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B-ESSENTIAL AND B-WEYL SPECTRA OF SUM OF TWO COMMUTING BOUNDED OPERATORS

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In this paper, we devote our research to the B-essential spectra of the sum of two bounded linear operators defined on a Banach space by means of the B-essential spectra of each of the two operators where their products are finite rank operators.

Key words : B-Fredholm operators; B-essential spectra; finite rank operators.

1. Introduction

Let X be a Banach space. We denote by $\mathcal{L}(X)$ be the algebra of bounded linear operators on X. For $U \in \mathcal{L}(X)$ we will denote by $N(U)$ the null space of U and $R(U)$ the range of U. The nullity $\alpha(U)$ of U is defined as the dimension of $N(U)$ and the deficiency $\beta(U)$ of U is defined as the codimension of $R(U)$ in X. If the range $R(U)$ of U is closed and $\alpha(U) < \infty$ (resp. $\beta(U) < \infty$), then U is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator denoted by $\Phi_+(X)$ $\big($resp. $\Phi_-(X)\big)$. A semi-Fredholm operator is an upper or a lower semi-Fredholm operator denoted by $\Phi_\pm(X)$. If both $\alpha(U)$ and $\beta(U)$ are finite then U is called a Fredholm operator denoted by $\Phi(X)$. The index of U is defined by $i(U) = \alpha(U) - \beta(U)$. The subset of all finite rank operators of $\mathcal{L}(X)$ is denoted by $\mathfrak{F}_0(X)$.

An operator $U \in \mathcal{L}(X)$, called a quasi-Fredholm operator if $dis(U) = d$, $R(U^d) \cap N(U)$ and $R(U) + N(U^d)$ are a closed and complemented subspace of X, where $d \in \mathbb{N}$ and $dis(U) = \inf\big\{n \in \mathbb{N},\ \forall m \in \mathbb{N}\ m \geq n \Rightarrow \big[R(U^n) \cap N(U)\big] \subset \big[R(U^m) \cap N(U)\big]\big\}$. The set of quasi-Fredholm operators denoted by $QF(X)$.

The operator U is called a B-Fredholm, upper (resp. lower) semi B-Fredholm operator, if there exists an integer n such that the range space $R(U^n)$ is closed and U_n is a Fredholm, upper (resp. lower) semi-Fredholm operator, where U_n is the restriction of U to $R(U^n)$ considered as a map from $R(U^n)$ into $R(U^n)$ (in particular $U_0 = U$), (see [2]). We define the index of a B-Fredholm operator U as the index of the Fredholm operator U_n where n is any integer such that $R(U_n)$ is closed and U_n is a Fredholm operator. In fact, $i(U) = \alpha(U_n) - \beta(U_n)$, where $\alpha(U_n)$ is the dimension of the kernel $\ker(U_n)$ of U_n, and $\beta(U_n)$ is the codimension of the range $R(U_n) = R(U_{n+1})$ of U_n into $R(U_n)$. In Proposition 2.1 [2], it has been shown that the index of an operator is independent of the integer n. We denote by $BF(X)$, $SBF_+(X)$ and $SBF_-(X)$ the class of all B-Fredholm, upper semi B-Fredholm and lower semi B-Fredholm operators respectively. The B-Fredholm resolvent of U is defined by $\rho_{BF}(U) = \big\{\lambda \in \mathbb{C} : U - \lambda \in BF(X)\big\}$. In [2], Berkani proved that the class $BF(X)$ contains the class $\Phi(X)$ of Fredholm operators as a proper subclass and each B-Fredholm operator is in the class $QF(X)$ of quasi-Fredholm operators in the sense of Labrousse (see [10]).

The upper (resp. lower) semi B-Fredholm spectrum and B-Fredholm spectrum of U are defined by

$$\sigma_{B1}(U) = \big\{\lambda \in \mathbb{C} \text{ such that } U - \lambda \notin \ SBF_+(X)\big\}.$$

$$\sigma_{B2}(U) = \big\{\lambda \in \mathbb{C} \text{ such that } U - \lambda \notin \ SBF_-(X)\big\}.$$

$$\sigma_{BF}(U) = \big\{\lambda \in \mathbb{C} \text{ such that } U - \lambda \notin \ BF(X)\big\}.$$

An operator $U \in \mathcal{L}(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(U)$ of U is defined by

$$\sigma_{BW}(U) = \{\lambda \in \mathbb{C} : \ U - \lambda \text{ is not a } B\text{-Weyl operator}\}.$$

In the case of a normal operator U acting on a Hilbert space H, Berkani proved that

$$\sigma_{BW}(U) = \sigma(U) \backslash E(U)$$

where $E(U)$ is the set of all eigenvalues of U which are isolated in the spectrum of U (see [3, Theorem 4.5]).

