Indian J. Pure Appl. Math., **47**(3): 501-521, September 2016

DOI: 10.1007/s13226-016-0200-9

RIEMANN PROBLEM IN NON-IDEAL GAS DYNAMICS

K. Ambika and R. Radha

School of Mathematics and Statistics, University of Hyderabad, Hyderabad 46, *India*
e-mails: ambikakohirkar@gmail.com, repakar@yahoo.com

(*Received* 13 *November* 2014; *after final revision* 22 *July* 2015;
accepted 18 *December* 2015)

In this paper we consider the Riemann problem for gas dynamic equations governing a one dimensional flow of van der Waals gases. The existence and uniqueness of shocks, contact discontinuities, simple wave solutions are discussed using R-H conditions and Lax conditions. The explicit form of solutions for shocks, contact discontinuities and simple waves are derived. The effects of van der Waals parameter on the shock and simple waves are studied. A condition is derived on the initial data for the existence of a solution to the Riemann problem. Moreover, a necessary and sufficient condition is derived on the initial data which gives the information about the existence of a shock wave or a simple wave for a 1-family and a 3-family of characteristics in the solution of the Riemann problem.

Key words : Van der Waals gas; shock wave; simple wave; contact discontinuity; Riemann problem.

1. Introduction and Preliminaries

The Riemann problem for a system of conservation laws in gas dynamics, magnetodynamics, shallow water theory etc. has attracted considerable attention of the researchers in connection with the theoretical and numerical aspects of its solution. The first existence theorem for solutions to the initial value problem for nonlinear hyperbolic systems of equations was given by Glimm [3] in his fundamental paper.

The Riemann problem corresponding to the shock-tube problem for Euler equations, which is a basic physical problem in gas dynamics, was given by Courant and Friedrichs [2].

Lax [7] solved the Riemann problem with left and right initial data separated at the origin, where the left and right initial data were chosen sufficiently close to each other.

The Riemann problem for Euler equations does not admit a solution expressible in a closed form even for ideal gases. This has motivated several researchers to develop iterative schemes to determine the different waves issuing from an initial discontinuity. Two methods were proposed by Godunov [4] based on a fixed point scheme. A different approach was given by Smoller [15] in which for a given initial data, a solution is derived in the implicit form for an ideal gas. The Riemann problem was attempted by Colella [1] and Saurel [12] for real gases using Riemann solvers. Roe [11] developed a Riemann solver for an ideal polytropic gas. The Riemann problem for van der Waals isothermal fluid was attempted by Hattori [6]. Shearer [13] solved the Riemann problem uniquely for conservation laws of mixed type in a class of admissible solutions. The shock tube problem was solved by Guardone and Vigevano [5] for van der Waals gas with an approximate Riemann solver, using an extention of Roe linearisation procedure from ideal gas to van der waals gas. Quartapelle *et al.*, [8] discussed the Riemann problem for a polytropic gas (based on the energy function) using Hugoniot curves.

The Riemann problem and the elementary wave interactions of an isentropic system in magnetogasdynamics were studied by Raja Sekhar and Sharma [9, 10] in which they have shown that the presence of magnetic field makes both the shock and the rarefaction stronger as compared to the situation in the absence of the magnetic field. Chun Shen [14] proved that the limiting solutions of the Riemann problem for isentropic magnetogasdynamics equations converge to the corresponding solution of the transport equation in the absence of pressure and magnetic fields.

The main motivation of the present work is to study the existence and uniqueness of shocks, contact discontinuity, simple wave solutions for van der Waals gases and to give a solution to the Riemann problem. R-H conditions and Lax conditions are derived for the system representing gas dynamics equations for van der Waals gases through which the solution for shock waves, simple waves and contact discontinuities are discussed. The Riemann problem is considered for an arbitrary initial data and a condition is derived on Riemann data for the existence of a solution either in terms of shocks or simple waves or both. In continuation, the interaction of two weak shocks are studied and analyzed.

