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ANNIHILATOR CONDITION OF A PAIR OF DERIVATIONS IN PRIME AND SEMIPRIME RINGS[1]

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Let n be a fixed positive integer, R be a prime ring, D and G two derivations of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(D(u)u^n - u^n G(u)) = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then one of the following holds:

1. $D = G = 0$, unless R satisfies s_4;
2. char $(R) \neq 2$, R satisfies s_4, n is even and $D = G$;
3. char $(R) \neq 2$, R satisfies s_4, n is odd and D and G are two inner derivations induced by b, c respectively such that $b + c \in C$;
4. char $(R) = 2$ and R satisfies s_4.

We also investigate the case when R is a semiprime ring.

Key words : Prime ring; derivation; extended centroid; Martindale quotient ring.

1. INTRODUCTION

Throughout this paper, R always denotes a prime ring with center $Z(R)$, extended centroid C and Q is its Martindale quotient ring. For $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y]$

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and defined by $[x,y] = xy - yx$. An additive subgroup L of R is said to be a Lie ideal of R, if $[L, R] \subseteq L$. By D we mean a derivation of R, that is an additive mapping $D : R \to R$ satisfying $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A derivation D is called Q-inner if it is inner induced by an element, say $q \in Q$ as an adjoint, that is, $D(x) = [q, x]$ for all $x \in R$. A derivation which is not Q-inner is called a Q-outer derivation. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

A well known result proved by Posner [23], states that if the commutators $[D(x), x] \in Z(R)$ for all $x \in R$, then either $D = 0$ or R is commutative. Then many related generalizations of Posners result have been obtained by a number of authors in literature. Brešar proved in [5] that if $D(x)x - xG(x) \in Z(R)$ for all $x \in R$, then either $D = G = 0$ or R is commutative. Later Lee and Wong [21] studied the same situation of Brešar for all x in some noncentral Lie ideal L of R and obtained that either $D = G = 0$ or R satisfies s_4.

Recently, Argaç and De Filippis [1] obtained the following result:

Let R be a prime ring with char $(R) \neq 2$, L a non-central Lie ideal of R, D, G two derivations of R and $n \geq 1$ a fixed integer. If $(D(x)x - xG(x))^n = 0$ for all $x \in L$, then either $D = G = 0$ or R satisfies the standard identity s_4 and D, G are inner derivations, induced respectively by the elements a and b such that $a + b \in Z(R)$.

Recently, first author of this paper [10] studied the case with left annihilator condition. More precisely, he obtained the following result:

Let R be a prime ring, D and G two derivations of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(D(u)u - uG(u))^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then one of the following holds:
(i) $D = G = 0$ unless R satisfies s_4;
(ii) char $(R) = 2$ and R satisfies s_4;
(iii) char $(R) \neq 2$ and R satisfies s_4, D and G are two inner derivations induced by p, q respectively such that $p + q \in C$.

In [19], Lee and Zhou studied the situation for generalized derivations as follows:

Let R be a prime ring that is not commutative and such that $R \ncong M_2(GF(2))$, let G, H be two generalized derivations of R, and let m, n be two fixed positive integers. Then $G(x^m)x^n - x^n H(x^m) = 0$ for all $x \in R$ iff the following two conditions hold: (1) There exists $w \in Q$ such that

$G(x) = xw$ and $H(x) = wx$ for all $x \in R$; (2) either $w \in C$, or x^m and x^n are C-dependent for all $x \in R$.

Carini and De Filippis [6] studied the situation in a noncentral Lie ideal of a prime ring R with central values. More precisely, they proved the following theorem:

Let R be a prime ring, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \geq 1$ such that $H(u^n)u^n + u^nG(u^n) \in C$, for all $u \in L$, then either there exists $a \in U$ such that $H(x) = xa, G(x) = -ax$, or R satisfies the standard identity s_4. Moreover, in the last case the structures of the maps G, H are obtained.

