*Indian J. Pure Appl. Math.*, **47**(1): 97-110, March 2016

DOI: 10.1007/s13226-015-0169-9

# NEIGHBORHOOD CONTRACTION IN GRAPHS

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(*Received* 2 *May* 2014; *after final revision* 11 *May* 2015;
*accepted* 15 *October* 2015)

Let $G$ be a graph and $v$ be any vertex of $G$. Then the *neighborhood contracted graph* $G_v$ of $G$, with respect to the vertex $v$, is the graph with vertex set $V - N(v)$, where two vertices $u, w \in V - N(v)$ are adjacent in $G_v$ if either $w = v$ and $u$ is adjacent to any vertex of $N(v)$ in $G$ or $u, w \notin N[v]$ and $u, w$ are adjacent in $G$. The properties of the neighborhood contracted graphs are discussed in this paper. The neighborhood contraction in some special class of graphs, the domination in a graph and the neighborhood contracted graphs are discussed in the paper.

**Key words** : Neighborhood; degree; induced subgraph; contraction; domination.

## 1. INTRODUCTION

The graphs considered in this paper are simple and finite. The reader is referred to [3, 4, 7], for the notations and terminologies used here, unless specified otherwise.

In a graph $G$, *edge contraction* is an operation which removes an edge $x = uv$ from the graph by merging the two end vertices $u$ and $v$ of the edge. Using the concept of edge contraction the graph minors are defined. A graph $H$ is a minor of graph $G$ if a graph isomorphic to $H$ can be obtained from $G$ by contracting some edges, deleting some edges, and deleting some isolated vertices. For example, the graph $H$ is minor of graph $G$ as shown in the Figure 1.1.

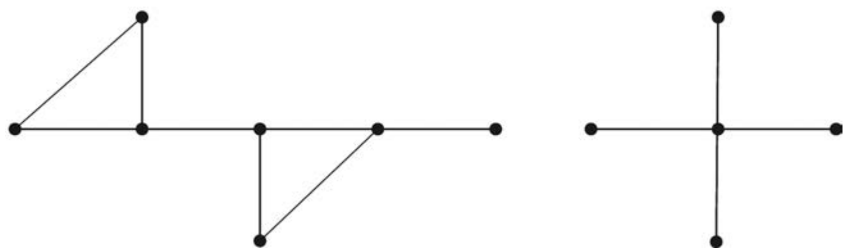

Figure 1.1: The graph $G$ and its minor $H$

The Figure 1.2 illustrates the construction of graph minor $H$ from graph $G$. First construct a subgraph of $G$ by deleting the dashed (staggered) edges and the resulting isolated vertex, and then contract the dotted edge.

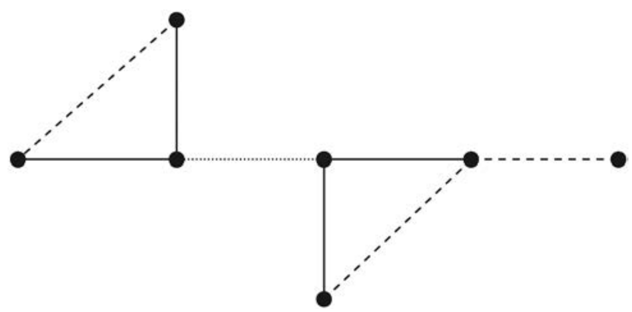

Figure 1.2: Illustration to obtain the graph minor

## 2. Motivation

Enough work has been carried out on graph minors ([1, 2, 5, 6]). However, the edge contractions limit the applications to simple reswitching of a network, where one may have to compromise on the protocols/complexities that would arise in the neighborhood of such edge. The computer networks are complex concepts when there is breakdown due to either physical damage or failure of switches etc. We encounter similar situations in electrical networks and social networks too. In order to handle such crisis, one needs to remodel the network where the neighborhood of the vertex has prevalent influence. This has prompted to consider the contraction of neighborhood of a vertex instead of an edge alone. Therefore in this paper, we introduce another special class of graph minors, which we call the neighborhood contracted graphs.

*Definition* 2.1 — Let $G$ be a graph and let $v$ be any vertex of $G$. Then the neighborhood contracted graph $G_v$ of $G$, with respect to the vertex $v$, is the graph with vertex set $V - N(v)$, where two vertices $u, w \in V - N(v)$ are adjacent in $G_v$ such that one of the following conditions hold.

1. $w = v$ and $u$ is adjacent to any vertex of $N(v)$ in $G$.

2. $u, w \notin N[v]$ and $u, w$ are adjacent in $G$.

