Vries and Boussinesq equations by using symmetries and then apply singularity analysis to the resultant ordinary differential equations. We employ all the possible combinations of symmetries to reduce the order of the equations.

Key words : Singularity analysis; symmetry analysis; Hirota bilinear forms; Korteweg-de Vries equation; Boussinesq equation.

1. Introduction

Symmetry analysis is very useful to find classes of particular solutions of linear and nonlinear differential equations. Many authors use this analysis to reduce the order of a differential equation and to find particular solutions in the case of partial differential equations and the solution in the case of ordinary differential equations. Singularity analysis is also very useful to determine whether the given equation is integrable in terms of an analytic function apart from isolated singulatities or not. This analysis finds common application. For the Korteweg-de Vries(K-dV) equation [13], developed as a model to explain mathematically the observation of John Scott Russell [22] half a century earlier, many authors have tried to find solutions both analytically and numerically by the use of many methods. This is also the case with the Boussinesq equation [3].

In this paper we analyze the Hirota[1] bilinear forms for the K-dV equation and the Boussinesq equation from the point of view of symmetry analysis to reduce the (1+1) evolution equations to ordinary differential equations. In addition to further reduction using symmetry we also examine the ordinary differential equations for integrability in terms of singularity analysis. A partial study of these equations has been made in terms of reduction using the similarity variable based on the common symmetry, $\Gamma = t\partial_t + kx\partial_x$ (k is a constant which differs for the two equations), of the two bilinear forms [23]. We complement that study by making reductions using the other Lie symmetries of the two equations. The symmetry reduction of the K-dV equation is undertaken in §2 and its singularity analysis is performed in §3. Similarly in §4 we report the symmetry reduction of the Boussinesq equation and its singularity analysis is in §5. We conclude with some observations in §6.

2. Reduction of the K- dV Equation

The bilinear form of the K-dV equation is [23]

$$FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2 - 4FF_{tx} + 4F_tF_x = 0, \tag{1}$$

where F is a function of x and t. The Lie point symmetries (1) are[2]

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_x \\ \Gamma_3 &= xF\partial_F \\ \Gamma_4 &= 3t\partial_t + x\partial_x \\ \Gamma_5 &= 6t\partial_x - x^2F\partial_F \\ \Gamma_6 &= A(t)F\partial_F, \end{aligned} \tag{2}$$

where $A(t)$ is a free function.

The symmetries Γ_1, Γ_2, Γ_3 and Γ_6 are not good for reduction by themselves. The first two eliminate one of the variables, t or x, from the dependency of the reduced dependent variable. The second two eliminate the dependent variable altogether! Although it is not completely apparent, the use of Γ_5 results in the reduced dependent variable being identically zero. In Tamizhmani *et al.*

[1] Hirota has made a substantial contribution to the field of nonlinear partial differential equations arising in the study of fluid flows. For a sampling see [7-12].

[2] We use the Mathematica Addon, Sym [4-6, 2], for the calculation of Lie symmetries throughout this paper. For the description of an algebra we make use of the Mubarakzyanov Classification Scheme [15-18] and representations of the algebras are based upon the papers of Patera *et al*. [19-21].

[23] reduction was performed using the similarity variable which follows from the invariants of Γ_4. We look for some combination of the symmetries to obtain a suitable candidate for reduction. The possibility we choose the linear combination of Γ_1, Γ_2 and Γ_3 and write

$$\Gamma = A\Gamma_1 + B\Gamma_2 + \Gamma_3 \tag{3}$$

and note that there is no need to put a third constant in front of Γ_3 because the symmetry is written up to a constant multiplier. It follows that

$$\Gamma = A\partial_t + B\partial_x + xF\partial_F \tag{4}$$

for which the associated Lagrange's system is

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dF}{xF} \tag{5}$$

so that the characteristics are

$$z = Ax - Bt \quad \text{and} \quad F = u(Ax - Bt)\exp\left[\frac{x^2}{2B}\right]. \tag{6}$$

Equation (1) becomes the fourth-order ordinary differential equation

$$uu'''' - 4u'u''' + 3u''^2 + \left(\frac{4B}{A^3} + \frac{12}{A^2B}\right)(uu'' - u'^2) + \frac{6u^2}{A^4B^2} = 0, \tag{7}$$

where u is a function of z and $'$ denotes the derivative with respect to z.

