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ON THE BILAPLACIAN PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

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In this paper, we present a study for a nonlinear problem governed by the biharmonic equation in the plane. Using Green's formula, the problem is converted into a system of nonlinear integral equations for the unknown data of the boundary. Existence and uniqueness of the solution of the system of nonlinear boundary integral equations is established.

Key words : Boundary integral equation method; Pseudo-differential operator; Nonlinear boundary conditions.

1. INTRODUCTION

In the paper [7, 8], a nonlinear boundary conditions for the Laplace equation is considered. Using the boundary integral method the problem is converted into a nonlinear integral equation for the unknown data of the boundary. For studying the solvability of the nonlinear equation the authors give some assumption on the nonlinearity part.

The purpose of the present paper is to see the feasibility of extending the approach in [7, 8] for the Laplace equation to the biharmonic equation.

The biharmonic equations are an important class of equations in both physics and engineering. In fluid dynamics, the so-called stream function satisfies the biharmonic equation. Many problems in elasticity can also be formulated in terms of the biharmonic equation where the fundamental physical quantities such as displacement, stress, and strain all satisfy the biharmonic equation. There have been extensive research activities on the biharmonic equation both theoretically and computationally (see, for example, [5, 9]).

The paper is organized as follows: In Section 2, we formulate the nonlinear boundary problem and present some preliminary results for the boundary value problems for the biharmonic equation. Section 3 contains the core materials for the basic boundary integral equations. Theorems 2 in Section 3 is the main results concerning existence and uniqueness of the nonlinear system of boundary integral equations (11).

1.1 *Definitions and Notations*

Definition 1 [1, 3, 5] — Let $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \ \leq m\}.$$

Definition 2 [1, 3, 5, 10] — Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the Sobolev space :

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); (1+|\xi|^2)^{\frac{s}{2}}|F[u]| \in L^2(\mathbb{R}^n)\}.$$

and the associated norm:

$$\|u\|_{H^s} = (\int_{\mathbb{R}^n} (1+|\xi|^2)^s |F[u]|^2 d\xi)^{\frac{1}{2}}.$$

with $F[.]$ the Fourier transform.

Definition 3 [1, 3, 5] — Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\Gamma := \partial\Omega$, we defined

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^n)\}, s \in \mathbb{R}$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ L^2(\Gamma), \quad s = 0 \\ (H^{-s}(\Gamma))' \ \text{(dual space)}, s < 0. \end{cases}$$

2. Formulation of the Problem

In the present work, we study the biharmonic equation, which is accompanied with a nonlinear boundary condition

$$\begin{cases} \Delta^2 u = 0\ , \quad x \in \Omega \\ Mu + f(x, u, \frac{\partial u}{\partial n}) = f_0(x)\ ,\ x \in \Gamma \\ Nu + g(x, u, \frac{\partial u}{\partial n}) = g_0(x)\ , \quad x \in \Gamma. \end{cases} \tag{1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^2$. We denote by $n = (n_1, n_2)$ the unit outward normal vector to Γ. The given functions $f_0 \in H^{-\frac{1}{2}}(\Gamma)$ and $g_0 \in H^{-\frac{3}{2}}(\Gamma)$ are defined on Γ, and

$f : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$, $g : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma)$, in general are some nonlinear mapping. The boundary operators M, N and Δ^2 are defined by

$$\Delta^2 u = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} + \nu\frac{\partial^2 u}{\partial x_2^2}\right) + 2(1-\nu)\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 u}{\partial x_2^2} + \nu\frac{\partial^2 u}{\partial x_1^2}\right)$$

$$Mu = \nu\Delta u + (1-\nu)\left[\frac{\partial^2 u}{\partial x_1^2}n_1^2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2}n_2^2\right] \tag{2}$$

$$Nu = -\frac{\partial}{\partial n}\Delta u + (1-\nu)\frac{d}{ds}\left\{\left(\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right)n_1 n_2 - \frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2)\right\} \tag{3}$$

Here ν is the poisson ratio, a real constant, and in application especially in the theory of elasticity we have $0 \leq \nu < 1$. The normal and tangential derivatives are given by

$$\frac{\partial}{\partial n} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2}$$

and

$$\frac{\partial}{\partial s} = -n_1\frac{\partial}{\partial x_2} + n_2\frac{\partial}{\partial x_1}$$

physically, Mu is the bending moment and Nu is the transverse force consisting of the shear force and twisting moment. For $u \in H^2(\Omega)$ we have that $u|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$ and $\frac{\partial u}{\partial n}|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$. Now integration by parts leads to the first Green formula in the form

$$\int_\Omega (\Delta^2 uv dx = a(u,v) - \int_\Gamma \left\{\frac{\partial v}{\partial n}Mu + vNu\right\}ds, \qquad \text{x} \in \Omega \tag{4}$$

where the bilinear form $a(u,v)$ is defined by

$$a(u,v) = \int_\Omega \left\{\nu\Delta u\Delta v + (1-\nu)\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\right)\right\}dx. \tag{5}$$