For example

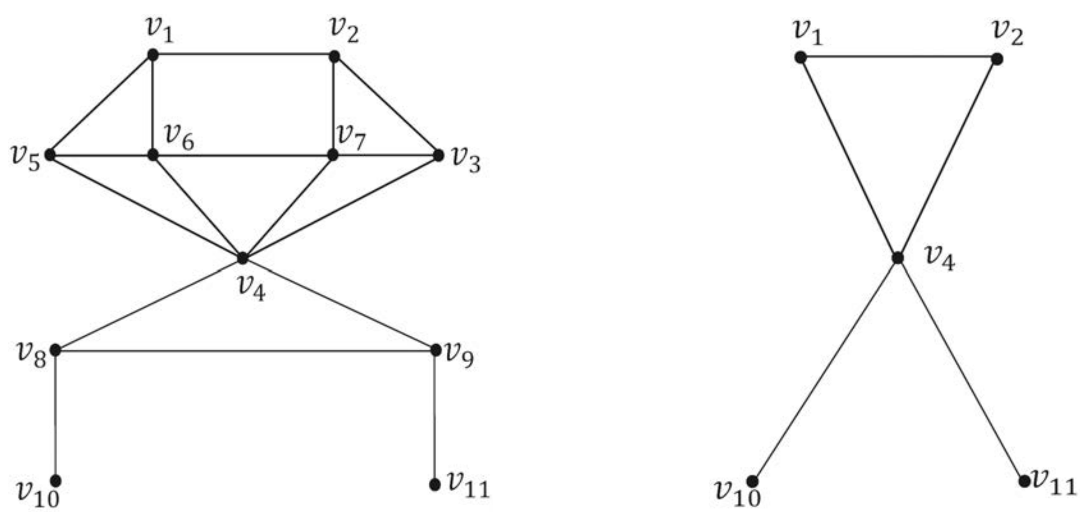


Figure 2.1: Graph $G$ and $G_{v_4}$

## 3. Some Results

*Lemma* 3.1 — Let $v$ be a vertex of a graph $G$. Then the degree of an arbitrary vertex $w$ in the graph $G_v$ is

$$deg_{G_v}(w) = \begin{cases} deg_G(w) & \text{if } w \in V - \cup_{u \in N(v)} N(u) \\ deg_G(w) - |N(w) \cap N(v)| + 1 & \text{if } w \in \cup_{u \in N(v)} N(u) - N[v]. \end{cases}$$

*Lemma* 3.2 — Let $G = (V, E)$ be a graph and let $v \in V$ be any vertex in the graph. Then the degree of $v$ in $G_v$ is

$$deg(v) = | \cup_{u \in N(v)} N(u) - N[v]|.$$

***Theorem* 3.3** — *Let $G = (V, E)$ be a graph and $v \in V$. Then $G_v$ is connected if and only if $G$ is connected.*

PROOF: Let $G_v$ be a connected graph and suppose that $G$ is a disconnected graph. Let $G^1, G^2, ...,$ $G^k$ be the components of $G$ and $v \in G^i$ $(1 \leq i \leq k)$. Then $N(v) \subseteq V(G^i)$. Clearly, $A = \cup_{u \in N(v)} N(u) - N[v]$ (by Lemma 3.2) forms a neighborhood of $v$ in $G_v$. Hence no vertex of a component other than $G^i$ will be adjacent to $v$ in $G_v$. Thus $G_v$ is also disconnected, a contradiction. So, $G$ must be connected graph.

Conversely, let $G$ be a connected graph and let $u, w$ be two vertices of $G_v$. Then $u$ and $w$ are also the vertices of $G$. Since $G$ is connected, there exists a path between $u$ and $w$ in $G$, say $uu_1u_2.....u_rw$.

If all these vertices are in $N[v]$ in $G$ then, $u$ and $w$ have a path in $G_v$ and hence $G_v$ must be connected.

If $\{u_1, u_2, ..., u_r\} \nsubseteq N[v]$, then since $G$ is connected, there exists a path between $u$ and some neighbor of $v$ and between $w$ and some neighbor of $v$ in $G$. Thus there exists a path between $u$ and $w$ in $G_v$ through $v$. Hence $G_v$ must be connected. $\square$

***Theorem* 3.4** — *Let $G = (V, E)$ be a graph and $u, v \in V$ be any two vertices of $G$ such that, $N(v) = \{v_1, v_2, ..., v_k\}$ and $N(u) = \{u_1, u_2, ..., u_r\}$ in $G$. Then $G_u \cong G_v$ if and only if $\langle V - N(u)\rangle \cong \langle V - N(v)\rangle$ and $\langle \cup_{i=1}^{k} N(v_i) - N[v]\rangle \cong \langle \cup_{j=1}^{r} N(u_j) - N[u]\rangle$.*

PROOF : Let $G_v \cong G_u$. By definition, $G_v$ is a graph with vertex set $V - N(v)$ and two vertices $u, w \in V - N[v]$ are adjacent in $G_v$ if and only if $u, w$ are adjacent in $G$. Thus $\langle V - N[u]\rangle \cong \langle V - N[v]\rangle$. Since $u$ and $v$ are isolated vertices in $\langle V - N(u)\rangle$ and $\langle V - N(v)\rangle$ respectively, we have $\langle V - N(u)\rangle \cong \langle V - N(v)\rangle$.