There are two possibilities for the Lie point symmetries of (7).

Case 1 : $6A + B^2 = 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u \\ \Gamma_4 &= z\partial_z + Kz^2u\partial_u. \end{aligned} \tag{8}$$

The Lie algebra of the four operators is $A_1 \oplus_s A_{3,1}$. The one-dimensional abelian subalgebra comprises Γ_4 and the three-dimensional subalgebra – commonly known as the Weyl-Heisenberg algebra - is composed of Γ_1, Γ_2 and Γ_3.

Case 2 : $6A + B^2 \neq 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u. \end{aligned} \tag{9}$$

The algebra is $A_{3,1}$.

Since $[\Gamma_2,\, \Gamma_i]_{LB} = 0,\; i = 1,\, 3,\, 4$, we take Γ_2 for reduction of the order of the equation. The characteristics of Γ_2 are

$$r = z \quad \text{and} \quad s = \frac{u'}{u}. \tag{10}$$

Under this nonlocal transformation we obtain the third-order nonlinear differential equation,

$$s''' + 6s'^2 + \left(\frac{4B}{A^3} + \frac{12}{A^2 B}\right) s' + \frac{6}{A^4 B^2} = 0 \tag{11}$$

from (7). Equation (11) has the Lie point symmetries

Case 1 : $6A + B^2 = 0$

$$\begin{aligned} \Gamma_1 &= \partial_r \\ \Gamma_2 &= \partial_s \\ \Gamma_3 &= -r\partial_r + \left(s - \frac{72}{B^5} r\right) \partial_s \end{aligned} \tag{12}$$

with the Lie algebra $A_{3,2}$.

Case 2 : $6A + B^2 \neq 0$

$$\begin{aligned} \Gamma_1 &= \partial_r \\ \Gamma_2 &= \partial_s. \end{aligned}$$

The Lie algebra is simply $2A_1$.

Since $[\Gamma_2,\, \Gamma_i]_{LB} = 0$ (or) $\Gamma_2,\; i = 1,\, 3$, we take Γ_2 for the next reduction of the order of (11). The characteristics are

$$p = r \quad \text{and} \quad Q = s' \tag{13}$$

so that (11) becomes the second-order differential equation,

$$Q'' + 6Q^2 + \left(\frac{4B}{A^3} + \frac{12}{A^2 B}\right) Q + \frac{6}{A^4 B^2} = 0, \tag{14}$$

where $'$ is now the derivative with respect to p. The Lie point symmetries of (14) are

$$\begin{aligned} \Gamma_1 &= \partial_p \\ \Gamma_2 &= p\partial_p + \left(\frac{72}{B^5} - 2Q\right) \partial_Q \end{aligned}$$

which has the algebra A_2 and is a representation of Lie's Type III algebra. The characteristics of Γ_1 give that

$$Q = T \quad \text{and} \quad Q' = \theta \tag{15}$$

and (14) become the first-order differential equation

$$\theta\theta' + 6T^2 + \left(\frac{4B}{A^3} + \frac{12}{A^2B}\right)T + \frac{6}{A^4B^2} = 0. \tag{16}$$

Equation (16) leads to an elliptic integral, but in Case 1 there is still the option of a further reduction of order using Γ_2 which in terms of the variables in (16) has the first extension

$$\Gamma_2^{[1]} = \left(\frac{72}{B^5} - 2T\right)\partial_T - 3\theta\partial_\theta - \theta'\partial_{\theta'} \tag{17}$$

for which the characteristics are

$$w = \frac{\theta^{\frac{2}{3}}}{\left(\frac{72}{B^5} - 2T\right)} \quad \text{and} \quad y = \frac{\theta'}{\theta^{\frac{1}{3}}}. \tag{18}$$