We note that the bilinear form $a(.,.)$ is well defined for functions in $H^2(\Omega)$. Now let $u \in H^2(\Omega, \triangle^2)$ where

$$H^2(\Omega, \triangle^2) = \{u \in H^2(\Omega); \triangle^2 u \in \widetilde{H}^{-2}(\Omega)\}$$

with $\widetilde{H}^{-2}(\Omega)$ denoting the dual space of $H^2(\Omega)$ and choose $v \in H^2(\Omega)$. Then the above Green formula holds and by duality argument one shows that $Mu \in H^{-\frac{1}{2}}(\Gamma)$ and $Nu \in H^{-\frac{3}{2}}(\Gamma)$ are well defined, where $H^{-\frac{1}{2}}(\Gamma)$ and $H^{-\frac{3}{2}}(\Gamma)$ are the dual spaces of $H^{\frac{1}{2}}(\Gamma)$ and $H^{\frac{3}{2}}(\Gamma)$, respectively.

This paper is mainly devoted to the formulation of nonlinear system of integral equations at the boundary related to (1) and we prove the existence and uniqueness.

3. Integral Equations Method

3.1 *Representative formulas and boundary operators*

For reformulating the problem (1) as a system of nonlinear boundary integral equations, we start with the Green representation formula of a weak solution in $H^2(\Omega)$

$$u(x) = \mathbb{V}(Mu, Nu)(x) - \mathbb{W}(u, \frac{\partial u}{\partial n})(x) \ \ , x \in \Omega \tag{6}$$

in term of simple and double layer potentiels. Here

$$\mathbb{V} : H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{3}{2}}(\Gamma) \to H^2(\Omega) \ \text{ and } \mathbb{W} : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^2(\Omega)$$

are continuous operators defined by

$$\mathbb{V}(Mu, Nu)(x) = \int_\Gamma \left\{ E(x,y)Nu(y) + \frac{\partial E}{\partial n_y}(x,y)Mu(y) \right\} ds_y, \ \ x \in \mathbb{R}^2 \setminus \Gamma,$$

$$\mathbb{W}(u, \frac{\partial u}{\partial n})(x) = \int_\Gamma \left\{ M_y E(x,y)\frac{\partial u}{\partial n}(y) + N_y E(x,y)u(y) \right\} ds_y, \ \ x \in \mathbb{R}^2 \setminus \Gamma$$

where

$$E\,(x,y) := \frac{1}{8\pi}\,|x-y|^2 \log|x-y| \tag{7}$$

is the fundamental solution of the biharmonic equation.

Letting $x \to \Gamma$ from inside Ω, and following the standard procedure in potential theory involving jump relations, we obtain the following integral equations on Γ.

$$u(x) = \int_\Gamma \left\{ E(x,y)Nu(y) + \left(\frac{\partial E}{\partial n_y}(x,y)Mu(y) \right) \right\} ds_y + \frac{1}{2}u(x) \tag{8}$$

$$- \int_\Gamma \left\{ (M_y E(x,y))\frac{\partial u}{\partial n_y} + (N_y E(x,y))u(y) \right\} ds_y$$

$$\frac{\partial u}{\partial n} = \int_\Gamma \left\{ \frac{\partial E(x,y)}{\partial n_x} Nu(y) + \left(\frac{\partial^2 E}{\partial n_x \partial n_y}(x,y)Mu(y) \right) \right\} ds_y + \frac{1}{2}\frac{\partial u}{\partial n}(x) \tag{9}$$

$$- \int_\Gamma \left\{ (\frac{\partial}{\partial n_x} M_y E(x,y))\frac{\partial u}{\partial n_y} + (\frac{\partial}{\partial n_x} N_y E(x,y))u(y) \right\} ds_y$$

In present time, in order to formulate the integral equations, we define the following operators at the boundary:

Definition 4 — Let $u \in C^{\infty}(\Gamma)$. We define the following operators for $x \in \Gamma$:

$$\begin{aligned}
K_{11}u(x) &= \int_{\Gamma} N_y E(x,y)u(y)ds_y \\
V_{12}\partial_n u(x) &= -\int_{\Gamma} M_y E(x,y)\frac{\partial u}{\partial n_y}ds_y \\
D_{21}u(x) &= \int_{\Gamma} \frac{\partial}{\partial n_x} N_y E(x,y)u(y)ds_y \\
K_{22}\partial_n u(x) &= \partial_n V_{12}\partial_n u(x) \\
V_{13}Mu(x) &= \int_{\Gamma} \frac{\partial E}{\partial n_y}(x,y)Mu(y)ds_y \\
V_{14}Nu(x) &= \int_{\Gamma} E(x,y)Nu(y)ds_y \\
V_{23}Mu(x) &= \partial_n V_{13}Mu(x) \\
V_{24}Nu(x) &= \partial_n V_{14}Nu(x)
\end{aligned}$$

The mapping properties of the integrals operators are collected in the following lemma.

Lemma 1 [3, 4, 5, 9] — The operators defined by :

$$\begin{aligned}
K_{11} &: H^s(\Gamma) \longrightarrow H^s(\Gamma), \;\; D_{21} : H^s(\Gamma) \longrightarrow H^{s-1}(\Gamma) \\
V_{12} &: H^s(\Gamma) \longrightarrow H^{s+1}(\Gamma), \;\; V_{13} : H^s(\Gamma) \longrightarrow H^{s+3}(\Gamma) \\
V_{14} &: H^s(\Gamma) \longrightarrow H^{s+3}(\Gamma), \;\; V_{23} : H^s(\Gamma) \longrightarrow H^{s+1}(\Gamma) \\
V_{24} &: H^s(\Gamma) \longrightarrow H^{s+3}(\Gamma), \;\; K_{22} : H^s(\Gamma) \longrightarrow H^s(\Gamma)
\end{aligned}$$

are continuous.

3.2 *Representation of the problem* (1) *as integral equations at the boundary* Γ

We consider $u \in H^2(\Omega, \Delta^2)$ satisfying the boundary conditions of the problem (1).
If we introduce in the system (8)(9) the given functions and the unknown functions on Γ such that:

$$\left\{ \begin{array}{l} u(x) = v(x) \quad \text{(unknown function)} \\ \partial_n u(x) = w(x) \quad \text{(unknown function)} \end{array} \right. , x \in \Gamma$$

and

$$\left\{ \begin{array}{l} Mu(x) = -f(x, u, \frac{\partial u}{\partial n}) + f_0(x), \; x \in \Gamma \\ Nu(x) = -g(x, u, \frac{\partial u}{\partial n}) + g_0(x), \; x \in \Gamma \end{array} \right. \tag{10}$$

the equation (8)and (9) may be written as

$$\left\{ \begin{array}{l} (\frac{1}{2}I + K_{11})v - V_{12}w + V_{14}g(x,v,w) + V_{13}f(x,v,w) = V_{14}g_0 + V_{13}f_0 \\ -D_{21}v + (\frac{1}{2}I - K_{22})w + V_{24}g(x,v,w) + V_{23}f(x,v,w) = V_{24}g_0 + V_{23}f_0. \end{array} \right.$$

This system of equations can be written in a matrix form as follows

$$\begin{bmatrix} \frac{1}{2}I + K_{11} & -V_{12} \\ -D_{21} & \frac{1}{2}I - K_{22} \end{bmatrix} \begin{bmatrix} v(x) \\ w(x) \end{bmatrix} + \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix} \begin{bmatrix} g(x,v,w) \\ f(x,v,w) \end{bmatrix}$$

$$= \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix} \begin{bmatrix} g_0(x) \\ f_0(x)) \end{bmatrix}$$

$$L(U) + V(F(U)) = V(F_0) \text{ on } \Gamma \tag{11}$$

Where

$$L = \begin{bmatrix} \frac{1}{2}I + K_{11} & -V_{12} \\ -D_{21} & \frac{1}{2}I - K_{22} \end{bmatrix}, V = \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix}, F = \begin{bmatrix} g(x,v,w) \\ f(x,v,w) \end{bmatrix}$$

and

$$U = \begin{pmatrix} v \\ w \end{pmatrix}, F_0 = \begin{pmatrix} g_0 \\ f_0 \end{pmatrix}.$$

3.3 *Solvability of the nonlinear system of integral equations (11)*

For studying the solvability of the nonlinear equation (11), we give some assumptions to be made here.

(**H1**) The functions $f(.,.,.)$ and $g(.,.,.)$ are a Caratheodory functions.