Furthermore, in [4], Berkani gived two following new generalized versions of the classical Weyl's theorem where U is a bounded linear operator acting on a Banach space X.

The version generalized Weyl's theorem is:

$$\sigma_{BW}(U) = \sigma(U) \backslash E(U).$$

where $E(U)$ is the set of all eigenvalues of U which are isolated in the spectrum of U

and the version II of the generalized Weyl's theorem is:

$$\sigma_{BW}(U) = \sigma(U) \backslash \Pi(U)$$

where $\Pi(U)$ is the set of all the poles of the resolvent of U.

Now, we give the following lemmas, proved by Berkani.

Lemma 1.1 ([3, Proposition 3.3]) — Let $U \in \mathcal{L}(X)$ be a B-Fredholm operator and let F be a finite rank operator. Then $U + F$ is a B-Fredholm operator and $i(U + F) = i(U)$. $\diamondsuit$

Lemma 1.2 — Let X be a Banach space. Let $U,\ V,\ T,\ S \in \mathcal{L}(X)$ be commuting operators, satisfying $UT + SV = I$. Then

(i) ([2, Proposition 3.2]) $UV \in BF(X)$ if and only if U and V are B-Fredholm operators on X.

(ii) ([6, Proposition 4.3]) $UV \in SBF_{+}(X)$ if and only if U and V are upper semi B-Fredholm operators on X.

(iii) ([6, Proposition 4.3]) $UV \in SBF_{-}(X)$ if and only if U and V are lower semi B-Fredholm operators on X.

(iv) ([5, Theorem 1.1]) If U and V are B-Fredholm operators, then UV is a B-Fredholm operator and $i(UV) = i(U) + i(V)$. $\diamondsuit$

The condition $UT+SV = I$, in Lemma 1.2, is very important to prove that $i(UV) = i(U)+i(V)$, $\big($See [5]$\big)$.

We state the following theorems proved in [1].

***Theorem* 1.1** ([1, Theorem 2.3]) — *Let $U \in \mathcal{C}(X)$. If $0 \in \rho(U)$. Then for $\lambda \neq 0$, we have*
(i) $\lambda \in \sigma_{ei}(U)$ *if and only if* $\dfrac{1}{\lambda} \in \sigma_{ei}(U^{-1})$, *for* $i = 1, 2, 3, 4, 5, 7, 8$.

(i) *If* $\mathbb{C} \backslash \sigma_{e5}(U)$ *and* $\mathbb{C} \backslash \sigma_{e5}(U^{-1})$ *are connected, then* $\lambda \in \sigma_{e6}(U)$ *if and only if* $\dfrac{1}{\lambda} \in \sigma_{e6}(U^{-1})$.

***Theorem* 1.2** ([1, Theorem 2.4]) — *Let U and V be two bounded linear operators on a Banach space X.*

(i) If $UV \in \mathcal{F}(X)$ *then*

$$\sigma_{ei}(U + V) \backslash \{0\} \subset [\sigma_{ei}(U) \cup \sigma_{ei}(V)] \backslash \{0\}, \quad i = 4, 5.$$

If, further, $VU \in \mathcal{F}(X)$, then

$$\sigma_{e4}(U + V) \backslash \{0\} = [\sigma_{e4}(U) \cup \sigma_{e4}(V)] \backslash \{0\}.$$

Moreover, if the complement of $\sigma_{e4}(U)$ *is connected, then*

$$\sigma_{e5}(U+V)\backslash\{0\} = [\sigma_{e5}(U) \cup \sigma_{e5}(V)]\backslash\{0\}.$$

(ii) *If the hypotheses of* (i) *is satisfied and if* $\mathbb{C}\backslash\sigma_{e5}(U+V)$, $\mathbb{C}\backslash\sigma_{e5}(U)$ *and* $\mathbb{C}\backslash\sigma_{e5}(V)$ *are connected, then*

$$\sigma_{e6}(U+V)\backslash\{0\} = [\sigma_{e6}(U) \cup \sigma_{e6}(V)]\backslash\{0\}.$$

(iii) *If* $UV \in \mathcal{F}_+(X)$ *then*

$$\sigma_{ei}(U+V)\backslash\{0\} \subset [\sigma_{ei}(U) \cup \sigma_{ei}(V)]\backslash\{0\}, \quad i = 1,\ 7.$$

If, further, $VU \in \mathcal{F}_+(X)$, *then*

$$\sigma_{e1}(U+V)\backslash\{0\} = [\sigma_{e1}(U) \cup \sigma_{e1}(V)]\backslash\{0\}.$$