Equation (16) becomes the algebraic equation

$$y + \tfrac{3}{2}\tfrac{1}{w^2} = 0 \tag{19}$$

when we make the substitution $A = -B^2/6$. Consequently we have achieved Lie's aim to reduce the differential equation to an algebraic equation. In principle the solution of (7) is found by performing a series of quadratures which reverse the reductions of order culminating in (19). The quadrature of (16) gives

$$\theta^2 + 4\left(T - \frac{36}{B^5}\right)^3 = J_1, \tag{20}$$

where J_1 is the constant of integration, in the instance of Case 1. (The result for Case 2 is slightly messier). The subsequent quadrature gives

$$p - p_0 = \int \frac{\mathrm{d}Q}{\sqrt{J_1 - 4\left(Q - \frac{36}{B^5}\right)^3}}, \tag{21}$$

where p_0 is the new constant of integration. As (21) leads to an elliptic function, one would not expect the further quadratures to be possible in closed form.

3. Singularity Analysis

Tamizhmani *et al.* have already analyzed the Hirota bilinear equation of (1) with the substitution of $z = \dfrac{x}{t^{\frac{1}{3}}}$ for the independent variable as follows [23]

$$FF_{zzzz} - 4F_zF_{zzz} + 3F_{zz}^2 + \frac{4}{3}(zFF_{zz} + FF_z - zF_z{}^2) = 0. \tag{22}$$

Our analysis below differs from the analysis presented in their paper. We firstly change the nonautonomous equation to an autonomous equation by solving (22) for z and then differentiating with respect to z. We obtain a fifth-order autonomous equation which reduces to

$$3P'P'''' - 3P''P''' + 18P'^2P'' - 4PP'' + 8P''^2 = 0 \tag{23}$$

on the substitution $P(z) = \dfrac{F'}{F}$. The dominant terms of the above equation are

$$3P'P'''' - 3P''P''' + 18P'^2P'' \tag{24}$$

from which it is evident that the leading-order behaviour is $-w^{-1}$, where $w = (z - z_0)$. We substitute $P(z) = -w^{-1} + mw^{-1+r}$ into (24) and equate the coefficient of m to zero to obtain the resonances $r = -1,\ 1,\ 2$ and 6. We now substitute the expansion $P(z) = -\dfrac{1}{w} + a_0 + a_1w + a_2w^2 + a_3w^3 + a_4w^4 + a_5w^5 + a_6w^6$ into (22) and equate coefficients of separate powers of w to obtain that $a_0,\ a_1$ and a_5 are arbitrary. Equation (23) passes the Painlevé test and as a consequence the reduced form of the Hirota bilinear equation of K-dV, (22), is integrable.

4. Reduction of the Boussinesq Equation

The bilinear form of the Boussinesq equation is [23]

$$FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2 + 3(FF_{tt} + 4F_t^2) = 0, \tag{25}$$

where F is a function of x and t. The Lie point symmetries of (25) are

$$\begin{aligned} \Gamma_1 &= \partial_t & \Gamma_5 &= tF\partial_F \\ \Gamma_2 &= \partial_x & \Gamma_6 &= txF\partial_F \\ \Gamma_3 &= xF\partial_F & \Gamma_7 &= 2t\partial_t + x\partial_x \\ \Gamma_4 &= F\partial_F. \end{aligned}$$

Under the operation of taking the Lie brackets of the symmetries the Lie algebra is $A_{3,5}^{\frac{1}{2}} \oplus_s 4A_1$ in which the three elements of the former subalgebra are Γ_1, Γ_2 and $\frac{1}{2}\Gamma_7$ and the latter comprises Γ_3, Γ_4, Γ_5 and Γ_6.