Consider the gas dynamics equations governing the continuous motion of a one dimensional flow in a van der Waals gas in the following form

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \\ &\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = 0, \\ &\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3 + \rho u e + p u\right) = 0. \end{aligned} \tag{1.1}$$

Here ρ is the fluid density, p the pressure, u the velocity and $e = \dfrac{p(1-b\rho)}{(\gamma-1)\rho}$ the internal energy in van der Waals gas, with b being the van der Waals gas parameter related to the omitted volume of the gas. The system (1.1) admits three characteristic wave fronts whose speeds $\dfrac{dx}{dt}$ are $\lambda_1 = u - F$, $\lambda_2 = u$ and $\lambda_3 = u + F$ such that $\lambda_1 < \lambda_2 < \lambda_3$ where $F(p,\rho) = \sqrt{\dfrac{\gamma p}{\rho(1-b\rho)}}$ is the sound speed where γ is the constant, and these three characteristic wave fronts carry the information from the given data.

2. R-H Conditions

As a natural property of the hyperbolic systems, the system (1.1) admits shocks as a consequence of intersection of characteristic curves of the same family.

If a shock, say a 1-shock (respectively; a 3-shock), is to be introduced due to the interaction of the characteristics of the family with speed $\dfrac{dx}{dt} = \lambda_1 = u - F$ (respectively; $\dfrac{dx}{dt} = \lambda_3 = u + F$), then the characteristics left of the shock $u_\ell - F_\ell$ (respectively; $u_\ell + F_\ell$) and right of the shock $u_r - F_r$ (respectively; $u_r + F_r$) satisfy Lax [7] conditions. Thus we have $s < u_\ell - F_\ell,\ u_r - F_r < s < u_r$ (respectively; $u_\ell < s < u_\ell + F_\ell,\quad u_r + F_r < s$) where s is the speed of a 1-shock (respectively, a 3-shock).

Thus we have for a 1-shock

$$s - u_\ell < -F_\ell < 0,\ \ -F_r < s - u_r < 0, \tag{2.1}$$

and for a 3-shock

$$0 < s - u_\ell < F_\ell,\ \ 0 < F_r < s - u_r. \tag{2.2}$$

In view of the sound speed $F(p,\rho)$ being positive, the Lax conditions (2.1)-(2.2) lead to the following conditions for a 1-shock

$$(s-u_\ell)^2 > F_\ell^2, \qquad (s-u_r)^2 < F_r^2, \tag{2.3}$$

and for a 3-shock

$$(s-u_\ell)^2 < F_\ell^2, \qquad (s-u_r)^2 > F_r^2. \tag{2.4}$$

Across a shock curve, the flow variables ρ_ℓ, u_ℓ and p_ℓ left of the shock and ρ_r, u_r and p_r right of the shock are related by the R-H conditions for the system (1.1) as follows:

$$\begin{aligned}
&s\left(\rho_\ell-\rho_r\right) = \left(\rho_\ell u_\ell - \rho_r u_r\right),\\
&s\left(\rho_\ell u_\ell-\rho_r u_r\right) = \left(p_\ell + \rho_\ell u_\ell^2 - p_r - \rho_r u_r^2\right),\\
&s\left(\frac{1}{2}\rho_\ell u_\ell^2 + \rho_\ell e_\ell - \frac{1}{2}\rho_r u_r^2 - \rho_r e_r\right) = \left(\frac{1}{2}\rho_\ell u_\ell^3 + \rho_\ell u_\ell e_\ell + p_\ell u_\ell - \frac{1}{2}\rho_r u_r^3 - \rho_r u_r e_r - p_r u_r\right),
\end{aligned}$$

where s is the shock speed, $e_\ell = \dfrac{p_\ell(1-b\rho_\ell)}{(\gamma-1)\rho_\ell}$ and $e_r = \dfrac{p_r(1-b\rho_r)}{(\gamma-1)\rho_r}$. Thus

$$\begin{aligned}
&\rho_\ell\left(s-u_\ell\right) - \rho_r\left(s-u_r\right) = 0,\\
&p_\ell - p_r - \rho_r\left(s-u_r\right)\left(u_\ell-u_r\right) = 0,\\
&\frac{1}{2}\left(s-u_\ell\right)^2 - \frac{1}{2}\left(s-u_r\right)^2 + \frac{p_\ell(1-b\rho_\ell)}{(\gamma-1)\rho_\ell} - \frac{p_r(1-b\rho_r)}{(\gamma-1)\rho_r} + \frac{p_\ell}{\rho_\ell} - \frac{p_r}{\rho_r} = 0.
\end{aligned} \tag{2.5}$$