Moreover, if $\mathbb{C}\backslash\sigma_{e4}(U)$ *is connected, then*

$$\sigma_{e7}(U+V)\backslash\{0\} = [\sigma_{e7}(U) \cup \sigma_{e7}(V)]\backslash\{0\}.$$

(iv) *If* $UV \in \mathcal{F}_-(X)$ *then*

$$\sigma_{ei}(U+V)\backslash\{0\} \subset [\sigma_{ei}(U) \cup \sigma_{ei}(V)]\backslash\{0\}, \quad i = 2,\ 8.$$

If, further, $VU \in \mathcal{F}_-(X)$, *then*

$$\sigma_{e2}(U+V)\backslash\{0\} = [\sigma_{e2}(U) \cup \sigma_{e2}(V)]\backslash\{0\}.$$

Moreover, if the complement of $\sigma_{e4}(U^*)$ *is connected, then*

$$\sigma_{e8}(U+V)\backslash\{0\} = [\sigma_{e8}(U) \cup \sigma_{e8}(V)]\backslash\{0\}.$$

(v) *If* $UV \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ *then*

$$\sigma_{e3}(U+V)\backslash\{0\} \subset [\sigma_{e3}(U) \cup \sigma_{e3}(V)] \cup [\sigma_{e1}(U) \cap \sigma_{e2}(V)] \cup [\sigma_{e2}(U) \cap \sigma_{e1}(V)]\backslash\{0\}.$$

Moreover, if $VU \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ *then*

$$\sigma_{e3}(U+V)\backslash\{0\} = [\sigma_{e3}(U) \cup \sigma_{e3}(V)] \cup [\sigma_{e1}(U) \cap \sigma_{e2}(V)] \cup [\sigma_{e2}(U) \cap \sigma_{e1}(V)]\backslash\{0\}.$$

The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidmann essential spectra [7], $\sigma_{e3}(.)$ is the Kato essential spectrum [9], $\sigma_{e4}(.)$ is the Wolf essential spectrum [7, 16], $\sigma_{e5}(.)$ is the Schechter essential spectrum [7, 13, 14], $\sigma_{e6}(.)$ denotes the Browder essential spectrum [7, 8, 11, $\sigma_{e7}(.)$ was introduced by Rakočević in [12] and designated the essential approximate point spectrum and $\sigma_{e8}(.)$ is the essential defect spectrum and was introduced by Schmoeger [15]. Note that all these sets are closed and if X is a Hilbert space and U is a self-adjoint operator on X, then all these sets coincide.

The main theme of the paper is to extend the results of Theorem 2.3 and 2.4 in [1] to B-Fredholm and upper (resp. lower) semi B-Fredholm operators in a Banach space X.

Firstly, we give the relation between the B-essential spectra of U and U^{-1}. Finally, we investigate the B-essential spectra of the sum of two bounded linear operators defined on a Banach space by means of the B-essential spectra of each of the two operators where their products are finite rank operators.

We organize the paper in the following way: In section 2 is devoted to the different B-essential spectra of bounded linear operators on a Banach space. The main results of this section are Theorems 2.1, 2.2 and 2.3.

2. MAIN RESULTS

***Theorem* 2.1** — *Let $U,$ T and S be commuting operators on a Banach space X. If $0 \in \rho(U)$ then, for every $\lambda \neq 0$ satisfying $U^{-1}S + TU = I + \lambda^{-1}S$, we have*

$$\lambda \in \sigma_{BW}(U) \text{ if and only if } \frac{1}{\lambda} \in \sigma_{BW}(U^{-1}). \tag{2.1}$$

PROOF : Let $\lambda \neq 0$. Since $0 \in \rho(U)$, then we can write

$$U - \lambda I = -\lambda(U^{-1} - \lambda^{-1})U \tag{2.2}$$

Let $\lambda \notin \sigma_{BW}(U)$. Then, we have $(U - \lambda I) \in BF(X)$ and $i(U - \lambda I) = 0$. By Eq. (2.2) we obtain $(U^{-1} - \lambda^{-1})U \in BF(X)$. On the other hand, we have

$$U^{-1}S + TU = I + \lambda^{-1}S.$$

Hence, we have that

$$(U^{-1} - \lambda^{-1})S + TU = I \tag{2.3}$$

Applying Lemma l.2 (i), we have $(U^{-1}-\lambda^{-1})\in BF(X)$ and $U\in BF(X)$. Again, using Lemma l.2 (iv), we obtain

$$i\big((U^{-1}-\lambda^{-1})U\big)=i(U^{-1}-\lambda^{-1})+i(U)=0.$$

Since $0\in\rho(U)$, then $U\in BF(X)$ and $i(U)=0$. Therefore $i(U^{-1}-\lambda^{-1})=0$. Consequently, we deduce that $\frac{1}{\lambda}\notin\sigma_{BW}(U^{-1})$. Therefore,

$$\frac{1}{\lambda}\in\sigma_{BW}(U^{-1}) \text{ implies that } \lambda\in\sigma_{BW}(U).$$

