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NORMALITY AND SHARING FUNCTIONS[1]

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In this paper, we proved some normality criteria for a family of meromorphic functions, where a holomorphic function is shared by every function from the family and a linear differential polynomial generated by the members of the family.

Key words : Meromorphic functions; holomorphic functions; normal families; sharing functions.

1. Introduction and Main Results

We denote the complex plane by $\mathbb{C}$, and the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by Δ. Let us recall the definition of normality. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on $\mathcal{D}$. The family $\mathcal{F}$ is said to be normal in $\mathcal{D}$, if every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically uniformly on compact subsets of $\mathcal{D}$, to a meromorphic function or ∞.* {[12], P.33, 71; [1], P.220, 225}.

Let f and g be meromorphic functions in a domain D and $a \in \mathbb{C}$. Let zeros of $f - a$ are zeros of $g - a$ (ignoring multiplicity), we write $f = a \Rightarrow g = a$. Hence $f = a \iff g = a$ means that $f - a$ and $g - a$ have the same zeros (ignoring multiplicity). If $f - a = 0 \iff g - a = 0$, then we say that f and g share the value $z = a$ IM. { [17], p. 108}

In this paper, we use the following standard notations of value distribution theory,

$$T(r, f), m(r, f), N(r, f), \ldots.$$

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We denote $S(r, f)$ for any function satisfying

$$S(r, f) = o\{T(r, f)\}, \text{ as } r \to +\infty,$$

possibly outside of a set with finite measure.

In [10] Mues and Steinmetz proved, *if a non-constant meromorphic function f in the plane, shares three distinct complex numbers a_1, a_2, a_3 with its first order derivative f' then $f \equiv f'$.*

Schwick [13] was the first who gave a connection between normality and shared values and proved a theorem related to above result of [10] which says that *the family $\mathcal{F}$ of meromorphic functions on a domain $\mathcal{D}$ is normal if f and f' share a_1, a_2, a_3 for every $f \in \mathcal{F}$, where a_1, a_2, a_3 are distinct complex numbers.* Since then many results of normality criteria concerning sharing values have been obtained. In [7], Fang and Xu proved the following theorem which states that: *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D and let a, b be two distinct finite complex numbers such that $b \neq 0$. If for any $f \in \mathcal{F}$, f and f' share $z = a$ IM and $f(z) = b$ whenever $f'(z) = b$ then $\mathcal{F}$ is normal in D.*

In 2001, Chen and Fang [3] generalized the result of Schiwck and proved the following theorem which states that: *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$, let k (≥ 2) be an integer and a, b, c are complex numbers such that $a \neq b$. If for all $f \in \mathcal{F}$, f and $f^{(k)}$ share a, b and the zeros of $f - c$ are of multiplicity at least $k + 1$, then $\mathcal{F}$ is normal in $\mathcal{D}$.*

The above result was further generalized by Han and Gu [8] as: *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$, a, b, c are complex numbers such that $a \neq b$. If for each $f \in \mathcal{F}$ the zeros of $f - c$ are of multiplicity at least $k + 1$, and $f(z) = a$ whenever $f^{(k)}(z) = a$, $f(z) = b$ whenever $f^{(k)}(z) = b$, then $\mathcal{F}$ is normal in $\mathcal{D}$.*

One may ask whether one can replace the values a, b and c by holomorphic functions $a(z)$, $b(z)$, and $c(z)$ and $f^{(k)}$ by a differential polynomial in f. In this article, we investigate this situation by replacing the values a, b and c by holomorphic functions $a(z), b(z)$ and $c(z)$ respectively and $f^{(k)}$ by a linear differential polynomial $a_k(z)f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + \ldots + a_1(z)f'(z) + a_0(z)f(z)$, where $a_0(z), \ldots, a_k(z)$ are holomorphic functions, with $a_k(z) \neq 0 \; \forall z \in \mathcal{D}$. We define $D_k(f(z)) := a_k(z)f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + \ldots + a_1(z)f'(z) + a_0(z)f(z)$.

Here are our main results.