All possible Lie brackets of the Lie point symmetries give us many possible routes for reduction of (25) to an ordinary differential equation. Γ_7 has already been used for such a reduction of (25) [23]. We consider

Case 1 : Since $[\Gamma_3, \Gamma_i]_{LB} = 0,\ \Gamma_4$ (or) $\Gamma_3,\ i = 1,\ 2,\ ..,\ 7$, we take take the combination of Γ_1, Γ_2 and Γ_3

$$\Gamma = A\Gamma_1 + B\Gamma_2 + \Gamma_3. \tag{26}$$

It follows that

$$\Gamma = A\partial_t + B\partial_x + xF\partial_F \tag{27}$$

for which the associated Lagrange's system is

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dF}{xF} \tag{28}$$

so that the characteristics are

$$z = Ax - Bt \quad \text{and} \quad F = u(Ax - Bt)\exp\left[\frac{x^2}{2B}\right]. \tag{29}$$

Equation (25) becomes the fourth-order ordinary differential equation

$$uu'''' - 4u'u''' + 3u''^2 + \left(\frac{3B^2}{A^3} + \frac{12}{A^2B}\right)(uu'' - u'^2) + \frac{6u^2}{A^4B^2} = 0, \tag{30}$$

where u is a function of z and $'$ denotes the derivative with respect to z.

Equation (30) has the Lie point symmetries

Case 1 : $8A^2 + B^3 = 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u \\ \Gamma_4 &= z\partial_z + Kz^2u\partial_u, \end{aligned} \tag{31}$$

where $K = -\dfrac{8}{B^4}$, which has the same algebra as in Case 1 of the similar stage of reduction of the K-dV equation.

Case 2 : $8A^2 + B^3 \neq 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u. \end{aligned} \tag{32}$$

Not surprisingly the algebra is $A_{3,1}$.

We note that all subsequent algebras are as for corresponding cases of the K-dV equation and further mention is not made.

Since $[\Gamma_2, \Gamma_i]_{LB} = 0, i = 1, 3, 4$, we take Γ_2 for reduction of the order of the equation. The characteristic of Γ_2 are

$$r = z \text{ and } s = \frac{u'}{u}. \tag{33}$$

On substitution of (33) into (30) we obtain the third-order nonlinear differential equation

$$s''' + 6s'^2 + \left(\frac{3B^2}{A^4} + \frac{12}{A^2B}\right)s' + \frac{6}{A^4B^2} = 0. \tag{34}$$

Equation (34) has the Lie symmetries

Case 1 : $8A^2 + B^3 = 0$

$$\begin{aligned}\Gamma_1 &= \partial_r\\ \Gamma_2 &= \partial_s\\ \Gamma_3 &= -r\partial_r + \left(s + \frac{16}{B^4}r\right)\partial_s.\end{aligned} \tag{35}$$

Case 2 : $8A^2 + B^3 \neq 0$

$$\begin{aligned}\Gamma_1 &= \partial_r\\ \Gamma_2 &= \partial_s.\end{aligned}$$

Since $[\Gamma_2,\, \Gamma_i]_{LB} = 0$ (or) $\Gamma_2,\, i = 1,\, 3$, we take Γ_2 for reduction of the order of the equation. The characteristics are

$$p = r \quad \text{and} \quad Q = s'. \tag{36}$$

Equation (34) becomes the second-order ordinary differential equation

$$Q'' + 6Q^2 + \left(\frac{3B^2}{A^4} + \frac{12}{A^2B}\right)Q + \frac{6}{A^4B^2} = 0, \tag{37}$$

where $'$ is the derivative with respect to p. The Lie symmetries of (37) are

$$\begin{aligned}\Gamma_1 &= \partial_p\\ \Gamma_2 &= p\partial_p - \left(\tfrac{16}{B^4} + 2Q\right)\partial_Q\end{aligned}$$

(obviously the second symmetry exists only for Case 1). The characteristics of Γ_1 are

$$Q = T \quad \text{and} \quad Q' = \theta. \tag{38}$$

Equation (37) becomes the first-order ordinary differential equation

$$\theta\theta' + 6T^2 + \left(\tfrac{3B^2}{A^4} + \tfrac{12}{A^2B}\right) T + \frac{6}{A^4B^2} = 0. \tag{39}$$

The integration of (39) leads to an elliptic integral.