Suppose $\langle \cup_{i=1}^{k} N(v_i) - N[v]\rangle \ncong \langle \cup_{j=1}^{r} N(u_j) - N[u]\rangle$. Let $A = \cup_{i=1}^{k} N(v_i) - N[v]$ and $B = \cup_{j=1}^{r} N(u_j) - N[u]$. Since $\cup_{i=1}^{k} N(v_i) - N[v] \subseteq V - N(v)$ and $\cup_{j=1}^{r} N(u_j) - N[u] \subseteq V - N(u)$, the graphs $\langle V - N(u)\rangle$ and $\langle V - N(v)\rangle$ differ only by the adjacencies in $\langle A\rangle$ and $\langle B\rangle$. If $\langle A\rangle \ncong \langle B\rangle$ then $\langle V - N(u)\rangle \ncong \langle V - N(v)\rangle$, a contradiction. Thus $\langle \cup_{i=1}^{k} N(v_i) - N[v]\rangle \cong \langle \cup_{j=1}^{r} N(u_j) - N[u]\rangle$. Conversely, let $x_p, y_p$ be two adjacent vertices of the graph $G_v$. We have the following possibilities.

*Case* 1 : Both $x_p, y_p \in V - N[v]$ in $G$.

Since $\langle V - N(v)\rangle \cong \langle V - N(u)\rangle$, there exist two adjacency preserving vertices, $x_q, y_q \in V - N(u)$ corresponding to $x_p$ and $y_p$. Since $u$ is an isolated vertex of $\langle V - N(u)\rangle$ and $x_q$ and $y_q$ are adjacent in $\langle V - N(u)\rangle$, and hence adjacent in $G_u$.

*Case* 2 : One of $x_p$ and $y_p$ is $v$ in $G$.

Note that, $x_p$ and $y_p$ are adjacent in $G_v$ and hence, $y_p$ must be adjacent to some vertex of $N(v)$ in $G$. That is $y_p \in \cup_{i=1}^{k} N(v_i) - N[v]$. Since $\langle \cup_{i=1}^{k} N(v_i) - N[v] \rangle \cong \langle \cup_{j=1}^{r} N(u_j) - N[u] \rangle$, there exists a vertex $y_q \in \cup_{j=1}^{r} N(u_j) - N[u]$ corresponding to $y_p$. Since $y_q$ is adjacent to some neighbor of $u$ in $G$, $y_q$ must be adjacent to $u$ in $G_u$.

Thus combining both the cases, $G_v \cong G_u$. □

***Theorem* 3.5** — $G_v \cong K_2$ *if and only if* $G$ *is connected and* $deg(v) = n - 2$ *in* $G$, *where* $n > 2$ *is the order of the graph* $G$.

PROOF : Suppose $G_v \cong K_2$ and let $V(G_v) = \{v, w\}$ Then $v$ should be adjacent to all other vertices except $w$. Hence $deg(v) = n - 2$ in $G$.

Conversely, let $G$ be a connected graph and $deg(v) = n - 2$ in $G$. Then there exists exactly one vertex $u \in V - N[v]$ and $u$ must be adjacent to some neighbor of $v$ in $G$. Thus $|V(G_v)| = 2$ and $u$ must be adjacent to $v$ in $G_v$. Thus $G_v \cong K_2$. □

*Corollary* 3.6 — The neighborhood contracted graph $G_v$ of a graph with respect to every vertex $v$ is $K_2$ if and only if $G$ is $(n-2)-$regular graph on $n$ vertices.

*Remark* 3.7 : Let $v$ be any vertex of a graph $G$. Then $G_v$ is a complete graph if and only if $\langle V - N[v] \rangle$ is a complete graph such that every vertex of $V - N[v]$ is adjacent to some neighbor of $v$ in $G$.

*Note* 3.8 : Let $G = (V, E)$ be a graph and $v \in V$. Then it is always possible to obtain a graph $H$ by subdividing the edges incident to $v$ in $G$, so that, $H_v \cong G$. But the graph $H$ so obtained is not unique.

***Theorem* 3.9** — *Let* $G = (V, E)$ *be a connected graph. To reduce the graph* $G$ *into a trivial graph, at most* $\lfloor \frac{n}{2} \rfloor$ *neighborhood contractions are required.*

PROOF : Let $G$ be a connected graph. If $G \cong K_2$, then neighborhood contraction with respect to any vertex will reduce the graph $G$ into a trivial graph. Suppose $G \ncong K_2$ and $G$ is connected. Then $n \geq 3$ and $\Delta(G) \geq 2$. Let $v$ be a vertex of maximum degree. Then $G_v$ will contain at most $n - 2$ vertices. If $G_v$ is not a trivial graph, then repeat the procedure with the maximum degree vertex of $G_v$. In each step at least 2 vertices of $G$ are reduced. Hence, at most $\lfloor \frac{n}{2} \rfloor$ neighborhood contractions are required to obtain the trivial graph. □

*Corollary* 3.10 — Let $G = (V, E)$ be a graph with $G^i = (V_i, E_i)$ as its components for $i = 1, 2, 3, ..., k$. Then with at most $\sum_{i=1}^{k} \frac{|V_i|}{2}$ neighborhood contractions a totally disconnected graph is obtained.