In [7], Chang *et al.* proved the following:

Let n be a fixed positive integer. Let R be a noncommutative $(n + 1)$!-torsion free prime ring. Suppose that there exist Jordan derivations D and $G : R \to R$ such that $D(x)x^n - x^nG(x) = 0$ for all $x \in R$. Then we have $D = 0$ and $G = 0$.

In the present paper, our aim is to study the same situation of derivations with left annihilator condition in a prime ring R that is, $a(D(x)x^n - x^nG(x)) = 0$ for all $x \in L$, where L is a noncentral Lie ideal of R.

Let Q be the Martindale quotient ring of a prime ring R and C the center of Q, which is called extended centroid of R. Note that Q is also a prime ring with C a field. We denote by $T = Q *_C C\{X\}$, the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of the noncommuting indeterminates $x_1, x_2, \ldots$. The elements of T are called generalized polynomial with coefficients in Q. Nontrivial generalized polynomial means a nonzero element of T. For more details about these objects we refer to [3, 14].

2. The Results on Prime Rings

First we fix the following remarks:

Remark 2.1 : Let R be a prime ring and L a noncentral Lie ideal of R. If char $(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $dim_C RC > 4$ i.e., char $(R) = 2$ and R does not satisfy s_4, then by [18, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either char $(R) \neq 2$ or R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Remark 2.2 : It is well known that each derivation of a prime ring R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q. We refer to [3, 20] for more details.

We begin with lemmas

Lemma 2.3 — Let n be a fixed positive integer, R be a prime ring with extended centroid C and $a, b, c \in R$. If $a \neq 0$ such that $a(b[x_1, x_2]^{n+1} - [x_1, x_2]b[x_1, x_2]^n - [x_1, x_2]^n c[x_1, x_2] + [x_1, x_2]^{n+1}c) = 0$ for all $x_1, x_2 \in R$, where $n \geq 1$ is a fixed integer, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $b, c \in C$.

PROOF : Assume that R does not satisfy any nontrivial GPI. If R is commutative, trivially R satisfies a nontrivial GPI which is a contradiction. So, R must be noncommutative. Let $T = Q *_C C\{X_1, X_2\}$, the free product of Q and $C\{X_1, X_2\}$, the free C-algebra in noncommuting indeterminates X_1 and X_2. Then, since $a(b[x_1, x_2]^{n+1} - [x_1, x_2]b[x_1, x_2]^n - [x_1, x_2]^n c[x_1, x_2] + [x_1, x_2]^{n+1}c) = 0$ is a GPI for R, we see that

$$a(b[X_1, X_2]^{n+1} - [X_1, X_2]b[X_1, X_2]^n - [X_1, X_2]^n c[X_1, X_2] + [X_1, X_2]^{n+1}c) = 0 \qquad (1)$$

in $T = Q *_C C\{X_1, X_2\}$. If $c \notin C$, then c and 1 are linearly independent over C. Thus, (1) implies

$$a([X_1, X_2]^{n+1}c) = 0$$

in T. This implies $c = 0$, a contradiction. Hence we conclude that $c \in C$. Then (1) reduces to

$$a(b[X_1, X_2] - [X_1, X_2]b)[X_1, X_2]^n = 0 \qquad (2)$$

in T. Again by the same way, this implies that $b \in C$. □

Lemma 2.4 — Let n be a positive integer, R be a noncommutative prime ring with extended centroid C and $a, b, c \in R$. Suppose that $a \neq 0$ such that $a(b[x_1, x_2]^{n+1} - [x_1, x_2]b[x_1, x_2]^n - [x_1, x_2]^n c[x_1, x_2] + [x_1, x_2]^{n+1}c) = 0$ for all $x_1, x_2 \in R$, where $n \geq 1$ is a fixed integer. Then one of the following holds:

1. $b, c \in C$, unless R satisfies s_4;

2. char $(R) \neq 2$, R satisfies s_4, n is even and $b - c \in C$;

3. char $(R) \neq 2$, R satisfies s_4, n is odd and $b + c \in C$;

4. char $(R) = 2$ and R satisfies s_4.