The first extension of the second symmetry of Case 1 is

$$\Gamma_2^{[1]} = \left(\tfrac{3B^2}{A^4} - 2T\right)\partial_T - 3\theta\partial_\theta - \theta'\partial_{\theta'}. \tag{40}$$

The associated Lagrange's system is

$$\frac{dT}{\left(\tfrac{3B^2}{A^4} - 2T\right)} = \frac{d\theta}{-3\theta} = \frac{d\theta'}{-\theta'} \tag{41}$$

so that the characteristics are

$$w = \frac{\theta^{\frac{2}{3}}}{\left(\tfrac{3B^2}{A^4} - 2T\right)} \quad \text{and} \quad y = \frac{\theta'}{\theta^{\frac{1}{3}}}. \tag{42}$$

Equation (39) becomes

$$y + 6w^2 = 0 \tag{43}$$

when we make the substitution $A^2 = -B^3/8$. Consequently we have again achieved Lie's aim to reduce the differential equation to an algebraic equation. The solution of (25) is found by performing a series of quadratures which reverse the reductions of order culminating in (43). The reversal leads back to (39) and its quadrature in this case gives

$$\theta^2 + 4\left(T + \frac{8}{B^3}\right)^3 = J_1, \tag{44}$$

where J_1 is the constant of integration. (The result for Case 2 is slightly messier). The subsequent quadrature gives

$$p - p_0 = \int \frac{\mathrm{d}Q}{\sqrt{J_1 - 4\left(T + \frac{8}{B^3}\right)^3}}, \tag{45}$$

where p_0 is the new constant of integration. As the quadrature of (45) leads to an elliptic function, one would not expect the further quadratures to be possible in closed form.

Case 2 : Next we take another possible combination of the symmetries, namely $\Gamma = A\Gamma_1 + B\Gamma_2 + \Gamma_4$. It follows that

$$\Gamma = A\partial_t + B\partial_x + F\partial_F \tag{46}$$

which has the associated Lagrange's system

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dF}{F} \tag{47}$$

and so the characteristics are

$$z = Ax - Bt \quad \text{and} \quad F = u(Ax - Bt)e^{\frac{x}{B}}. \tag{48}$$

Equation (25) becomes the fourth-order ordinary differential equation

$$uu'''' - 4u'u''' + 3u''^2 + \frac{3B^2}{A^4}(uu'' - u'^2) = 0, \tag{49}$$

where u is a function of z and $'$ denotes the derivative with respect to z. The Lie point symmetries of (49) with the condition $B \neq 0$ are

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u. \end{aligned} \tag{50}$$

Since $[\Gamma_2, \Gamma_i]_{LB} = 0$, $i = 1, 3$, we take Γ_2 for reduction of the order of the equation. The characteristic of Γ_2 are

$$r = z \quad \text{and} \quad s = \frac{u'}{u}. \tag{51}$$

On substitution of (51) into (49) we obtain the third-order nonlinear differential equation,

$$s''' + 6s'^2 + \frac{3B^2}{A^4}s' = 0. \tag{52}$$

Subject to $B \neq 0$ (52) has the Lie symmetries

$$\begin{aligned} \Gamma_1 &= \partial_r \\ \Gamma_2 &= \partial_s. \end{aligned}$$

We take Γ_2 for reduction of the order of the equation. The characteristics are

$$p = r \quad \text{and} \quad Q = s' \tag{53}$$

and (52) is reduced to the second-order ordinary differential equation

$$Q'' + 6Q^2 + \frac{3B^2}{A^4}Q = 0, \tag{54}$$

where $'$ is the derivative with respect to p. The Lie symmetries of (54) are

$$\begin{aligned}\Gamma_1 &= \partial_p\\ \Gamma_2 &= p\partial_p - 2Q\partial_Q.\end{aligned}$$