The above system involves seven variables, namely, ρ_r, u_r, p_r, ρ_ℓ, u_ℓ, p_ℓ and the shock speed s.

2.1 *1-Shock Wave*

From the relations (2.5), it follows that

$$s = u_\ell \pm F_\ell\sqrt{\frac{(\gamma-1)\left(\beta p_r + p_\ell\right)}{2\gamma p_\ell}}, \tag{2.6a}$$

$$\frac{\rho_r}{\rho_\ell} = \frac{(\beta p_r + p_\ell)}{\left(1+\frac{2b\rho_\ell}{\gamma-1}\right)p_r + \left(\beta - \frac{2b\rho_\ell}{\gamma-1}\right)p_\ell}, \tag{2.6b}$$

$$u_r = u_\ell \pm \sqrt{\frac{2}{\gamma(\gamma-1)}}\frac{F_\ell(1-b\rho_\ell)(p_r-p_\ell)}{\sqrt{(\beta p_r + p_\ell)p_\ell}}, \tag{2.6c}$$

where $F_\ell = \sqrt{\dfrac{\gamma p_\ell}{\rho_\ell(1-b\rho_\ell)}}$, $F_r = \sqrt{\dfrac{\gamma p_r}{\rho_r(1-b\rho_r)}}$ and $\beta = (\gamma+1)/(\gamma-1)$.

In view of (2.1) and (2.3), the equation (2.6a), implies that $s - u_\ell < 0$, $p_r > p_\ell$. Hence as a consequence of (2.6b) and (2.6c) we have $\rho_r > \rho_\ell$ and $u_r < u_\ell$. Thus we introduce a variable $\xi < 0$ such that $p_r/p_\ell = \mathrm{e}^{-\xi}$. Then the solution for the 1-shock is in the following form

$$\frac{p_r}{p_\ell} = \mathrm{e}^{-\xi}, \quad \xi < 0, \tag{2.7a}$$

$$s_1 = u_\ell - F_\ell \sqrt{\frac{(\gamma - 1)\left(\beta \mathrm{e}^{-\xi} + 1\right)}{2\gamma}}, \tag{2.7b}$$

$$\frac{\rho_r}{\rho_\ell} = \frac{\left(\beta \mathrm{e}^{-\xi} + 1\right)}{\beta + \mathrm{e}^{-\xi} - \frac{2b\rho_\ell}{\gamma - 1}\left(1 - \mathrm{e}^{-\xi}\right)}, \tag{2.7c}$$

$$u_r = u_\ell + F_\ell \sqrt{\frac{2}{\gamma(\gamma - 1)}} \frac{\left(1 - b\rho_\ell\right)\left(1 - \mathrm{e}^{-\xi}\right)}{\sqrt{\left(\beta \mathrm{e}^{-\xi} + 1\right)}}, \tag{2.7d}$$

where $s = s_1$ is the speed of the 1-shock wave. Thus if u_ℓ, p_ℓ, ρ_ℓ and u_r, p_r, ρ_r satisfy Lax conditions (2.1) for the 1-shock, then the jump conditions are given by equations (2.7a) – (2.7d). Conversely, when $p_r > p_\ell$, if the jump conditions are given by the equations (2.7b) – (2.7d) then the 1- shock satisfies Lax conditions (2.1), i.e., for the 1-shock, the jump conditions (2.7b) – (2.7d) hold when $p_r > p_\ell$ if and only if Lax conditions (2.1) hold. Hence, a discontinuity is a 1-shock wave if and only if $p_r > p_\ell$.