PROOF : By assumption, R satisfies generalized polynomial identity

$$g(x_1,x_2)=a(b[x_1,x_2]^{n+1}-[x_1,x_2]b[x_1,x_2]^n-[x_1,x_2]^nc[x_1,x_2]+[x_1,x_2]^{n+1}c). \qquad (3)$$

By Lemma 2.3, $b,c\in C$ which gives the conclusion (i) unless R satisfies a nontrivial GPI. Now, assume that R satisfies a nontrivial GPI. Since R and Q satisfy same generalized polynomial identities (see [9]), Q satisfies $g(x_1,x_2)$. Moreover, if C is infinite, we have $g(x_1,x_2)=0$ for all $x_1,x_2\in Q\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q\otimes_C\overline{C}$ are prime and centrally closed [12], we may replace R by Q or $Q\otimes_C\overline{C}$ according to C finite or infinite. Thus we may assume that $C=Z(R)$ and R is C-algebra centrally closed, which satisfies $g(x_1,x_2)=0$. By Martindale's theorem [22], R is then a primitive ring and hence is isomorphic to a dense ring of linear transformations of a vector space V over C.

Suppose that $\dim_C V\geq 3$.

We shall show that for any $v\in V$, v and cv are linearly C-dependent. Suppose that v and cv are linearly independent for some $v\in V$. Since $\dim_C V\geq 3$, there exists $u\in V$ such that v,cv,u are linearly C-independent set of vectors. By density, there exist $x_1,x_2\in R$ such that

$$x_1v=0,\ \ x_1cv=0,\ \ x_1u=cv;\ \ x_2v=cv,\ \ x_2cv=u,\ \ x_2u=0.$$

Then $[x_1,x_2]v=0$, $[x_1,x_2]cv=cv$ and so $0=a(b[x_1,x_2]^{n+1}-[x_1,x_2]b[x_1,x_2]^n-[x_1,x_2]^nc[x_1,x_2]+[x_1,x_2]^{n+1}c)v=acv$.

This implies that if $acv\neq 0$, by contradiction, we can say that v and cv are linearly C-dependent. Now choose $v\in V$ such that v and cv are linearly C-independent.

Then $acv=0$. Set $W=Span_C\{v,cv\}$. Let $ac\neq 0$. Then there exists $w\in V$ such that $acw\neq 0$ and then $ac(v-w)=-acw\neq 0$. By the previous argument we have that w,cw are linearly C-dependent and $(v-w),c(v-w)$ too. Thus there exist $\alpha,\beta\in C$ such that $cw=\alpha w$ and $c(v-w)=\beta(v-w)$. Then $cv=\beta(v-w)+cw=\beta(v-w)+\alpha w$ i.e., $(\alpha-\beta)w=cv-\beta v\in W$. Now $\alpha=\beta$ implies that $cv=\beta v$, a contradiction. Hence $\alpha\neq\beta$ and so $w\in W$.

Again, if $u\in V$ with $acu=0$ then $ac(w+u)\neq 0$. So, $w+u\in W$ forcing $u\in W$. Thus it is observed that $w\in V$ with $acw\neq 0$ implies $w\in W$ and $u\in V$ with $acu=0$ implies $u\in W$. This implies that $V=W$ i.e., $\dim_C V=2$, a contradiction. Hence we conclude that $ac=0$.

Again since v,cv,u are linearly C-independent set of vectors, by density, there exist $x_1,x_2\in R$ such that

$$x_1v=0,\ \ x_1cv=0,\ \ x_1u=v+cv;\ \ x_2v=cv,\ \ x_2cv=u,\ \ x_2u=0.$$

Then $[x_1, x_2]v = 0$, $[x_1, x_2]cv = v + cv$ and so $0 = a(b[x_1, x_2]^{n+1} - [x_1, x_2]b[x_1, x_2]^n - [x_1, x_2]^n c[x_1, x_2] + [x_1, x_2]^{n+1}c)v = av$. Since $a \neq 0$, by above argument this leads to a contradiction.