The existence of Γ_2 is not anticipated and it is clearly an hidden symmetry of Type II [1, 14]. The characteristics of Γ_1 are

$$Q = T \quad \text{and} \quad Q' = \theta. \tag{55}$$

Equation (54) becomes the first-order ordinary differential equation

$$\theta\theta' + 6T^2 + \frac{3B^2}{A^4}T = 0. \tag{56}$$

The second quadrature of (56) leads to an elliptic integral. As in Case 1 reduction to an algebraic equation is possible due to the existence of the hidden symmetry for (54), but Γ_2 does not give the elliptic integral.

$$\begin{aligned}\Gamma_2 &= p\partial_p - 2Q\partial_Q + (-2Q' - Q')\partial_{Q'}\\ &= p\partial_p - 2Q\partial_Q - 3Q'\partial_{Q'}\\ &= p\partial_p - 2T\partial_T - 3\theta\partial_\theta\end{aligned} \tag{57}$$

The extension of Γ_2 is

$$\begin{aligned}\Gamma_2^{[1]} &= p\partial_p - 2T\partial_T - 3\theta\partial_\theta + (-3\theta' + 2\theta')\partial_{\theta'}\\ &= p\partial_p - 2T\partial_T - 3\theta\partial_\theta - \theta'\partial_{\theta'}.\end{aligned} \tag{58}$$

This implies that

$$\frac{dT}{2T} = \frac{d\theta}{3\theta} = \frac{d\theta'}{\theta'} \tag{59}$$

gives the characteristics

$$w = \frac{\theta^{\frac{2}{3}}}{T} \quad \text{and} \quad y = \frac{\theta'}{\theta^{\frac{1}{3}}}. \tag{60}$$

Equation (56) becomes

$$y\theta^{\frac{4}{3}} + \frac{6\theta^{\frac{2}{3}}}{w^2} + \frac{3B^2\theta^{\frac{1}{3}}}{A^4 w} = 0. \tag{61}$$

As we have seen above, after two quadratures one obtains an elliptic integral and further quadratures are stymied.

CASE 3 : Consider another possible combination

$$\Gamma = A\partial_t + B\partial_x + tF\partial_F \tag{62}$$

for which the associated Lagrange's system is

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dF}{tF} \tag{63}$$

so that the characteristics are

$$z = Ax - Bt \quad \text{and} \quad F = u(Ax - Bt)e^{\frac{t^2}{2A}}. \tag{64}$$

Under this reduction (25) becomes the fourth-order ordinary differential equation

$$uu'''' - 4u'u''' + 3u''^2 + \frac{3B^2}{A^4}(uu'' - u'^2) + \frac{3u^2}{A^5} = 0, \tag{65}$$

where u is a function of z and $'$ denotes the derivative with respect to z. The Lie point symmetries of (65) are

Case 1 : $8A^3 + B^4 \neq 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u \\ \Gamma_4 &= z\partial_z + Kz^2u\partial_u \end{aligned} \tag{66}$$

where $K = \pm\dfrac{4(-1)^{\frac{1}{3}}}{B^{\frac{10}{3}}}$.

Case 2 : $8A^3 + B^4 = 0$

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= zu\partial_u. \end{aligned} \tag{67}$$

Since $[\Gamma_2, \Gamma_i]_{LB} = 0$, $i = 1, 3, 4$ for Case 1 (the 4 is omitted for Case 2), we take Γ_2 for reduction of the order of the equation. The characteristic of Γ_2 are

$$r = z \quad \text{and} \quad s = \frac{u'}{u}. \tag{68}$$

On substitution of (68) into (65) we obtain the third-order nonlinear differential equation

$$s''' + 6s'^2 + \frac{3B^2}{A^4}s' + \frac{3}{A^5} = 0. \tag{69}$$

The Lie symmetries of (69) are

Case 1 : $8A^3 - B^4 = 0$

$$\begin{aligned}\Gamma_1 &= \partial_r \\ \Gamma_2 &= \partial_s \\ \Gamma_3 &= r\partial_r + (Kr - s)\partial_s, \end{aligned} \tag{70}$$

where $K = \pm \dfrac{8(-1)^{\frac{1}{3}}}{B^{\frac{10}{3}}}$.