***Theorem* 1.1** — *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\mathcal{D}$ such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k, where k is a positive integer. Let $a(z)$, $b(z)$ be holomorphic functions in $\mathcal{D}$. If for each $f \in \mathcal{F}$,*

1. $b(z) \neq 0,$ *for all* $z \in \mathcal{D}$,

2. $a(z) \neq b(z),$ *and* $b(z) - D_k(a(z)) \neq 0,$

3. $f(z) = a(z)$ *if and only if* $D_k(f(z)) = a(z),$

4. $f(z) = b(z)$ *whenever* $D_k(f(z)) = b(z),$

then $\mathcal{F}$ *is normal in* D.

The following example shows that the hypothesis $a(z) \neq b(z)$ and $b(z) - D_k(a(z)) \neq 0$ can not be dropped in Theorem 1.1.

Example 1.2 : Let $D = \Delta$, let $k = 1$, $a(z) = b(z) = z^{k-1}$, $a_k(z) = 1$, $a_i(z) = 0$, for $i = 0, 1, \dots k-1$ and

$$\mathcal{F} = \left\{ e^{nz} - \frac{z^{k-1}}{n^k} + z^{k-1} : n = 1, 2, 3, \dots \right\}.$$

Then for any $f \in \mathcal{F}$ and

$$f = e^{nz} - \frac{z^{k-1}}{n^k} + z^{k-1}, \quad f^{(k)} = n^k e^{nz}.$$

Clearly, all conditions of Theorem 1.1 except (2) are satisfied. However, $\mathcal{F}$ is not normal in Δ. Also, this example confirms that $b(z) \neq 0$ is necessary in Theorem 1.1 as $f^{(k)}(z) \neq 0$.

Example 1.3 : Let $D = \Delta$, let $k = 1$, $b(z) = b$, (a non zero constant function) and $a(z) = ((-1)^{k+1} + 1)b$ and

$$\mathcal{F} = \left\{ b\frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!n(z - \frac{1}{n})} + a : n = 1, 2, 3, \dots \right\}.$$

Then, for every $f_n(z) \in \mathcal{F}$,

$$f_n(z) = b\frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!n(z - \frac{1}{n})} + a, \qquad f_n^{(k)}(z) = b - \frac{1}{n(z - \frac{1}{n})^{k+1}}.$$

Clearly, f_n and $f_n^{(k)}$ share a and $f_n^{(k)}(z) \neq b$, so that $f_n(z) = b$ whenever $f_n^{(k)}(z) = b$. But $\mathcal{F}$ is not normal in D. This example shows that Theorem 1.1 is not valid for a family of meromorphic functions.

***Theorem* 1.4** — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* $\mathcal{D}$, *let* k (> 0) *be an integer and let* $a(z),\ b(z),\ c(z)$ *be holomorphic functions such that* $a(z) \neq b(z)$ $\forall z \in \mathcal{D}$. *If for each* $f \in \mathcal{F}$

(a) *the zeros of* $f(z)-c(z)$ *are of multiplicity* $\geq k+1$,

(b) $f(z)=a(z)$ *whenever* $D_k(f(z))=a(z)$,

(c) $f(z)=b(z)$ *whenever* $D_k(f(z))=b(z)$,

then $\mathcal{F}$ *is normal in* $\mathcal{D}$.

By the following example we observe that the multiplicity restriction on zeros of $f(z)-c(z)$ is sharp in Theorem 1.4

Example 1.5 : Let $\mathcal{D}=\Delta=\{z : |z|<1\}$, let k be a positive integer, let $a_k=1$ and $a_i=0$ for $i=0,1,\dots,k-1$. Let $a(z)=z, b(z)=z+1, c(z)=0$ (a constant function) and let

$$\mathcal{F}=\left\{nz^k : n=1,2,3,\dots\right\}.$$

Then, for every $f_n(z)\in\mathcal{F}$, $f_n^{(k)}(z)=nk!$ in $\mathcal{D}$. So $f_n^{(k)}(z)\neq z$ in $\mathcal{D}$ and $f_n^{(k)}(z)\neq z+1$ in $\mathcal{D}$. Clearly all conditions of theorem, except (a), are satisfied but $\mathcal{F}$ is not normal in $\mathcal{D}$.

The following example shows that $a(z_0)$ and $b(z_0)$ can not be equal for some z_0 in $\mathcal{D}$.