Observe that, as the van der Waals gas parameter b increases, the sound speed increases. As a result, both the density ρ_r and the speed of the 1-shock s_1 decrease and the velocity u_r increases compared to the corresponding density, shock speed and velocity in an ideal gas (i.e., $b = 0$) for a given u_ℓ, ρ_ℓ and p_ℓ.

2.2 *Contact discontinuity*

Rewriting the relations (2.5), we have

$$p_r = p_\ell, \quad u_r = u_\ell, \quad s = u_\ell = u_r \text{ and } \rho_r \neq \rho_\ell. \tag{2.8}$$

Hence, by introducing a variable ξ, we have

$$s_2 = u_r = u_\ell, \tag{2.9a}$$

$$p_r = p_\ell, \tag{2.9b}$$

$$u_r = u_\ell, \tag{2.9c}$$

$$\rho_r = \rho_\ell \mathrm{e}^{\xi}, \quad -\infty < \xi < \infty, \tag{2.9d}$$

where $s = s_2$ is the speed of contact discontinuity.

2.3 3-*Shock wave*

The relations (2.5) can be rewritten as

$$s = u_r \pm F_r \sqrt{\frac{(\gamma-1)\,(\beta p_\ell + p_r)}{2\gamma p_r}} \tag{2.10a}$$

$$\frac{\rho_\ell}{\rho_r} = \frac{(\beta p_\ell + p_r)}{\left(1+\frac{2b\rho_r}{\gamma-1}\right) p_\ell + \left(\beta - \frac{2b\rho_r}{\gamma-1}\right) p_r}, \tag{2.10b}$$

$$u_\ell = u_r \pm \sqrt{\frac{2}{\gamma(\gamma-1)}}\,\frac{F_r(1-b\rho_r)(p_\ell - p_r)}{\sqrt{(\beta p_\ell + p_r)p_r}}. \tag{2.10c}$$

In view of (2.2) and (2.4), the equation (2.10*a*) implies that $p_r < p_\ell$ and $s - u_r > 0$. Hence, as a consequence of (2.10*b*) and (2.10*c*), we have $\rho_\ell > \rho_r$ and $u_r < u_\ell$. Thus we introduce a variable $\xi \leq 0$ such that $p_\ell/p_r = \mathrm{e}^{-\xi}$. Then the solution for the 3-shock is in the following form

$$\frac{p_\ell}{p_r} = \mathrm{e}^{-\xi}, \quad \xi < 0, \tag{2.11a}$$

$$s_3 = u_r + F_r \sqrt{\frac{(\gamma-1)\,(\beta + \mathrm{e}^{\xi})}{2\gamma \mathrm{e}^{\xi}}} \tag{2.11b}$$

$$\frac{\rho_\ell}{\rho_r} = \frac{(\beta + \mathrm{e}^{\xi})}{(1+\beta \mathrm{e}^{\xi}) + \frac{2b\rho_r}{\gamma-1}\,(1-\mathrm{e}^{\xi})}, \tag{2.11c}$$

$$u_\ell = u_r + F_r \sqrt{\frac{2}{\gamma(\gamma-1)}}\,\frac{(1-b\rho_r)(1-\mathrm{e}^{\xi})}{\sqrt{(\beta+\mathrm{e}^{\xi})\mathrm{e}^{\xi}}}, \tag{2.11d}$$

where $s = s_3$ is the speed of the 3-shock wave. Thus, if u_ℓ, p_ℓ, ρ_ℓ and u_r, p_r, ρ_r satisfy Lax conditions (2.2) for the 3-shock, then the jump conditions are given by equations (2.11*a*) – (2.11*d*). Conversely, when $p_\ell > p_r$, if the jump conditions are given by the equations (2.11*b*)–(2.11*d*), then the 3- shock satisfies the Lax conditions (2.2), i.e. for the 3-shock, jump conditions (2.11*b*) – (2.11*d*) when $p_\ell > p_r$ hold if and only if Lax conditions (2.2) hold. Hence, a discontinuity is a 3-shock wave if and only if $p_\ell > p_r$.