Hence, v and cv are linearly C-dependent for all $v \in V$. Thus for each $v \in V$, $cv = \alpha_v v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $cv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed. Now let $r \in R$, $v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c, r]v = 0$ for all $v \in V$ i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V, $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$.

Therefore, from (3) we have that R satisfies generalized polynomial identity

$$g(x_1, x_2) = a[b, [x_1, x_2]][x_1, x_2]^n = 0. \tag{4}$$

Let v and bv are linearly independent over C. Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, cv, w are linearly C-independent set of vectors. By density, there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \ \ x_1 bv = v, \ \ x_1 w = -v + 2bv; \ \ x_2 v = bv, \ \ x_2 bv = w, \ \ x_2 w = 0.$$

Then $[x_1, x_2]v = v$, $[x_1, x_2]bv = -v + bv$ and so $0 = a[b, [x_1, x_2]][x_1, x_2]^n v = av$. Then again by above arguments, since $a \neq 0$, this leads a contradiction. Hence we conclude that v and bv are linearly dependent for all $v \in V$, implying $b \in C$.

If $\dim_C V = 2$, then $R \cong M_2(C)$, that is, R satisfies s_4. If char $(R) = 2$, we get our conclusion (4). So let char $(R) \neq 2$. Now we use the fact $[x, y]^2 \in Z(M_2(C))$ for all $x, y \in M_2(C)$, and consider the following two cases:

- Let n be an even integer. Then we have from (3) that R satisfies

$$a[x_1, x_2]^n[b - c, [x_1, x_2]] = 0. \tag{5}$$

By Lemma 2.1 in [11], since char $(R) \neq 2$, we have $b - c \in C$, which is our conclusion (2).

- Let n be an odd integer. Then we have from (3) that R satisfies

$$a[x_1, x_2]^n[b + c, [x_1, x_2]] = 0. \tag{6}$$

By Lemma 2.1 in [11], since char $(R) \neq 2$, $b + c \in C$, which is our conclusion (3). □

***Theorem* 2.5** — *Let n be a fixed integer, R be a prime ring, D and G two derivations of R, L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(D(u)u^n - u^nG(u)) = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then one of the following holds:*

1. *$D = G = 0$, unless R satisfies s_4;*
2. *char $(R) \neq 2$, R satisfies s_4, n is even and $D = G$;*
3. *char $(R) \neq 2$, R satisfies s_4, n is odd and D and G are two inner derivations induced by b, c respectively such that $b + c \in C$;*
4. *char $(R) = 2$ and R satisfies s_4.*

PROOF : If char $(R) = 2$ and R satisfies s_4, we obtain our conclusion (4). So, let either char $(R) \neq 2$ or R does not satisfy s_4. Since L is a noncentral Lie ideal of R, by Remark 1, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence, by our assumption we have,

$$a(D([x_1, x_2])[x_1, x_2]^n - [x_1, x_2]^nG([x_1, x_2])) = 0 \tag{7}$$

for all $x_1, x_2 \in I$.

Now we divide the proof in the two cases:

Case I : Let D and G are both Q-inner derivations of R i.e., $D(x) = [b, x]$ for all $x \in R$ and $G(x) = [c, x]$ for all $x \in R$, where $b, c \in Q$. Then from (7), we obtain that

$$a(b[x_1, x_2]^{n+1} - [x_1, x_2]b[x_1, x_2]^n - [x_1, x_2]^nc[x_1, x_2] + [x_1, x_2]^{n+1}c) = 0 \tag{8}$$

for all $x_1, x_2 \in I$. By Chuang [9], this GPI is also satisfied by Q and hence by R. By Lemma 2.4, if R does not satisfy s_4, $b, c \in C$, that is, $D = G = 0$, which is our conclusion (1). If char $(R) \neq 2$ and R satisfies s_4, then by Lemma 2.4, $b + c \in C$ when n is odd and $b - c \in C$ when n is even. If n is even, then $b - c \in C$ implies $D = G$. Thus conclusions (2) and (3) are obtained.