Example 1.6 : Let $k(>0)$ be an integer, let $\mathcal{D}=\Delta=\{z : |z|<1\}$, let $a_k=1$ and $a_i=0$ for $i=0,1,\dots,k-1$. Let $a(z)=z, b(z)=2z, c(z)=0$ (a constant function). $\{$So at $z_0=0$ $a(z_0)=b(z_0)\}$ and let

$$\mathcal{F}=\left\{nz^{k+1} : n=1,2,3,\dots\right\}.$$

Then, f_n has zero of multiplicity $k+1$ and $f_n^{(k)}(z)=n(k+1)!z$. Clearly all other conditions of theorem are satisfied but $\mathcal{F}$ is not normal in $\mathcal{D}$.

The following example shows that the condition that $f(z)=a(z)$ whenever $D_k(f(z))=a(z)$ can not be dropped.

Example 1.7 : $\{$[4], [5]$\}$ Let $k(>0)$ be an integer, let $\mathcal{D}=\Delta=\{z : |z|<1\}$, $a_k=1$ and $a_i=0$ for $i=0,1,\dots,k-1$, let $a(z)=0, b(z)=1, c(z)=0$ and let $\mathcal{F}=\{f_n : n=1,2,3,\dots\}$, where

$$f_n(z)=\frac{\left(z+\frac{1}{(k+1)n}\right)^{k+1}}{k!\left(z+\frac{1}{n}\right)}=\frac{z^k}{k!}+p_{k-2}(z)+\frac{\left(-\frac{k}{n}\right)^{k+1}}{(k+1)^{k+1}k!\left(z+\frac{1}{n}\right)},$$

where $p_{k-2}(z)$ is a polynomial of degree $k-2$. Then, for every $f_n\in\mathcal{F}$, f_n has zero of multiplicity $k+1$, and

$$f_n^{(k)}(z)=1-\frac{\left(\frac{k}{n}\right)^{k+1}}{(k+1)^{k+1}\left(z+\frac{1}{n}\right)^{k+1}},$$

Thus $f_n^{(k)}(z) \neq 1$, this shows that $f(z) = b(z)$ whenever $D_k(f(z)) = b(z)$. Also $f_n^{(k)}$ has exactly $k+1$ distinct zeros and f_n has only one zero. So condition $f(z) = a(z)$ whenever $D_k(f(z)) = a(z)$ is not satisfied. Since f_n takes on the values 0 and ∞ in any fixed neighborhood of 0, if n is sufficiently large. So $\mathcal{F}$ fails to be normal in $\mathcal{D}$.

2. Some Lemmas

In order to prove our results we need the following Lemmas:

Lemma 2.1 {[20], p. 216; [19], p. 814} (Zalcman's Lemma) — Let $\mathcal{F}$ be a family of meromorphic functions in the unit disk Δ, with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least k. If $\mathcal{F}$ is not normal at z_0 in Δ, then for $0 \leq \alpha < k$, there exist

1. a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,
2. a sequence of functions $f_n \in \mathcal{F}$,
3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges to a non-constant meromorphic function g on $\mathbb{C}$. Moreover g is of order at most two.

Lemma 2.2 {[14], p. 21; [2], p. 360} — Let $f(z)$ be a transcendental meromorphic function of finite order. Let k be a positive integer. If the zeros of $f(z)$ are of multiplicity at least $k+1$, then $f^{(k)}(z) - b$ has infinitely many zeros for any non-zero complex number b.

Lemma 2.3 {[14], p. 22} — Let $k(>0)$ be an integer, let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + \dfrac{q(z)}{p(z)}$, where $a_i, (i = 0, 1, 2, \ldots, n)$ are constants with $a_n \neq 0, q(z)$ and $p(z)$ are co-prime polynomials with deg $q(z) <$ deg $p(z)$. If $f^{(k)}(z) \neq 1$, then

$$f(z) = \frac{z^k}{k!} + \ldots + a_0 + \frac{1}{(az+b)^n},$$

where $a \neq 0$. If all zeros of $f(z)$ have multiplicity at least $k+1$, then

$$f(z) = \frac{(cz+d)^{k+1}}{az+b},$$

where $c(\neq 0), d$ are constants.