(**H2**) We assume that for all $x \in \Gamma$,

$$f(x,.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\,,\, g(x,.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

are differentiable and the derivatives are bounded satisfying

$$0 < \alpha \leq \frac{\partial f}{\partial v} \leq l_1 < +\infty\,,\, |\frac{\partial f}{\partial w}| \leq \beta$$
$$0 < \alpha \leq \frac{\partial g}{\partial w} \leq l_2 < +\infty\,,\, |\frac{\partial g}{\partial v}| \leq \beta$$

for some constants α, β and l_i, $i = 1, 2$ with $\alpha > \beta$.

Remark 1 : The functions $f(.,.,.)$ and $g(.,.,.)$ are a Caratheodory functions (**H1**) (i.e.) $f(.,v,w)$ and $f(.,v,w)$ are measurable for all $(v,w) \in \mathbb{R} \times \mathbb{R}$ and $f(x,.,.), g(x,.,.)$ are continuous for almost all $x \in \Gamma$.

Remark 2 : The assumption (**H2**) implies that the Nemytski operator

$$F : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Gamma) \times L^2(\Gamma)$$

is Lipschitz continuous and strongly monotonous.

From the Lagrange's mean value theorem, there exists ζ_1, ζ_2 such that:

$$\begin{aligned} f(v,w) - f(v',w') &= (f(v,w) - f(v',w)) + (f(v',w) - f(v',w')) \\ &= \frac{\partial f(\zeta_1, w)}{\partial v}(v - v') + \frac{\partial f(v', \zeta_2)}{\partial w}(w - w') \end{aligned}$$

then

$$\begin{aligned} (v - v')(f(v,w) - f(v',w')) &= \frac{\partial f(\zeta_1, w)}{\partial v}(v - v')^2 + \frac{\partial f(v', \zeta_2)}{\partial w}(w - w')(v - v') \\ &\geq \alpha \left|v - v'\right|^2 - \beta(v - v').(w - w') \\ &\geq \alpha \left|v - v'\right|^2 - \frac{1}{2}\beta \left|v - v'\right|^2 - \frac{1}{2}\beta \left|w - w'\right|^2 \end{aligned}$$

such that

$$a.b \leq \frac{1}{2} |a|^2 + \frac{1}{2} |b|^2 .$$

In the same manner one can show that

$$(w - w')(g(v,w) - g(v',w')) \geq -\frac{1}{2}\beta \left|w - w'\right|^2 - \frac{1}{2}\beta \left|v - v'\right|^2 + \alpha \left|w - w'\right|^2 .$$

Finally combining these inequalities we obtain

$$\begin{aligned} (U - U', F(U) - F(U')) &= ((v - v'), f(v,w) - f(v',w')) \\ &\quad +((w - w'), g(v,w) - g(v',w')) \\ &\geq (\alpha - \beta)(\left\|v - v'\right\|^2 + \left\|w - w'\right\|^2) \\ &\geq (\alpha - \beta) \left\|U - U'\right\|_0^2 \end{aligned}$$

with $U = (v,w), U' = (v',w') \in L^2(\Gamma) \times L^2(\Gamma)$. Hence the operator $F(U)$ is strongly monotonous.

For the continuity of the Nemytski operator $F(U)$

$$\begin{aligned} |f(v,w) - f(v',w')| &= |(f(v,w) - f(v',w)) + (f(v',w) - f(v',w'))| \\ &= |\frac{\partial f(\zeta_1, w)}{\partial v}(v - v') + \frac{\partial f(v', \zeta_2)}{\partial w}(w - w')| \\ &\leq l_1 \left|v - v'\right| + \beta \left|w - w'\right| \\ &\leq Max\{l_1, \beta\}(\left|v - v'\right| + \left|w - w'\right|) \\ &\leq l_1(\left|v - v'\right| + \left|w - w'\right|) \end{aligned}$$

then we obtain

$$\left\| f(U) - f(U') \right\| \quad \leq \quad l_1 \left\| U - U' \right\|_0 .$$

In the same manner one can show that

$$\left\| g(U) - g(U') \right\| \leq l_2 \left\| U - U' \right\|_0$$

then we get

$$\left\| F(U) - F(U') \right\|_0 \leq (l_1 + l_2) \left\| U - U' \right\|_0$$

which proves that the operator F is Lipschitz continuous for $U = (v, w), U' = (v', w') \in L^2(\Gamma) \times L^2(\Gamma)$

Based on this property we can consider the solvability of (11)

***Theorem* 2** — *Let assumptions (**H1**) and (**H2**) hold. Then, for every* $F_0 = (g_0, f_0) \in H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ *there exists a unique* $U = (v, w) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ *such that*

$$L(U) + V(F(U)) = V(F_0) \text{ on } \Gamma.$$

PROOF : The proof follows from the well-known theorem by Browder and Minty on monotone operators [2, 6].