Observe that as the van der Waals parameter b increases, both the density ρ_ℓ and the speed of the 3-shock s_3 increase, and the velocity u_ℓ decreases compared to the corresponding density, shock speed and velocity in an ideal gas (i.e., $b = 0$) for a given u_r, ρ_r and p_r.

3. Simple Wave Solutions

To determine all the possible simple wave solutions of the system (1.1), we consider the density, velocity and pressure of a gas to be of the form $\rho = R(\sigma)$, $u = U(\sigma)$ and $p = P(\sigma)$ respectively, where R, U and P are functions of a single variable σ which depends on x and t. Thus the system (1.1), on simplification, reduces to

$$\begin{aligned} &\frac{dR}{d\sigma}\left(\frac{\partial\sigma}{\partial t} + U\frac{\partial\sigma}{\partial x}\right) + R\frac{dU}{d\sigma}\frac{\partial\sigma}{\partial x} = 0, \\ &\frac{dU}{d\sigma}\left(\frac{\partial\sigma}{\partial t} + U\frac{\partial\sigma}{\partial x}\right) + \frac{1}{R}\frac{dP}{d\sigma}\frac{\partial\sigma}{\partial x} = 0, \\ &\frac{dP}{d\sigma}\left(\frac{\partial\sigma}{\partial t} + U\frac{\partial\sigma}{\partial x}\right) + \frac{\gamma P}{(1-bR)}\frac{dU}{d\sigma}\frac{\partial\sigma}{\partial x} = 0. \end{aligned} \tag{3.1}$$

If $\frac{dR}{d\sigma}$ is non zero, then the above system can be written as

$$\begin{aligned} &\left(-\frac{dU}{d\sigma}R\frac{dU}{d\sigma} + \frac{1}{R}\frac{dR}{d\sigma}\frac{dP}{d\sigma}\right)\frac{\partial\sigma}{\partial x} = 0, \\ &\left(-\frac{dP}{d\sigma}R\frac{dU}{d\sigma} + \frac{\gamma P}{1-bR}\frac{dR}{d\sigma}\frac{dU}{d\sigma}\right)\frac{\partial\sigma}{\partial x} = 0, \end{aligned} \tag{3.2}$$

which leads to

$$\begin{aligned} &\frac{dP}{d\sigma} = \frac{\gamma P}{R(1-bR)}\frac{dR}{d\sigma}, \\ &\left(\frac{dU}{d\sigma}\right)^2 = \frac{\gamma P}{R^3(1-bR)}\left(\frac{dR}{d\sigma}\right)^2, \end{aligned} \tag{3.3}$$

or

$$\frac{dP}{d\sigma} = \frac{dU}{d\sigma} = 0. \tag{3.4}$$

Without loss of generality, one can choose $R(\sigma) = \sigma = \rho$. By solving the equations $(3.3)_{1,2}$, we get

$$\begin{aligned} &p = P(\rho) = C_1\left(\frac{\rho}{1-b\rho}\right)^{\gamma}, \\ &u = U(\rho) = C_2 \pm \frac{2}{\gamma-1}\left(\frac{\gamma p(1-b\rho)}{\rho}\right)^{1/2}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Observe that $p/P(\rho)$, $u + U(\rho)$ (respectively, $u - U(\rho)$) are the invariant solutions, i.e., constant solutions of the system (1.1) along the characteristics $\frac{dx}{dt} = \lambda_1$ (respectively, λ_3). In fact, the term $p/P(\rho)$ is equal to the entropy of the system (1.1) for the following reasons. Since $p = \rho RT/(1-b\rho)$ is the equation of state (T being the temperature and R being the universal gas constant) and the internal energy is $e = p(1-b\rho)/((\gamma-1)\rho)$, we have from the thermodynamical equation $T\mathrm{d}S = \mathrm{d}e + p\mathrm{d}\left(\frac{1}{\rho}\right)$ that $S = c_V \log(pP(\rho)) + \text{constant}$.