PROOF : The proof follows from Theorem 3.3 and Theorem 3.9. □

***Theorem* 3.11** — *Let $v$ be any vertex of a graph $G$. Then $G_v \cong (\overline{G})_v$ if and only if both the following conditions are true.*

1. $\overline{\langle N(v) \rangle} \cong \langle V - N[v] \rangle$ *and*

2. *every vertex of $N(v)/V - N[v]$ is adjacent to some vertices, but not to all vertices of $V - N[v]/N(v)$ in $G$.*

PROOF : Suppose $G_v \cong (\overline{G})_v$. Then the induced subgraphs of the vertex sets of $G_v$ and $(\overline{G})_v$ must be isomorphic. In particular, $\langle V - N(v) \rangle \cong \overline{\langle N[v] \rangle}$. Since $v$ is an isolated vertex in both the induced subgraphs, $\langle V - N[v] \rangle \cong \overline{\langle N(v) \rangle}$ in $G$.

Let $N_G(v) = A$ and $V - N_G[v] = B$. Suppose $A$ contains a vertex $u$ which is adjacent to every vertex of $B$. Then every vertex of $V - N[v]$ is adjacent to some neighbor of $v$ in $G$. Thus every vertex of $B$ is adjacent to $v$ in $G_v$ and hence $v$ is a full degree vertex in $G_v$. But $B$ is the neighborhood of $v$ in $\overline{G}$ and $u \in A$ is not adjacent to any vertex of $B$. Thus $\overline{G}$ contains a vertex $u$ which is not adjacent to any neighbor of $v$ in $\overline{G}$. Thus $(\overline{G})_v$ contains a vertex $u$ which is not adjacent to $v$ in $(\overline{G})_v$ and hence $v$ is not a full degree vertex in $(\overline{G})_v$. Since $\overline{\langle N(v) \rangle} \cong \langle V - N[v] \rangle$, the above leads to $G_v \ncong (\overline{G})_v$, a contradiction.

Similarly, if $B$ contains a vertex $w$ which is adjacent to every vertex of $B$, we arrive at a contradiction.

Suppose $A$ contains a vertex $u$ which is not adjacent to any vertex of $B$. Then by the argument as above, we can prove that, $v$ is not a full degree vertex of $G_v$, whereas, $v$ is a full degree vertex of $(\overline{G})_v$, which leads to a contradiction. Thus the second condition holds.

Conversely, let us assume that $\overline{\langle N(v) \rangle} \cong \langle V - N[v] \rangle$. Then $\langle N(v) \rangle_{G_v} \cong \langle N(v) \rangle_{(\overline{G})_v}$. Since every vertex of $V - N[v]/N(v)$ is adjacent to some vertex of $N(v)/V - N[v]$ in $G$, $v$ is a full degree vertex in both $G_v$ and $(\overline{G})_v$. Hence $G_v \cong (\overline{G})_v$. □

## 4. Neighborhood Contraction in Some Special Classes of Graphs

*Note* 4.1 : The following are the simple observations:

1. $G_v \cong G$ if and only if $deg(v) = 0$ in $G$.

2. $G_v$ is a trivial graph if and only if $deg(v) = n - 1$ in $G$.

3. Let $K_{m,n}$ be a complete bipartite graph with partite sets $V_1$ and $V_2$ with $|V_1| = m$ and $|V_2| = n$. Then
$$(K_{m,n})_v = \begin{cases} K_{1,m-1} & if \ \ v \in V_1. \\ K_{1,n-1} & if \ \ v \in V_2. \end{cases}$$

4. Let $G = W_n = C_{n-1} + K_1$ be a wheel graph on $n$ vertices. Then
$$G_v = \begin{cases} K_1 & if \ \ deg(v) = n-1. \\ P_{n-3} + K_1 & otherwise. \end{cases}$$

5. $(P_n)_v = \begin{cases} P_{n-1} & if \ deg(v) = 1. \\ P_{n-2} & if \ deg(v) = 2. \end{cases}$

**Trees**

*Proposition* 4.2 — If $G$ is a bipartite graph, then $G_v$ is also bipartite.

Proof : Let $v$ be a vertex of a bipartite graph $G$ with partite sets $V_1$ and $V_2$ and let $v \in V_1$. Then $v$ is adjacent to some vertices of $V_2$. The vertices of $V_1$ which are adjacent to the neighbors of $v$ are adjacent to $v$ in $G_v$. Since $V_1$ is an independent set of $G$, the induced subgraph $\langle V_1 \rangle$ forms an acyclic graph. The vertices of $B_1 = v \cup (V_2 - N(v))$ forms one independent set and the remaining vertices of $V_1$ form another independent set $B_2$ of $G_v$. Also, the edges of $G_v$ will have one end vertex in $B_1$ and the other end vertex in $B_2$. Thus $G_v$ is a bipartite graph. □

***Theorem* 4.3** — *If $G$ is a tree then $G_v$ is also a tree.*

Proof : Let $u, w$ be two vertices of the graph $G_v$ such that they lie on a cycle and $P_1$ and $P_2$ be two paths between $u$ and $w$ in $G_v$. We have two cases.

*Case* 1 : $u$ and $w$ are adjacent to $v$ in $G_v$.
Since $P_1$ and $P_2$ are the paths between $u$ and $w$ in $G_v$, the vertices of $P_1$ and $P_2$ lie in $V - N(v)$ in $G$. Hence there exist two paths between $u$ and $w$ in $G$ also, a contradiction.

*Case* 2 : $u = v$ and $w \in V - N(v)$ in $G$.