Since the simple layer potential operator on Γ

$$V : H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$$

is an isomorphism it is sufficient to consider the unique solvability of equation

$$TU := V^{-1}LU + FU = F_0 \text{ on } \Gamma. \tag{12}$$

We shall prove that the operator

$$T : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous and strongly monotonous.

i - in the first we show that T is continuous:

It is clear from the continuity of the mapping properties of the simple layer operator V and L by the Lemma 1, that

$$V^{-1}L : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous. And from (H2)

$$F : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous. Hence the boundary integral operator

$$T : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous.

ii - In the second we show that T is strongly monotonous operator.

Let $\mu = (\mu_1, \mu_2) \in H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\mu(x) := V^{-1}LU(x)$$

for all $U(x) = (v, w) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, is the $(M\varphi, N\varphi)$ of the biharmonic function

$$\begin{aligned}\varphi(x) &= \mathbb{V}(M\varphi(x), N\varphi(x)) - \mathbb{W}(\varphi(x), \frac{\partial\varphi}{\partial n}) \\ &= \mathbb{V}\mu(x) - \mathbb{W}U(x)\end{aligned}$$

for $x \in \Omega$, this means that $\varphi(x)$ satisfies the problem

$$\begin{cases} \Delta^2\varphi(x) = 0 \quad , \quad x \in \Omega \\ \varphi(x) = v \ \ , \ \ x \in \Gamma \\ \frac{\partial\varphi}{\partial n} = w \ \ , \ \ x \in \Gamma \end{cases}$$

Then Green's theorem yields

$$\begin{aligned}(V^{-1}LU, U) &= \int_\Gamma \mu_1 v ds + \int_\Gamma \mu_2 w ds \\ &= \int_\Gamma N\varphi\varphi ds + \int_\Gamma M\varphi\frac{\partial\varphi}{\partial n} ds \\ &= a(\varphi, \varphi).\end{aligned}$$

Hence, the linearity of $V^{-1}L$ implies that for all $U, U' \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$

$$\begin{aligned}(V^{-1}L(U - U'), U - U') &= a(\varphi - \varphi', \varphi - \varphi') \\ &= \int_\Omega \left\{ \nu(\Delta(\varphi - \varphi'))^2 + (1 - \nu)\sum_{i,j=1}^{2}\left(\frac{\partial^2(\varphi - \varphi')}{\partial x_i \partial x_j}\right)^2 \right\} dx.\end{aligned}$$

In an other hand, we have

$$\|\varphi - \varphi'\|_{H^2(\Omega)} \le c \|U - U'\|_{H^{-\frac{3}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)} \le c \|U - U'\|_0$$

Hence

$$\begin{aligned}
(TU - TU', U - U') &= (FU - FU', U - U') + (V^{-1}L(U - U'), U - U') \\
&\ge (\alpha - \beta) \|U - U'\|_0^2 + a(\varphi - \varphi', \varphi - \varphi') \\
&\ge (\alpha - \beta) \|U - U'\|_0^2 \\
&\ge \frac{1}{c^2}(\alpha - \beta) \|\varphi - \varphi'\|_{H^2(\Omega)}^2 \\
&\ge \frac{1}{c^2}(\alpha - \beta) \|U - U'\|_{H^{\frac{3}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)}^2
\end{aligned}$$

by the trace theorem [3, 5]. Which completes the proof. ■

Summarizing the above analysis we have proved the following result.

***Theorem* 3** — *Let $f_0 \in H^{-\frac{1}{2}}(\Gamma)$, and $g_0 \in H^{-\frac{3}{2}}(\Gamma)$ be given. Assume that the conditions (**H1**) and (**H2**) are satisfied, then the problem (1) has an unique weak solution $u \in H^2(\Omega)$.*

PROOF Finding weak solution $u \in H^2(\Omega)$ of problem (1) is composed in three steps. Firstly, a system of nonlinear boundary integral equations (11) is solved for $(u, \frac{\partial u}{\partial n})$ on the boundary Γ. In the next step, boundary differential operators (Mu, Nu) are deduced from formula (10). The last step consists on determining $u(x)$ at any point $x \in \Omega$ by formula (6). ■

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