Case II : Next assume that D and G are not both Q-inner derivations of R, but they are C-dependent modulo inner derivations of R. Suppose $D = \lambda G + ad_b$, that is, $D(x) = \lambda G(x) + [b, x]$ for all $x \in R$, where $\lambda \in C$, $b \in Q$. Then G cannot be an inner derivation of R. From (7), we obtain that I satisfies

$$a\Big(\lambda G([x_1, x_2])[x_1, x_2]^n + [b, [x_1, x_2]][x_1, x_2]^n - [x_1, x_2]^nG([x_1, x_2])\Big) = 0.$$

Since G is not inner derivation of R, by Kharchenko's theorem [17], we have that

$$a\Big(\lambda([u,x_2]+[x_1,v])[x_1,x_2]^n+[b,[x_1,x_2]][x_1,x_2]^n-[x_1,x_2]^n([u,x_2]+[x_1,v])\Big)=0. \quad (9)$$

for all $x_1, x_2, u, v \in I$. By Chuang [9], this GPI is also satisfied by Q and hence by R. In particular for $u = v = 0$, we have that R satisfies

$$a[b,[x_1,x_2]][x_1,x_2]^n = 0.$$

By Lemma 2.4, we get that $b \in C$. Hence, we get from (9) that R satisfies

$$a\Big(\lambda([u,x_2]+[x_1,v])[x_1,x_2]^n-[x_1,x_2]^n([u,x_2]+[x_1,v])\Big)=0. \quad (10)$$

Assuming $v = 0$ and $u = x_1$, we have that R satisfies

$$a(\lambda-1)[x_1,x_2]^{n+1}=0 \quad (11)$$

that is

$$a'[x_1,x_2]^{n+1}=0, \quad (12)$$

where $a' = a(\lambda - 1)$. For any two fixed $x, y \in R$, set $w = [x, y]^{n+1}$. Then $a'w = 0$. From (12), we can write $a'[u, wva']^{n+1} = 0$ for all $u, v \in R$. Since $a'w = 0$, it reduces to $a'(uwva')^{n+1} = 0$. This can be written as $(wva'u)^{n+2} = 0$ for all $u, v \in R$. By Levitzki's Lemma [15, Lemma 1.1], $wva'u = 0$ for all $u, v \in R$. Since R is prime, either $a' = 0$ or $w = 0$. If $w = 0$, then $[x, y]^{n+1} = 0$ for all $x, y \in R$. In this case by Herstein [13, Theorem 2], R is commutative, contradicting the fact that $0 \neq L$ is noncentral. Thus $a' = a(\lambda - 1) = 0$. Since $a \neq 0$, this implies $\lambda = 1$ and hence $D = G$. Now (10) gives

$$a\Big(([u,x_2]+[x_1,v])[x_1,x_2]^n-[x_1,x_2]^n([u,x_2]+[x_1,v])\Big)=0 \quad (13)$$

for all $u, v, x_1, x_2 \in R$. Since R is noncommutative, we may choose $p \in R$ such that $p \notin C$. Then replacing u with $[p, x_1]$ and v with $[p, x_2]$ in (13), we get

$$a\Big([p,[x_1,x_2]][x_1,x_2]^n-[x_1,x_2]^n[p,[x_1,x_2]]\Big)=0 \quad (14)$$

for all $x_1, x_2 \in R$. Then by Lemma 2.4, we conclude that since $p \notin C$, char $(R) \neq 2$, R satisfies s_4 and n is even. Thus conclusion (2) is obtained.

The situation when $G = \lambda D + ad_c$ is similar.