Lemma 2.4 {[17], p. 43; [18], p. 110} (Hayman's Inequality) — Let $k(> 0)$ be an integer, suppose that $f(z)$ is a transcendental meromorphic function in the complex plane. Then

$$T(r,f) < \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) N\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r,f).$$

3. PROOF OF THEOREM 1.1

PROOF : Suppose that $\mathcal{F}$ is not normal at $z_0 \in \Delta$, then by Lemma 2.1, there exist

1. a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,
2. a sequence of functions $f_n \in \mathcal{F}$ and
3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta) - a(z_n + \rho_n\zeta)]$ converges locally uniformly to a non-constant entire function g. Moreover g is of order at most one.

Now we claim,

(i) $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = \varphi(z_0)$, where $\varphi(z) = \dfrac{a(z) - D_k(a(z))}{a_k(z)}$.

(ii) $g^{(k)}(\zeta) \neq B$, where $B = \dfrac{b(z_0) - D_k(a(z_0))}{a_k(z_0)}$. Note that B is a constant.

Since

$$g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta) - a(z_n + \rho_n\zeta)] \to g(\zeta). \tag{3.1}$$

We have

$$g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) - a^{(k)}(z_n + \rho_n\zeta) \to g^{(k)}(\zeta). \tag{3.2}$$

Now suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem, there exists $\zeta_n;\ \zeta_n \to \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a(z_n + \rho_n\zeta_n)] = 0.$$

Thus

$$f_n(z_n + \rho_n\zeta_n) = a(z_n + \rho_n\zeta_n).$$

So, by condition (3) of theorem, we have $D_k(f_n(z_n + \rho_n\zeta_n)) = a(z_n + \rho_n\zeta_n)$. Also

$$\begin{aligned}
\frac{D_k(f_n(z_n + \rho_n\zeta_n))}{a_k(z_n + \rho_n\zeta_n)} &= f_n^{(k)}(z_n + \rho_n\zeta_n) + \frac{a_{k-1}(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} f_n^{(k-1)}(z_n + \rho_n\zeta_n) + \\
&\quad \ldots + \frac{a_0(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} f(z_n + \rho_n\zeta_n) \\
&= f_n^{(k)}(z_n + \rho_n\zeta_n) + \frac{a_{k-1}(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} \left[\rho_n g_n^{(k-1)}(\zeta_n)\right. \\
&\quad \left. + a^{(k-1)}(z_n + \rho_n\zeta_n)\right] + \ldots \\
&\quad + \frac{a_0(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} \left[\rho_n^k g_n(\zeta_n) + a(z_n + \rho_n\zeta_n)\right] \\
&\to g^{(k)}(\zeta_0) + a^{(k)}(z_0) + \frac{a_{k-1}(z_0)}{a_k(z_0)} a^{(k-1)}(z_0) + \ldots + \frac{a_o(z_0)}{a_k(z_0)} a(z_0).
\end{aligned} \tag{3.3}$$

Therefore it follows that,

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n\to\infty}\left[\frac{D_k(f_n(z_n+\rho_n\zeta_n))}{a_k(z_n+\rho_n\zeta_n)}\right] - a^{(k)}(z_0) - \frac{a_{k-1}(z_0)}{a_k(z_0)}a^{(k-1)}(z_0) - \ldots - \frac{a_o(z_0)}{a_k(z_0)}a(z_0) \\ &= \lim_{n\to\infty}\frac{a(z_n+\rho_n\zeta_n)}{a_k(z_n+\rho_n\zeta_n)} - a^{(k)}(z_0) - \frac{a_{k-1}(z_0)}{a_k(z_0)}a^{(k-1)}(z_0) - \ldots - \frac{a_o(z_0)}{a_k(z_0)}a(z_0) \\ &= \frac{a(z_0) - D_k a(z_0)}{a_k(z_0)} = \varphi(z_0). \end{aligned}$$

Hence we have proved $g^{(k)}(\zeta) = \varphi(z_0)$ whenever $g(\zeta) = 0$. On the other hand, if $g^{(k)}(\zeta_0) = \varphi(z_0)$ then there exists $\zeta_n;\ \zeta_n \to \zeta_0$, such that

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n+\rho_n\zeta_n) - a^{(k)}(z_n+\rho_n\zeta_n) = \varphi(z_0).$$

We have to show that

$$g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = \lim_{n\to\infty}\left[f_n(z_n+\rho_n\zeta_n) - a(z_n+\rho_n\zeta_n)\right] = f(z_0) - a(z_0) = 0.$$