Since we assumed that there exist two paths between $u$ and $w$ in $G_v$, $w$ is not adjacent to any neighbor of $v$ in $G$. Thus we can find two vertices $v_1$ and $v_2$ on the paths $P_1$ and $P_2$ respectively such that, $v_1$ and $v_2$ are adjacent to some neighbors of $v$ in $G$. Hence there exist two paths between $u = v$ and $w$ in $G$, a contradiction.

Thus there exists a unique path between any two vertices of $G_v$ and so $G_v$ is acyclic. Since $G$ is connected, $G_v$ is also connected. Thus $G_v$ is a tree. $\square$

***Theorem* 4.4** — *Let $v$ be any vertex of a graph $G = (V, E)$. Then $G_v$ is acyclic if and only if $\langle V - N(v)\rangle$ is acyclic and no two vertices of $\cup_{u\in N(v)}N(u) - N[v]$ are adjacent in $G$.*

PROOF : If $G_v$ is acyclic then clearly $\langle V - N(v)\rangle$ is acyclic as $V - N(v)$ is the vertex set of $G_v$. Suppose $u_i, u_j \in \cup_{u\in N(v)}N(u) - N[v]$ are adjacent in $G$. Then $u_i$ and $u_j$ are the neighbors of $v$ in $G_v$ and hence $u_i, u_j$ and $v$ form a cycle in $G_v$, a contradiction.

Conversely, since $\langle V - N(v)\rangle$ is acyclic in $G$, $\langle V - N(v)\rangle$ is acyclic in $G_v$ too. Also, since no two vertices of $\cup_{u\in N(v)}N(u) - N[v]$ are adjacent in $G$, they are not adjacent in $G_v$. Since $\cup_{u\in N(v)}N(u) - N[v]$ is the neighborhood of $v$ in $G_v$, the vertices of $N[v]$ will not form any cycle in $G_v$. Thus $G_v$ is acyclic. $\square$

**Regular Graphs**

***Theorem* 4.5** — *Let $v \in V$ be any vertex of a graph $G = (V, E)$. Then $G_v$ is $r$-regular if and only if $| \cup_{u\in N(v)} N(u) - N[v]| = r$ and for any $w \in V - N[v]$,*

$$deg_{\langle V-N[v]\rangle}(w) = \begin{cases} r-1 & \text{if } \; w \in \cup_{u\in N(v)}N(u) - N[v] \\ r & \text{if } \; w \in V - \cup_{u\in N(v)}N(u) \end{cases} \tag{1}$$

PROOF : Let $G_v$ be an $r$-regular graph. Thus $|N(v)| = | \cup_{u\in N(v)} N(u) - N[v]| = r$ (by Lemma 3.2). Also, $deg_G(w) = deg_{G_v}(w) = r$ for any $w \in V - \cup_{u\in N(v)}N(u)$. Hence $deg(w) = r$ in $\langle V - N[v]\rangle$ if $w \in V - \cup_{u\in N(v)}N(u)$. Let $w \in N(v)$ in $G_v$. Then $deg(w) = r$ in $G_v$. Thus $w$ must be adjacent to some neighbor of $v$ in $G$ and hence $w \in \cup_{u\in N(v)}N(u) - N[v]$ and $deg(w) = r - 1$ in $\langle V - N[v]\rangle$.

Conversely, let $w \in V(G_v)$. Then by (1) and by the definition of $G_v$, $deg(w) = r$ in $G$ for any $w \neq v$. If $w = v$, then since $r = |\cup_{u\in N(v)}N(u) - N[v]| = |N_{G_v}(v)|$, $deg(w) = r$. Thus $deg(w) = r$ for any $w \in V(G_v)$ and hence $G_v$ is an $r$-regular graph. $\square$

*Corollary* 4.6 — Let $G$ be a $k-$regular graph on $n$ vertices. Then $G_v$ is a $k-$regular graph for a vertex $v$ in $G$ if and only if $|\cup_{u\in N(v)} N(u) - N[v]| = k$ and for any $w \in V - N[v]$,

$$deg_{\langle V-N[v]\rangle}(w) = \begin{cases} k-1 & \text{if } w \in \cup_{u\in N(v)}N(u) - N[v] \\ k & \text{if } w \in V - \cup_{u\in N(v)}N(u) \end{cases} \tag{2}$$

***Theorem* 4.7** — *Let $G$ be a $k-$regular graph. Then for every vertex $v$ of $G$, $G_v$ is $k-$regular if and only if one of the following holds.*

1. $G \cong \overline{K}_n$ or

2. $G \cong C_n$ where $n \geq 5$

PROOF : Let $G_v$ be a $k-$regular graph with respect to every vertex $v$ of a $k-$regular graph $G$. Let $N(v) = \{v_1, v_2, ..., v_k\}$ with $k \neq 2$. If $k = 0$ then, $G$ is a totally disconnected graph and hence $G_v \cong G \cong \overline{K}_n$ for every vertex $v$ of $G$. Suppose $k \geq 1$.

If $k = 1$ then, $G \cong mK_2$ for some positive integer $m$. Then $G_v \cong K_1 \cup (m-1)K_2$, a contradiction to the fact that, $G_v$ is $k$-regular. Thus let $k \geq 2$.