Case 2 : $8A^3 - B^4 \neq 0$

$$\begin{aligned}\Gamma_1 &= \partial_r \\ \Gamma_2 &= \partial_s\end{aligned}$$

Since $[\Gamma_2, \Gamma_i]_{LB} = 0$ (or) $\Gamma_2, i = 1, 3$ for Case 1, we take Γ_2 for reduction of the order of the equation. The characteristics are

$$p = r \quad \text{and} \quad Q = s' \tag{71}$$

and (69) becomes the second-order ordinary differential equation

$$Q'' + 6Q^2 + \left(\frac{3B^2}{A^4}\right) Q + \frac{3}{A^5} = 0, \tag{72}$$

where $'$ is the derivative with respect to p. The symmetries of the above equation are

$$\begin{aligned}\Gamma_1 &= \partial_p \\ \Gamma_2 &= p\partial_p - (K + 2Q)\,\partial_Q.\end{aligned}$$

The characteristics of Γ_1 are

$$Q = T \quad \text{and} \quad Q' = \theta \tag{73}$$

and (72) becomes the first-order ordinary differential equation

$$\theta\theta' + 6T^2 + \left(\frac{3B^2}{A^4}\right) T + \frac{3}{A^5} = 0. \tag{74}$$

The second quadrature of (74) leads to an elliptic integral.

$$\begin{aligned}\Gamma_2 &= p\partial_p + (K - 2Q)\,\partial_Q + (-2Q' - Q')\partial_{Q'} \\ &= p\partial_p + (K - 2Q)\,\partial_Q - 3Q'\partial_{Q'} \\ &= p\partial_p + (K - 2T)\,\partial_T - 3\theta\partial_\theta \end{aligned} \tag{75}$$

The extension of Γ_2 is

$$\begin{aligned}\Gamma_2^{[1]} &= p\partial_p + (K - 2T)\,\partial_T - 3\theta\partial_\theta + (-3\theta' + 2\theta')\partial_{\theta'} \\ &= p\partial_p + (K - 2T)\,\partial_T - 3\theta\partial_\theta - \theta'\partial_{\theta'}. \end{aligned} \tag{76}$$

The associated Lagrange's system is

$$\frac{dT}{(K-2T)} = \frac{d\theta}{-3\theta} = \frac{d\theta'}{-\theta'} \tag{77}$$

and the characteristics are

$$w = \frac{\theta^{\frac{2}{3}}}{(K-2T)} \quad \text{and} \quad y = \frac{\theta'}{\theta^{\frac{1}{3}}}. \tag{78}$$

As before we obtain

$$\theta^2 + 4T^3 + \left(\frac{48T^2}{B^{\frac{10}{3}}}\right)T^2 + \frac{192T}{B^{\frac{20}{3}}} + J_1 = 0 \tag{79}$$

when we make the substitution $A^3 = -B^4/8$, where J_1 is the constant of integration, in the instance of Case 1. (The result for Case 2 is slightly messier). The subsequent quadrature gives

$$p - p_0 = \int \frac{\mathrm{d}Q}{\sqrt{J_1 - 4T^3 - \left(\frac{48}{B^{\frac{10}{3}}}\right)T^2 - \frac{192T}{B^{\frac{20}{3}}}}}, \tag{80}$$

where p_0 is the new constant of integration. As (80) leads to an elliptic function, one would not expect the further quadratures to be possible in closed form.