Now, assume that $g^{(k)}(\zeta_0) = \varphi(z_0)$ and consider

$$\begin{aligned} &\frac{D_k(f_n(z_n+\rho_n\zeta_n)) - a(z_n+\rho_n\zeta_n)}{a_k(z_n+\rho_n\zeta_n)} \\ &\quad \to g^{(k)}(\zeta_0) + a^{(k)}(z_0) + \frac{a_{k-1}(z_0)}{a_k(z_0)}a^{(k-1)}(z_0) + \ldots + \frac{a_o(z_0)}{a_k(z_0)}a(z_0) - \frac{a(z_0)}{a_k(z_0)} \\ &\quad = g^{(k)}(\zeta_0) - \frac{a(z_0) - D_k(a(z_0))}{a_k(z_0)} = g^{(k)}(\zeta_0) - \varphi(z_0). \end{aligned}$$

Now, by using assumption (3) of the theorem we get $f_n(z_n+\rho_n\zeta_n) = a(z_n+\rho_n\zeta_n)$, so is $g(\zeta_0) = 0$. This shows that $g(\zeta) = 0$ if and only if $g^{(k)}(\zeta) = \varphi(z_0)$.

From (3.3) we deduce that

$$\begin{aligned} &\frac{D_k(f_n(z_n+\rho_n\zeta)) - b(z_n+\rho_n\zeta)}{a_k(z_n+\rho_n\zeta)} \\ &\quad \to g^{(k)}(\zeta) + a^{(k)}(z_0) + \frac{a_{k-1}(z_0)}{a_k(z_0)}a^{(k-1)}(z_0) + \ldots + \frac{a_o(z_0)}{a_k(z_0)}a(z_0) - \frac{b(z_0)}{a_k(z_0)} \\ &\quad = g^{(k)}(\zeta) - \frac{b(z_0) - D_k(a(z_0))}{a_k(z_0)} = g^{(k)}(\zeta) - B. \end{aligned} \tag{3.4}$$

Next we prove that $g^{(k)}(\zeta) \neq B$. Suppose that there exists ζ_0 satisfying $g^{(k)}(\zeta_0) = B$. Then, by Hurwitz's theorem, there exists a sequence $\zeta_n;\ \zeta_n \to \zeta_0$ and by (3.4)

$$\{D_k(f(z_n+\rho_n\zeta_n))\} - b(z_n+\rho_n\zeta_n) = 0.$$

From the assumption, we have $f_n(z_n+\rho_n\zeta_n)=b(z_n+\rho_n\zeta_n)$. Then we get

$$\begin{aligned} g(\zeta_0) &= \lim_{n\to\infty}\rho_n^k[f_n(z_n+\rho_n\zeta_n)-a(z_n+\rho_n\zeta_n)] \\ &= \lim_{n\to\infty}\rho_n^k[b(z_n+\rho_n\zeta_n)-a(z_n+\rho_n\zeta_n)]=\infty, \end{aligned}$$

which is a contradiction. So $g^{(k)}(\zeta)\neq B$. Hence we get

$$g^{(k)}(\zeta)=B+e^{A\zeta+D}, \tag{3.5}$$

where A and D are two constants. We claim that $A=0$. Suppose that $A\neq 0$, then

$$g(\zeta)=\frac{B\zeta^k}{k!}+\frac{e^{A\zeta+D}}{A^k}+\frac{c_1\zeta^{k-1}}{(k-1)!}+\ldots+c_{k-1}\zeta+c_k, \tag{3.6}$$

where $c_1,c_2,\ldots,c_k$ are constants. Let $g^{(k)}(\zeta)=\varphi(z_0)$ then by (3.5), (3.6) and $g(\zeta)=0$ if and only if $g^{(k)}(\zeta)=\varphi(z_0)$, we get

$$\frac{B\zeta^k}{k!}+\frac{c_1\zeta^{k-1}}{(k-1)!}+\ldots+c_k+\frac{B-\varphi(z_0)}{A^k}=0.$$

This is a polynomial of degree k in ζ this polynomial has k solutions, which contradicts the fact that $g^{(k)}$ has infinitely many solutions. Thus we have,

$$g^{(k)}(\zeta)=B+e^D$$

and

$$g(\zeta)=(B+e^D)\frac{\zeta^k}{k!}+\frac{c_1\zeta^{k-1}}{(k-1)!}+\ldots+c_k.$$

Since g is non-constant, this contradicts $g(\zeta)=0$ if and only if $g^{(k)}(\zeta)=\varphi(z_0)$. Thus $\mathcal{F}$ is normal in D. This completes the proof of theorem. □