Next assume that D and G are C-independent modulo inner derivations of R. Since neither D nor G is inner, by Kharchenko's theorem [17], we have from (7) that I satisfies

$$a(([u,x_2]+[x_1,v])[x_1,x_2]^n-[x_1,x_2]^n([x,x_2]+[x_1,y]))=0. \tag{15}$$

In particular, I satisfies the blended component

$$a[u,x_2][x_1,x_2]^n=0. \tag{16}$$

By [9], this GPI is also satisfied by Q and hence by R. In particular, assuming $u=x_1$, we get R satisfies $a[x_1,x_2]^{n+1}=0$. Since R is noncommutative, as above this leads $a=0$, a contradiction. □

***Theorem* 2.6** — *Let n be a positive integer, R be a noncommutative prime ring with char $(R)\neq 2$, D and G two derivations of R. Suppose that there exists $0\neq a\in R$ such that $a(D(x)x^n-x^nG(x))=0$ for all $x\in R$, where $n\geq 1$ is a fixed integer. Then $D=G=0$.*

PROOF : By Theorem 2.5, we have only to consider the case when R satisfies s_4. If $D=G=0$, we are done. So, let one of them must be nonzero. By Theorem 2.5, we have the following two cases:

Case I : n is even and $D=G$.

Let D be Q-inner, that is $D(x)=[b,x]$ for all $x\in R$ and $b\in Q$. Then R satisfies

$$a(bx^{n+1}-xbx^n-x^nbx+x^{n+1}b)=0. \tag{17}$$

Since R satisfies s_4, there exists a field K such that $R\subseteq M_2(K)$ and $M_2(K)$ satisfies (l7). Since $M_2(F)$ is a dense ring of K-linear transformations over a vector space V, we assume that there exists $v\neq 0$, such that $\{v,bv\}$ is linear K-independent. By the density of $M_2(K)$, there exist $r\in M_2(K)$ such that $rv=0, rbv=bv$. Then

$$0=a(br^{n+1}-rbr^n-r^nbr+r^{n+1}b)v=abv.$$

Of course for any $u\in V$, $\{u,v\}$ linearly K-dependent implies $abu=0$. Let $ab\neq 0$. Then there exists $w\in V$ such that $abw\neq 0$ and so $\{w,v\}$ are linearly K-independent. Also $ab(w+v)=abw\neq 0$ and $ab(w-v)=abw\neq 0$. By the above argument, it follows that w and bw are linearly K-dependent, as are $\{w+v,b(w+v)\}$ and $\{w-v,b(w-v)\}$. Therefore there exist $\alpha_w,\alpha_{w+v},\alpha_{w-v}\in K$ such that

$$bw=\alpha_w w,\quad b(w+v)=\alpha_{w+v}(w+v),\quad b(w-v)=\alpha_{w-v}(w-v).$$

In other words we have

$$\alpha_w w+bv=\alpha_{w+v}w+\alpha_{w+v}v \tag{18}$$

and

$$\alpha_w w - bv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{19}$$

By comparing (18) with (19) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{20}$$

and

$$2bv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{21}$$

By (20), and since $\{w, v\}$ are K-independent and $char(K) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (21) it follows $2bv = 2\alpha_w v$. This leads a contradiction with the fact that $\{v, bv\}$ is linear K-independent.

Therefore, we conclude that $ab = 0$. Again, by the density of $M_2(K)$, there exist $r \in M_2(K)$ such that $rv = 0, rbv = v + bv$. Then

$$0 = a(br^{n+1} - rbr^n - r^n br + r^{n+1}b)v = av.$$

By above argument, this implies $a = 0$, a contradiction.

Let D be not Q-inner, then by Kharchenko's Theorem [17], R satisfies

$$a(yx^n - x^n y) = 0. \tag{22}$$

Thus $R \subseteq M_2(K)$ and $M_2(K)$ satisfies (22). Assuming $y = e_{12}, x = e_{11}$, we get $0 = -ae_{12}$, implying $a_{11} = a_{21} = 0$. Again, assuming $y = e_{21}, x = e_{11}$, we have $a_{22} = a_{12} = 0$, that is $a = 0$, a contradiction.