Note that each vertex of $N[v]$ is not adjacent to every vertex of $N[v]$, for otherwise, $\langle N[v]\rangle \cong K_k$ and since $G$ is $k-$regular, $N[v]$ is a component of $G$. Hence $G_v$ is a graph with $K_1$ as one of its components, a contradiction. Let us assume that, $k > 2$.

*Claim* : $|N(v_i) \cap (V - N[v])| = 1$ for all $v_i \in N(v)$ in $G$.

Suppose $v_1 \in N(v)$ such that, $v_1$ is adjacent to at least two vertices $w_i, w_j$ of $V - N[v]$ in $G$. Then $v_1$ can be adjacent to at most $k-2$ vertices of $N[v]$. Since $G_{v_1}$ is $k-$regular, these $k-2$ vertices must be adjacent to $k$ vertices of $V - N[v_1]$ in $G$. Since the $k-2$ neighbors of $v_1$ lie in $N[v]$, the $k$ neighbors of these $k-2$ vertices and $w_i, w_j$ are adjacent to $v$ in $G_v$. Thus $deg(v) > k$ in $G_v$, a contradiction. Thus the claim holds.

Thus each $v_i \in N(v)$ is adjacent to a unique $w_i$ in $V - N[v]$. Since $G$ is $k-$regular, each $v_i \in N(v)$ must be adjacent to $k-1$ vertices of $N[v]$ with $k > 2$ and $w_i \in V - N[v]$ for $1 \leq i \leq k$. Thus in $G_{v_i}$, $v_i$ is adjacent to the $k-1$ neighbors of $w_i$, $k-2$ vertices $w_j$ which are adjacent to the neighbors of $v_i$ in $N[v]$ and a neighbor of $v$ to which $v_i$ is not adjacent in $G$. Thus $deg(v_i) > k$ in $G_{v_i}$, a contradiction. Thus $k = 2$. Thus $G \cong C_n$.

If $n = 3$ or $4$, then $G_v \cong K_1$ or $K_2$ accordingly for each $v$ in $C_3$ or $C_4$. Thus $G \cong C_n$ for $n \geq 5$.

The converse part is immediate from the definition of $G_v$. $\square$

## 5. Domination and Neighborhood Contraction

***Theorem* 5.1** — *Let $v$ be any vertex of a graph $G = (V, E)$. Then $\gamma(G) \geq \gamma(G_v)$.*

Proof : Let $D$ be a $\gamma$-set of a graph $G$ and $v \in V$. If $v$ is an isolated vertex of $G$ then $G_v \cong G$ and hence, $\gamma(G) = \gamma(G_v)$.

Suppose $deg(v) > 0$ and let $N(v) = \{v_1, v_2, ..., v_k\}$ in $G$. Note that the vertices which are adjacent to $v_i$ $(1 \leq i \leq k)$ in $G$ are adjacent to $v$ in $G_v$. Hence if $v_i \in D$ in $G$ for some $i$, then the vertices dominated by $v_i$ in $G$ are dominated by $v$ in $G_v$. Since every vertex of $V - D$ is adjacent to some vertex of $D$ in $G$, so is in $G_v$. Thus the vertices of $(D - N(v)) \cup \{v\}$ forms a dominating set of $G$. Thus $\gamma(G) \geq \gamma(G_v)$. $\square$

A vertex $v$ of a graph $G$ is said to be a *contraction critical vertex with respect to* $\gamma$ if $\gamma(G) > \gamma(G_v)$ and is said to be *contraction redundant with respect to* $\gamma$ if $\gamma(G) = \gamma(G_v)$.

***Theorem* 5.2** — *Let $v$ be a $\gamma$-fixed or $\gamma$-free vertex of a graph $G = (V, E)$. Then $v$ is a contraction redundant vertex of the graph if and only if one of the following conditions holds:*

1. $deg(v) = n - 1$.
2. *$v$ is an isolated vertex of a $\gamma$-set $D$ such that no neighbor of $v$ is adjacent to a vertex of $D$ in $G$ and the vertices of $D$ which are adjacent to the vertices of $\cup_{u \in N(v)} N(u) - N[v]$ must dominate some vertex of $G$ uniquely.*

Proof : Since $v$ is a $\gamma$-fixed or $\gamma$-free vertex of $G$, there exists at least one $\gamma$-set $D$ of $G$ such that, $v \in D$. Suppose $v$ is a contraction redundant vertex, then $\gamma(G) = \gamma(G_v)$.

Suppose $deg(v) < n - 1$ and $v$ is adjacent to some vertex $u$ of $D$ in $G$. Note that $u$ is not a vertex of $G_v$. If $u$ dominates some vertex of $G$ uniquely in $G$, then, since $u$ is adjacent to $v$ in $G$, the neighbors of $u$ are adjacent to $v$ in $G_v$. Since $D$ is a $\gamma$-set of $G$, the vertices of $D - \{u\}$ form a dominating set of $G_v$ and hence, $\gamma(G_v) < \gamma(G)$, a contradiction. Thus $v$ must be an isolated vertex of $D$ in $G$.