Case 4 : Consider the combination

$$\Gamma = \partial_t + C\partial_x \tag{81}$$

for which the associated Lagrange's system is

$$\frac{dt}{1} = \frac{dx}{C} = \frac{dF}{0}. \tag{82}$$

The characteristics are

$$z = x - Ct \quad \text{and} \quad F = u(x - Ct) \tag{83}$$

so that (25) becomes the fourth-order ordinary differential equation

$$uu'''' - 4u'u''' + 3u''^2 + 3C^2(uu'' + u'^2) = 0, \tag{84}$$

where u is a function of z and $'$ denotes the derivative with respect to z. Equation (84) has the Lie point symmetries ($C \neq 0$)

$$\begin{aligned} \Gamma_1 &= \partial_z \\ \Gamma_2 &= u\partial_u. \end{aligned}$$

Since $[\Gamma_2, \Gamma_1]_{LB} = 0$, we take Γ_2 for reduction of the order of the equation. The characteristic of Γ_2 are

$$r = z \quad \text{and} \quad s = \frac{u'}{u}. \tag{85}$$

On substitution of (85) into (84) we obtain the third-order nonlinear differential equation

$$s''' + 6s'^2 + 3C^2\left(s' + s^2\right) = 0 \tag{86}$$

which has the sole Lie point symmetry

$$\Gamma_1 \quad = \quad \partial_r.$$

The characteristics are

$$p = s \quad \text{and} \quad Q = s' \tag{87}$$

and (86) is reduced to the second-order ordinary differential equation

$$Q''Q^2 + QQ'^2 + 6Q^2 + 3C^2\left(Q + 2p^2\right) = 0, \tag{88}$$

where $'$ is the derivative with respect to p. There are no Lie point symmetries for the above equation and, as it is an equation of the second order, Lie's algorithm cannot be applied in closed form. Therefore we cannot make a further reduction.

5. Singularity Analysis

Tamizhmani *et al.* [23] have already analyzed the Hirota bilinear equation of (25) with the substitution of $z = \dfrac{x}{t^{\frac{1}{2}}}$ for the independent variable to obtain the fourth-order ordinary differential equation

$$FF_{zzzz} - 4F_zF_{zzz} + 3F_{zz}^2 + \frac{3}{4}(z^2FF_{zz} + 2zFF_z - z^2{F_z}^2) = 0. \tag{89}$$

Our analysis bellow differs from the analysis presented in their paper. On substitution of $P(z) = \dfrac{F'}{F}$ into (89) we arrive at

$$P''' + 6P'^2 + \frac{3}{4}(z^2P' + 2zP) = 0. \tag{90}$$

The dominant terms of the above equation are

$$P''' + 6P'^2 \approx 0. \tag{91}$$

The leading-order behaviour, subsequent substitution of $P(z) = -w^{-1} + mw^{-1+r}$, where $w = (z - z_0)$, into (91) and collection of the lowest power of w leads us to the resonances $r = -1,\ 1$ and 6.

We substitute the expansion, $P(z) = \dfrac{\alpha}{w} + a_0 + a_1 w + a_2 w^2 + a_3 w^3 + a_4 w^4 + a_5 w^5 + a_6 w^6$, into (90) and collect the coefficients of separate powers of w. We obtain that a_0 and a_5 are arbitrary. So (90) passes the Painlevé test. Hence the reduced form of the Hirota bilinear equation of the Boussinesq equation, (89), is integrable.

6. Conclusion

We have investigated the symmetry and singularity properties of the Hirota bilinear form of the Korteweg-de Vries equation and the Boussinesq equation for the propagation of long waves in shallow water. Initially we determined the Lie point symmetries of the equation and then examined the reduction of the partial differential equation to an ordinary differential equation and subsequent reduction of order to the extent possible. Under certain circumstances it was possible to reduce the original partial differential equation to an algebraic equation. An interesting feature was the similarity in the algebraic structures found for the ordinary differential equations occurring in the process of reduction of order. By taking a combination of symmetries which included a term containing the dependent variable we obtained similarity reductions which are different from the results of [23]. Further more, choices of the linear combinations of the vector fields allow us to construct a nontrivial similarity functions rather than the obvious choice of a simple traveling wave. Equations (22) and (89) are more complicated to perform singularity analysis. To make it simpler, first, we have transformed these equations to a simpler forms by introducing the change of variable $P(z) = \dfrac{F'}{F}$. We have shown now that it becomes easy to do singularity analysis.

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