4. PROOF OF THEOREM 1.4

PROOF : Since normality is a local property, we assume that $\mathcal{D}=\Delta=\{z:|z|<1\}$. Suppose $\mathcal{F}$ is not normal in $\mathcal{D}$. Without loss of generality we assume that $\mathcal{F}$ is not normal at the point $z_0=0$ in Δ. Then by Lemma 2.1, there exist

1. a sequence of complex numbers $z_n\to z_0$, $|z_n|<r<1$,
2. a sequence of functions $f_n\in\mathcal{F}$ and

3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \dfrac{f_n(z_n + \rho_n\zeta) - c(z_n + \rho_n\zeta)}{\rho_n^k}$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$ in $\mathbb{C}$ and the zeros of $g(\zeta)$ are of multiplicity at least $k+1$. Moreover g is of order at most two.

Case 1 : When $c(z_0) = a(z_0)$, we claim that

(i) $g^{(k)}(\zeta) \neq B$, where $B = \dfrac{b(z_0) - D_k(c(z_0))}{a_k(z_0)}$ is a constant.

(ii) $g^{(k)}(\zeta) = A \Rightarrow g(\zeta) = 0$, where $A = \dfrac{a(z_0) - D_k(c(z_0))}{a_k(z_0)}$ is a constant.

Clearly, $g^{(k)}(\zeta) \not\equiv B$ as zeros of $g(\zeta)$ are of multiplicity at least $k+1$. Suppose $g^{(k)}(\zeta_0) = B$, then by Hurwitz's theorem there exist $\zeta_n;\ \zeta_n \to \zeta_0$ such that

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) - c^{(k)}(z_n + \rho_n\zeta_n) = B.$$

Now, consider

$$\begin{aligned}
&\frac{D_k(f(z_n + \rho_n\zeta_n) - b(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} \\
&= f_n^{(k)}(z_n + \rho_n\zeta_n) + \frac{D_{k-1}(f_n(z_n + \rho_n\zeta_n)) - b(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} \\
&= f_n^{(k)}(z_n + \rho_n\zeta_n) + \frac{a_{k-1}(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)}\left[\rho_n g_n^{(k-1)}(\zeta_n) + c^{(k-1)}(z_n + \rho_n\zeta_n)\right] \\
&\quad + \ldots + \frac{a_0(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)}\left[\rho_n^k g_n(\zeta_n) + c(z_n + \rho_n\zeta_n)\right] - \frac{b(z_n + \rho_n\zeta_n)}{a_k(z_n + \rho_n\zeta_n)} \\
&\longrightarrow g^{(k)}(\zeta_0) + \frac{a_{k-1}(z_0)}{a_k(z_0)}c^{(k-1)}(z_0) + \ldots + \frac{a_1(z_0)}{a_k(z_0)}c'(z_0) + \frac{a_0(z_0)}{a_k(z_0)}c(z_0) \\
&\quad - \frac{b(z_0)}{a_k(z_0)} + c^{(k)}(z_0) = g^{(k)}(\zeta_0) - \frac{b(z_0) - D_k(f(z_0))}{a_k(z_0)}.
\end{aligned} \tag{4.1}$$

Now, it follows from (4.1) and condition (c) of the Theorem 1.4 that

$$f(z_n + \rho_n\zeta_n) = b(z_n + \rho_n\zeta_n).$$

Thus,

$$\begin{aligned}
g_n(\zeta_n) &= \frac{f_n(z_n + \rho_n\zeta_n) - c(z_n + \rho_n\zeta_n)}{\rho_n^k} \\
&= \frac{b(z_n + \rho_n\zeta_n) - c(z_n + \rho_n\zeta_n)}{\rho_n^k} \longrightarrow \infty.
\end{aligned}$$

This means $g(\zeta_0)=\infty$, which contradicts $g^{(k)}(\zeta_0)=B$. Thus $g^{(k)}(\zeta)\neq B$.