To prove the other way around. Let $\lambda\neq 0$, we suppose that $\lambda^{-1}\notin\sigma_{BW}(U^{-1})$, then $(U^{-1}-\lambda^{-1})\in BF(X)$, and $i(U^{-1}-\lambda^{-1})=0$. Since $0\in\rho(U)$ then $U\in BF(X)$ and $i(U)=0$. Applying Lemma l.2 (i), we have $(U^{-1}-\lambda^{-1})U\in BF(X)$. Consequently, by (2.2), (2.3) and Lemma l.2 (iv), we have that

$$U-\lambda I\in BF(X) \text{ and } i(U-\lambda I)=0.$$

Therefore $\lambda\notin\sigma_{BW}(U)$. Hence

$$\lambda\in\sigma_{BW}(U) \text{ implies that } \frac{1}{\lambda}\in\sigma_{BW}(U^{-1}),$$

and the proof is complete. Q.E.D

Corollary 2.1 — Let $U,\ T$ and S be commuting operators on a Banach space X. If $0\in\rho(U)$ then, for every $\lambda\neq 0$ satisfying $U^{-1}S+TU=I+\lambda^{-1}S$, we have

$$\lambda\in\sigma_{BF}(U) \text{ if and only if } \frac{1}{\lambda}\in\sigma_{BF}(U^{-1}),$$

and

$$\lambda\in\sigma_{Bi}(U) \text{ if and only if } \frac{1}{\lambda}\in\sigma_{Bi}(U^{-1}) \text{ for } i=1,\ 2. \qquad \diamondsuit$$

***Theorem* 2.2** — *Let $U,\ V,\ T$ and S be commuting operators on a Banach space X, satisfying $UT+SV=I+\lambda(T+S)$, for every $\lambda\neq 0$. Then,*

(i) *if $UV\in\mathfrak{F}_0(X)$, then*

$$\sigma_{BF}(U+V)\backslash\{0\}=\big[\sigma_{BF}(U)\cup\sigma_{BF}(V)\big]\backslash\{0\}.$$

(ii) *if $UV\in\mathfrak{F}_0(X)$, then*

$$\sigma_{Bi}(U+V)\backslash\{0\}=\big[\sigma_{Bi}(U)\cup\sigma_{Bi}(V)\big]\backslash\{0\}. \text{ for } i=1,\ 2. \qquad \diamondsuit$$

PROOF : For $\lambda\in\mathbb{C}$. We have

$$(U-\lambda I)(V-\lambda I)=UV-\lambda(U+V-\lambda I) \tag{2.4}$$

(i) Let $\lambda \notin \sigma_{BF}(U) \cup \sigma_{BF}(V) \cup \{0\}$. Then, $(U - \lambda I) \in BF(X)$ and $(V - \lambda I) \in BF(X)$. Since, $UT + SV = I + \lambda(T + S)$, hence we obtain

$$UT + SV - \lambda(T + S) = (U - \lambda I)T + S(V - \lambda I) = I.$$

Applying Lemma l.2 (i), we have $(U - \lambda I)(V - \lambda I) \in BF(X)$. Since $UV \in \mathfrak{F}_0(X)$, by Eq. (2.4) and Lemma l.1, we have that $(U + V - \lambda I) \in BF(X)$ and hence $\lambda \in \sigma_{BF}(U + V)$. Thus, we obtain

$$\sigma_{BF}(U + V)\backslash\{0\} \subset \big[\sigma_{BF}(U) \cup \sigma_{BF}(V)\big]\backslash\{0\}. \tag{2.5}$$

To prove the inverse inclusion of Eq. (2.5). We suppose that $\lambda \notin \sigma_{BF}(U + V) \cup \{0\}$, then $(U + V - \lambda I) \in BF(X)$. Since $UV \in \mathfrak{F}_0(X)$ then applying Eq. (2.4) and Lemma l.1, we have

$$(U - \lambda I)(V - \lambda I) \in BF(X). \tag{2.6}$$

Applying Lemma l.2 (i), we have that $(U - \lambda I) \in BF(X)$ and $(V - \lambda I) \in BF(X)$. Therefore $\lambda \notin \sigma_{BF}(U) \cup \sigma_{BF}(V)$. This proved that

$$\big[\sigma_{BF}(U) \cup \sigma_{BF}(V)\big]\backslash\{0\} \subset \sigma_{BF}(U + V)\backslash\{0\}.$$