Suppose some neighbor $w$ of $v$ in $G$ is adjacent to some vertex $u \in D$. Then $u \in D$ is adjacent to $v$ in $G_v$. Since $v$ is an isolate of $D$ in $G$, no neighbor of $v$ dominates any vertex of $G$ uniquely, so is in $G_v$. Also, $v$ is dominated by $u$ in $G_v$ and hence $D - \{v\}$ forms a dominating set of $G_v$. Hence $\gamma(G_v) < \gamma(G)$, a contradiction.

Suppose there exists a vertex $w \in D$ such that $w$ is adjacent to some vertex $a \in \cup_{u \in N(v)} N(u) - N[v]$ in $G$ and does not dominate any vertex of $G$ uniquely. Then $a$ is adjacent to $w$ and $v$ in $G_v$. Since both $v$ and $w$ do not dominate any vertex of $G$ uniquely, $(D - \{v, w\} \cup \{a\})$ forms a dominating set of $G_v$. Thus $\gamma(G_v) < \gamma(G)$, a contradiction.

Conversely, let $deg(v) = n - 1$ in $G$. Then $G_v \cong K_1$ and hence $\gamma(G) = \gamma(G_v) = 1$. If $deg_G(v) = 0$ then $G \cong G_v$ and hence $\gamma(G) = \gamma(G_v)$. Suppose $deg_G(v) \geq 1$ and $v$ is an isolated vertex of a $\gamma$-set $D$ of $G$ such that, no neighbor of $v$ dominates any vertex of $D$ in $G$. Then $v$ dominates itself in $G_v$ and hence lies in a $\gamma$-set of $G_v$. Also note that $D$ is a $\gamma$-set of $G$, and the vertices of $D$ which are adjacent to the vertices of $\cup_{u \in N(v)} N(u) - N[v]$ dominate some vertex uniquely in $G$. The vertices of $D$ other than $v$ must lie in the $\gamma$-set of $G_v$. Thus $\gamma(G_v) = \gamma(G)$. □

***Theorem* 5.3** — *Let $v$ be a $\gamma$-totally free vertex of a graph $G = (V, E)$. Then $\gamma(G) = \gamma(G_v)$ if and only if $N(v) \cap D = \{u\}$ for any $\gamma$-set $D$ of $G$ such that one of the following conditions holds:*

1. *$u$ is an isolate of $D$ such that no neighbor of $v$ is adjacent to any vertex of $D$ in $G$.*
2. *the vertices of $D$ which are adjacent to the vertices of $N[v]$ must dominate some vertex of $G$ uniquely.*

PROOF : Let $v$ be a $\gamma$-totally free vertex of a graph $G = (V, E)$. Suppose $\gamma(G) = \gamma(G_v)$. Since $v$ is $\gamma$-totally free vertex of $G$, $v$ does not belong to any $\gamma$-set of the graph. Then $v$ is dominated by some vertex of a $\gamma$-set $D$ of $G$.

Suppose $|N(v) \cap D| \geq 2$ for some $\gamma$-set $D$ of $G$. Let $u, w \in N(v) \cap D$. Then the vertices dominated by $u$ and $w$ in $G$ are dominated by $v$ in $G_v$. Hence $(D - \{u, w\}) \cup \{v\}$ forms a dominating set of $G_v$, a contradiction. Thus $N(v) \cap D = \{u\}$ for any $\gamma$-set $D$ of $G$.

Suppose $u$ is an isolate of $D$. Then $v$ dominates all the vertices of $G$ which are dominated by $u$ in $G$. Then $v$ lies in a dominating set of $G_v$. Also note that, $v$ is an isolate of the $\gamma$-set of $G_v$ if no neighbor of $v$ is adjacent to any vertex of $D$.

Suppose there exists a vertex $w \in D$ such that, $w$ is adjacent to a vertex of $N(v)$ in $G$. Suppose $w$ does not dominate any vertex of $G$ uniquely. Then $w$ is adjacent to $v$ in $G_v$ and the vertices dominated by $w$ are dominated by some vertex of $D$. Thus $(D - \{u, w\}) \cup \{v\}$ forms a dominating set of $G_v$, a contradiction. Thus the vertices of $D$, which are adjacent to the vertices of $N[v]$, must dominate some vertex of $G$ uniquely.

Conversely, let $N(v) \cap D = \{u\}$ and $u$ is an isolate of $D$ such that no neighbor of $v$ is adjacent to any vertex of $D$ in $G$. Then $v$ is an isolated vertex in a $\gamma$-set of $G_v$. Hence $(D - \{u\}) \cup \{v\}$ forms a $\gamma$-set of $G_v$. Thus $\gamma(G) = \gamma(G_v)$. □

If the vertices of $D$ which are adjacent to the vertices of $N[v]$ in $G$, dominates some vertex of $G$ uniquely, then $v$ dominates some vertex of $G$ uniquely in $G_v$, which are dominated by $u$ in $G$. Since other vertices of $D$ also dominate some vertex of $G$ uniquely, they must lie in the $\gamma$-set of $G_v$. Thus $\gamma(G) = \gamma(G_v)$.

We now introduce some properties of vertices in respect of domination in connection with neighborhood contraction.