Now, we prove claim (ii). Suppose $g^{(k)}(\zeta_0)=A$, by Hurwitz's theorem, there exist $\zeta_n;\ \zeta_n\to\zeta_0$ such that

$$g_n^{(k)}(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)-c^{(k)}(z_n+\rho_n\zeta_n)=A.$$

Now, consider

$$\frac{D(f^{(k)}(z_n+\rho_n\zeta_n))-a(z_n+\rho_n\zeta_n)}{a_k(z_n+\rho_n\zeta_n)}\longrightarrow g^{(k)}(\zeta_0)-\frac{a(z_0)-D_k(c(z_0))}{a_k(z_0)}. \tag{4.2}$$

So, it follows from (4.2) and condition (b) of the Theorem 1.4 that

$$f(z_n+\rho_n\zeta_n)=a(z_n+\rho_n\zeta_n).$$

Thus,

$$\begin{aligned} g_n(\zeta_n) &= \frac{f_n(z_n+\rho_n\zeta_n)-c(z_n+\rho_n\zeta_n)}{\rho_n^k} \\ &= \frac{a(z_n+\rho_n\zeta_n)-c(z_n+\rho_n\zeta_n)}{\rho_n^k}=0. \end{aligned}$$

So $g(\zeta_0)=0$.

Subcase 1.1 : If $B\neq 0$, it is evident from Lemma 2.2, that $g(\zeta)$ is not a transcendental meromorphic function as $g^{(k)}(\zeta)\neq B$. Hence, $g(\zeta)$ is a non-constant rational function. Then, by Lemma 2.3, we have

$$g(\zeta)=\frac{B\zeta^k}{k!}+\ldots+b_0+\frac{1}{(\alpha\zeta+\beta)^n}. \tag{4.3}$$

Thus

$$g^{(k)}(\zeta)=B+\frac{D}{(\alpha\zeta+\beta)^{n+k}}, \tag{4.4}$$

where D is a constant, by (4.3) and (4.4) we conclude that the number of the zeros of $g(\zeta)$ is $k+n$, and by Hurwitz's theorem, multiplicity of zeros are at least $k+1$, so the number of distinct zeros of $g(\zeta)$ is at most $\dfrac{k+n}{k+1}$. It is simple to check that the zeros of $g^{(k)}(\zeta)-A$ are of multiplicity 1, so the number of distinct zeros of $g^{(k)}$ is $k+n$, which does not justify the claim (ii).

Subcase 1.2 : If $B=0$, then by, claim (i) $g^{(k)}(\zeta)\neq 0$. This shows that either $g(\zeta)\neq 0$ or the zeros of $g(\zeta)$ are of multiplicity at most k, which contradicts that the zeros of $g(\zeta)$ are of multiplicity at least $k+1$. So we have $g(\zeta)\neq 0$. By claim (ii) we get $g^{(k)}(\zeta)\neq A$, where $A\neq 0$. Since

$g(\zeta) \neq 0,\ g^{(k)}(\zeta) \neq A,\ A \neq 0$, this implies $N\left(r, \frac{1}{g}\right) = 0$ and $N\left(r, \frac{1}{g^{(k)}-A}\right) = 0$, so by Lemma 2.4 (Hayman's inequality) we get that $g(\zeta)$ is a constant, which is a contradiction.

Case 2 : When $c(z_0) \neq a(z_0)$ and $c(z_0) \neq b(z_0)$. By Lemma 2.1, there exist

1. a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,
2. a sequence of functions $f_n \in \mathcal{F}$ and
3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \dfrac{f_n(z_n + \rho_n\zeta) - c(z_n + \rho_n\zeta)}{\rho_n^k}$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$ in $\mathbb{C}$. The zeros of $g(\zeta)$ are of multiplicity at least $k+1$. Moreover g is of order at most two.

We claim that

(a) $g^{(k)}(\zeta) \neq B$, where $B = \dfrac{b(z_0) - D_k(c(z_0))}{a_k(z-0)}$ is a constant.

(b) $g^{(k)}(\zeta) \neq A$, where $A = \dfrac{a(z_0) - D_k(c(z_0))}{a_k(z_0)}$ is a constant.

Using the method of Case 1 these claims can be proven. By Lemma 2.2, we have $g(\zeta)$ is a non-constant rational function, so is $g^{(k)}(\zeta)$ and by claim (a) and (b), $g^{(k)}(\zeta)$ has two omitted values, which is a contradiction.

Case 3 : When $c(z_0) = b(z_0)$. This case is similar to case 1. □

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