Hence

$$\sigma_{BF}(U + V)\backslash\{0\} = \big[\sigma_{BF}(U) \cup \sigma_{BF}(V)\big]\backslash\{0\}.$$

(ii) For $i = 1$, the proof is similar to (i) if we replace $BF(X)$ and $\sigma_{BF}(.)$ by $SBF_+(X)$ and $\sigma_{B1}(.)$ respectively and apply part (ii) of Lemma l.2.

For $i = 2$, this assertion follows, similar as (i), it suffices to replace $BF(X)$ and $\sigma_{BF}(.)$ by $SBF_-(X)$ and $\sigma_{B2}(.)$ respectively and use part (iii) of Lemma l.2. Q.E.D

In the following theorem, we prove the result of Theorem 2.2 to B-Weyl spectrum.

***Theorem* 2.3** — *Let $U,\ V,\ T$ and S be commuting operators on a Banach space X, satisfying $UT + SV = I + \lambda(T + S)$, for every $\lambda \neq 0$. If $UV \in \mathfrak{F}_0(X)$, then*

$$\sigma_{BW}(U + V)\backslash\{0\} \subset \big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\}.$$

Moreover, if $i(U - \lambda I) = 0$, then

$$\sigma_{BW}(U + V)\backslash\{0\} = \big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\}. \qquad \diamondsuit$$

PROOF : Let $\lambda \notin \sigma_{BW}(U) \cup \sigma_{BW}(V) \cup \{0\}$. Then, we infer that $(U - \lambda I) \in BF(X)$, $i(U - \lambda I) = 0$, $(V - \lambda I) \in BF(X)$ and $i(V - \lambda I) = 0$. Since, $UT + SV = I + \lambda(T + S)$, hence we have that

$$UT + SV - \lambda(T+S) = (U-\lambda I)T + S(V-\lambda I) = I.$$

Applying Lemma l.2 (iv), we have $(U-\lambda I)(V-\lambda I) \in BF(X)$ and $i(U-\lambda I)(V-\lambda I)$ $= i(U-\lambda I) + i(V-\lambda I) = 0$. Since, $UV \in \mathfrak{F}_0(X)$, by Eq. (2.4) and Lemma l.1, we have that

$$(U+V-\lambda I) \in BF(X) \text{ and } i(U+V-\lambda I) = 0.$$

Therefore, $\lambda \notin \sigma_{BW}(U+V)$, and consequently

$$\sigma_{BW}(U+V)\backslash\{0\} \subset \big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\}. \tag{2.7}$$

To prove the inverse inclusion of Eq. (2.7). Suppose that $\lambda \in \sigma_{BW}(U+V) \cup \{0\}$, then $(U + V - \lambda I) \in BF(X)$ and $i(U+V-\lambda I) = 0$. Since $UV \in \mathfrak{F}_0(X)$, then applying Eq. (2.4), we have

$$(U-\lambda I)(V-\lambda I) \in BF(X) \text{ and } i(U-\lambda I)(V-\lambda I) = 0.$$

Now, using Lemma l.2 (i), we have $(U-\lambda I) \in BF(X)$ and $(V-\lambda I) \in BF(X)$. Since $i(U-\lambda I) = 0$ and $i(V-\lambda I) = 0$. So we conclude $\lambda \notin \sigma_{BW}(U) \cup \sigma_{BW}(V)$. Hence

$$\big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\} \subset \sigma_{BW}(U+V)\backslash\{0\}.$$

Therefore

$$\sigma_{BW}(U+V)\backslash\{0\} = \big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\}. \qquad \text{Q.E.D}$$

Remark 2.1 : (i) If $0 \in \overline{\Big[\big[\sigma_{BW}(U) \cup \sigma_{BW}(V)\big]\backslash\{0\}\Big]}$ then $0 \in \sigma_{BW}(U+V)$ and $\sigma_{BW}(U+V)$ $= \sigma_{BW}(U) \cup \sigma_{BW}(V)$.

The proof of this assertion follows from the fact that the B-essential spectrum is closed.

(ii) The same result of (i) is true for Theorem 2.2, if we replace $\sigma_{BW}(.)$ by $\sigma_{BF}(.)$ or $\sigma_{Bi}(.),$ for $i = 1, 2.$ $\diamondsuit$

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