*Definition* 5.4 — Let $G = (V, E)$ be a graph and $v$ be any vertex in $G$. Then the vertex $v$ is said to be

1. neighborhood contraction $\gamma-$fixed (n.c. $\gamma-$fixed) vertex if $v$ lies in every $\gamma-$set of $G_v$.
2. neighborhood contraction $\gamma-$free (n.c. $\gamma-$free) vertex if $v$ lies in some $\gamma-$sets of $G_v$ but not in all.
3. neighborhood contraction $\gamma-$totally free (n.c. $\gamma-$totally free) vertex if $v$ does not lie in any $\gamma-$set of $G_v$.

For example, consider the graph in the figure 5.1. The vertex $v_7$ is n.c. $\gamma-$fixed vertex, the vertex $v_4$ is n.c. $\gamma-$free vertex and the vertex $v_1$ is n.c. $\gamma-$ totally free vertex.

We have the following results.

***Theorem* 5.5** — *Let $v$ be a vertex of a graph $G = (V, E)$. Then $v$ is n.c. $\gamma-$totally free vertex if and only if*

1. *there exists a vertex $u \in V - N[v]$ such that $u$ dominates some vertex of $V - N[v]$ uniquely in $G$ and is adjacent to some vertex of $N(v)$ in $G$. and*

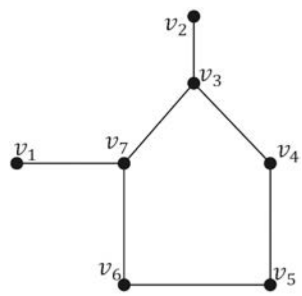


Figure 5.1: Illustration to n.c $\gamma-$fixed, n.c. $\gamma-$free and n.c. $\gamma-$totally free vertices.

2. *no neighbor of $v$ dominates any vertex of $G$ uniquely in $G$.*

PROOF : Suppose at least one condition of the theorem is not true and $v$ is n.c. $\gamma-$totally free vertex.

Suppose no neighbor of $N(v)$ dominates a vertex of $V - N[v]$ uniquely in $G$. Then any neighbor of $v$ does not dominate a vertex of $G_v$ uniquely in $G_v$. Thus $v$ may be a $\gamma-$free of $\gamma-$fixed vertex of $G_v$, a contradiction.

Suppose some neighbor of $v$ dominates a vertex of $V - N[v]$ uniquely in $G$. Then that vertex is uniquely dominated by $v$ in $G_v$. Then $v$ must lie in a dominating set of $G_v$, a contradiction. Hence both conditions must be true.

Conversely, suppose some neighbor $u$ of $N(v)$ dominates a vertex of $V - N[v]$ in $G$. Then $u$ is adjacent to $v$ in $G_v$ and $u$ dominates $v$ in $G_v$. Also no neighbor of $v$ dominates any vertex of $V-N[v]$ uniquely in $G$. Hence $v$ does not dominate any vertex of $G_v$ uniquely in $G_v$. Thus $v$ does not belong to any $\gamma-$set of $G_v$ and hence $v$ is a n.c. $\gamma-$totally free vertex. □

***Theorem* 5.6** — *Suppose $v$ is a vertex of a graph $G$ such that $\langle V - N[v]\rangle$ contains at least two isolated vertices $u, w$ such that $u, w$ are adjacent to some vertex of $N(v)$ in $G$. Then $v$ is a n.c. $\gamma-$fixed vertex.*

PROOF : Suppose $u, w \in V - N[v]$ such that $u, w$ are adjacent to some neighbor of $v$ in $G$. Then $u, w$ are adjacent to $v$ in $G_v$. Since $u, w$ are isolated vertices of $\langle V - N[v]\rangle$, $u$ and $w$ are uniquely dominated by $v$ in $G_v$. Thus $v$ must lie in every dominating set of $G_v$. Hence $v$ is a n.c. $\gamma-$fixed vertex. □

*Note* 5.7 : But the converse of the above result need not be true in general. For example, in the graph $G$ of figure 5.2, $\langle V - N[v_6]\rangle$ contains only one isolated vertex, which is adjacent to some neighbor of $v_6$. Still $v_6$ is n.c. $\gamma-$fixed vertex.

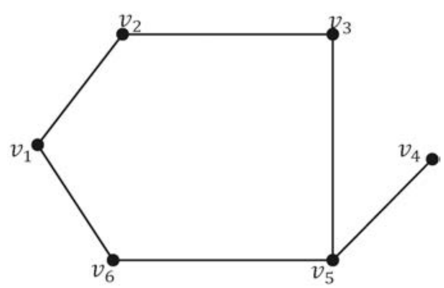


Figure 5.2: The graph $G$.

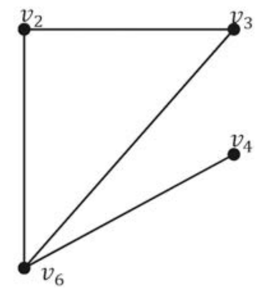


Figure 5.3: The graph $G_{v_6}$

*Remark* 5.8 : Let $v$ be a vertex of a graph $G$. Then by the definition of n.c. $\gamma-$free vertex, $v$ is a n.c. $\gamma-$free vertex if $G_v$ contains at least one $\gamma-$set $D_i$ containing $v$ and at least one $\gamma-$set $D_j$ with $v \notin D_j$.

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