Case II : n is odd and $D(x) = [b, x], G(x) = [c, x]$ for all $x \in R$ with $b + c \in C$.

In this case by hypothesis, R satisfies

$$a(bx^{n+1} - xbx^n + x^n bx - x^{n+1}b) = 0. \tag{23}$$

By the above argument of case-I, this implies $a = 0$, a contradiction. □

3. The Results on Semiprime Rings and Banach Algebras

In this section we extended Theorem 2.6 to the semiprime ring. Let R be a semiprime ring and U be its left Utumi quotient ring. Then $C = Z(U)$ is the extended centroid of R ([8, p-38]). It is well known that any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U [20, Lemma 2]. Let $M(C)$ be the set of all maximal ideals of C.

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 — ([2], Lemma 1 and Theorem 1). Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU : P \in M(C)$ *with* U/PU $2-$*torsion free*$\} = 0$.

***Theorem* 3.2** — *Let R be a 2-torsion free noncommutative semiprime ring, U the left Utumi quotient ring of R, d a nonzero derivation of R and $0 \neq a \in R$. If $ad(R) \subseteq Z(R)$, then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.*

PROOF : Since any derivation d can be uniquely extended to a derivation in U, and U and R satisfy the same differential identities (see [20]), then $ad(x) \in Z(U)$, for all $x \in U$. Let $P \in M(C)$ such that U/PU is 2-torsion free. Note that U is also 2-torsion free semiprime ring. PU is a prime ideal of U, which is $d-$invariant by Lemma 3.1 denote $\overline{U} = U/PU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$. For any $\overline{x} \in \overline{U}$, $\overline{a}\overline{d}(\overline{x}) \in Z(\overline{U})$. Since $Z(\overline{U}) = (C + PU)/PU = C/PU$, $\overline{a}\overline{d}(\overline{x}) \in (C + PU)/PU = C/PU$, for all $\overline{x} \in \overline{U}$. Moreover $\overline{U}$ is a prime ring. Then it is clear that either $\overline{d} = 0$; or $\overline{a} = 0$ in $\overline{U}$; or $[\overline{U}, \overline{U}] = 0$. This implies that, for any $P \in M(C)$, either $d(U) \subseteq PU$; or $a \in PU$; or $[U, U] \subseteq PU$. In any case $ad(U)[U, U] \subseteq PU$ for any $P \in M(C)$. By Lemma 3.1, $\bigcap\{PU : P \in M(C) \ \text{ with } \ U/PU \ \ 2 - \text{torsion free}\} = 0$. Thus $ad(U)[U, U] = 0$.

By using the theory of orthogonal completion for semiprime rings (see, [3, Chapter 3]), it follows that there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative. □

***Theorem* 3.3** — *Let n be a positive integer, R be noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and $0 \neq a \in R$. Let d, g be two nonzero derivations of R such that $a(d(x)x^n - x^n g(x)) = 0$ for all $x \in R$. Then there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = g(e_1U)$, $e_2a = 0$, and e_3U is commutative.*

PROOF : Since any derivation d can be uniquely extended to a derivation in U, and U and R satisfy the same differential identities (see [20]), then $a(d(x)x^n - x^n g(x)) = 0$, for all $x \in U$. Let $P \in M(C)$ such that U/PU is 2-torsion free. Note that U is also 2-torsion free semiprime ring. By Lemma 3.1 PU is a prime ideal of U, which is $d-$invariant. Denote $\overline{U} = U/PU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$. For any $\overline{x} \in \overline{U}$, we get $\overline{a}(\overline{d}(\overline{x})\overline{x}^n - \overline{x}^n\overline{g}(\overline{x})) = 0$. Moreover $\overline{U}$ is a prime ring so by Theorem 2.6 we get either $\overline{d} = 0$ and $\overline{g} = 0$ or $[\overline{U}, \overline{U}] = 0$, or $\overline{a} = 0$. In any case we both have $ad(U)[U, U] \subseteq PU$ and $ag(U)[U, U] \subseteq PU$ for all $P \in M(C)$. By Lemma 3.1, $\bigcap\{PU$:

$P \in M(C)$ *with* U/PU $2-$ *torsion free*$\} = 0$. Then $ad(U)[U,U] = 0$ and $ag(U)[U,U] = 0$. In particular $ad(R)[R,R] = 0$ and $ag(R)[R,R] = 0$. These imply that $0 = ad(R)[R^2,R] = ad(R)R[R,R] + ad(R)[R,R]R = ad(R)R[R,R]$. In particular $ad(R)R[R,ad(R)] = 0$. Therefore, $[ad(R),R]R[ad(R),R] = 0$ and similarly $[ag(R),R]R[ag(R),R] = 0$. Since R is semiprime, we obtain that $ad(R) \subseteq Z(R)$ and $ag(R) \subseteq Z(R)$. Hence, we get $a(d-g)(R) \subseteq Z(R)$. By Theorem 3.2, we have the conclusion. □

Corollary 3.4 — Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and $0 \neq a \in R$. Let d be a nonzero derivation of R such that $a(d(x)x^n + x^n d(x)) = 0$ for all $x \in R$. Then there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.

By a Banach algebra, we shall mean a complex normed algebra A whose underlying vector space is Banach space. The Jacobson radical of A is the intersection of all primitive ideals of A and is denoted by $rad(A)$.

***Theorem* 3.5** — *Let A be a Banach algebra, d, g a pair of continuous Jordan derivations of A, $0 \neq a \in A$ and n a fixed positive integer. If $a(d(x)x^n - x^n g(x)) \in rad(A)$ for all $x \in A$, then $d(A) \subseteq rad(A)$ and $g(A) \subseteq rad(A)$.*

PROOF : Let P be any primitive ideal of A. Since d, g are both continuous, $d(P) \subseteq P$ and $g(P) \subseteq P$ by Lemma 3.2 in [24]. Then d and g can be induced to a pair of Jordan derivations on Banach algebra A/P as follows :

$$\overline{d}(\overline{x}) = d(x) + P \text{ and } \overline{g}(\overline{x}) = g(x) + P$$

for all $\overline{x} \in A/P$ and $x \in A$. Since P is a primitive ideal of A, quotient algebra A/P is prime and semisimple. On the other hand, we should remark that the pair of Jordan derivations $\overline{d}$ and $\overline{g}$ on A/P are also a pair of derivations on A/P by Brešar [5]. Moreover, by a result of Johnson and Sinclair [16] every derivation on a semisimple Banach algebra is continuous. Hence there are no nonzero continuous derivations on commutative semisimple Banach algebras. Hence, we get $\overline{d} = 0$ and $\overline{g} = 0$ when A/P is commutative. It remains to show that $\overline{d} = 0$ and $\overline{g} = 0$ in case of when A/P is noncommutative. By the hypothesis we have

$$\overline{a}(\overline{d}(\overline{x})\overline{x}^n - \overline{x}^n \overline{g}(x)) = 0$$

for all $\overline{x} \in A/P$. It is clear that Theorem 2.6 that $\overline{d} = 0$ and $\overline{g} = 0$. In any case both $\overline{d} = 0$ and $\overline{g} = 0$. These imply that $d(A) \subseteq P$ and $g(A) \subseteq P$ for any primitive ideal P of A and hence we get

$d(A) \subseteq rad(A)$ and $g(A) \subseteq rad(A)$. □

Corollary 3.6 — Let A a semisimple Banach algebra, d, g be a pair of Jordan derivations on A and $0 \neq a \in A$. If $a(d(x)x^n - x^n g(x)) = 0$ for all $x \in A$, where n is a fixed positive integer, then $d = 0$ and $g = 0$.

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