In view of $p_\ell = P(\rho_\ell)$ and $u_\ell = U(\rho_\ell)$, the simple wave solution can be rewritten as

$$p = p_\ell \left(\frac{\rho(1-b\rho_\ell)}{\rho_\ell(1-b\rho)}\right)^{\gamma}, \tag{3.5}$$
$$u = u_\ell \pm \frac{2}{\gamma-1}\sqrt{\frac{\gamma p_\ell(1-b\rho_\ell)}{\rho_\ell}}\left\{1-\left(\frac{\rho(1-b\rho_\ell)}{\rho_\ell(1-b\rho)}\right)^{\frac{\gamma-1}{2}}\right\},$$

Since, for a 1-simple wave solution, the Lax conditions $u_\ell - F_\ell \le u_r - F_r$ are to be satisfied, we have, in view of (3.5) that

$$u_r - u_\ell = \frac{2}{\gamma-1}\sqrt{\frac{\gamma p_\ell(1-b\rho_\ell)}{\rho_\ell}}\left\{1-\left(\frac{\rho_r(1-b\rho_\ell)}{\rho_\ell(1-b\rho_r)}\right)^{\frac{\gamma-1}{2}}\right\},$$

where $F^2(p,\rho) = \gamma p/(\rho(1-b\rho))$, $F_r = F(p_r,\rho_r)$, $F_\ell = F(p_\ell,\rho_\ell)$ with $p_r = P(\rho_r)$, $p_\ell = P(\rho_\ell)$ and

$$F_r - F_\ell = \sqrt{\frac{\gamma p_\ell}{\rho_\ell(1-b\rho_\ell)}}\left\{\left(\frac{\rho_r}{\rho_\ell}\right)^{\frac{\gamma-1}{2}}\left(\frac{1-b\rho_\ell}{1-b\rho_r}\right)^{\frac{\gamma+1}{2}} - 1\right\}.$$

Thus, the Lax condition $F_r - F_\ell \le u_r - u_\ell$ for the 1-simple wave implies that

$$\left\{\left(\frac{\rho_r}{\rho_\ell}\right)^{\frac{(\gamma-1)}{2}}\left(\frac{1-b\rho_\ell}{1-b\rho_r}\right)^{\frac{(\gamma+1)}{2}} - 1\right\} \le \frac{2(1-b\rho_\ell)}{\gamma-1}\left\{1-\left(\frac{\rho_r(1-b\rho_\ell)}{\rho_\ell(1-b\rho_r)}\right)^{\frac{(\gamma-1)}{2}}\right\},$$

which can be rewritten as

$$\left(\frac{\rho_r}{\rho_\ell}\frac{(1-b\rho_\ell)}{(1-b\rho_r)}\right)^{(\gamma+1)/2} \le \left(\frac{\gamma+1-2b\rho_\ell}{\gamma+1-2b\rho_r}\right)\frac{\rho_r}{\rho_\ell}.$$

Let $\delta = \rho_r/\rho_\ell$. Then the above inequality reduces to the form

$$\left(\frac{(1-b\rho_\ell)}{(\delta^{-1}-b\rho_\ell)}\right)^{(\gamma+1)/2} \le \left(\frac{\gamma+1-2b\rho_\ell}{(\gamma+1)\delta^{-1}-2b\rho_\ell}\right).$$

The above inequality implies that $\delta \le 1$, i.e., $\rho_r \le \rho_\ell$. As a consequence, from (3.5), we obtain $p_r \le p_\ell$. Thus, for a 1-simple wave, the Lax condition $u_\ell - F_\ell \le u_r - F_r$ holds if and only if $\rho_r \le \rho_\ell$ or $p_r \le p_\ell$.

Similarly, one can conclude that the Lax condition $u_\ell + F_\ell \le u_r + F_r$ holds for a 3-simple wave if and only if $\rho_\ell \le \rho_r$ or $p_\ell \le p_r$.

Introducing $\xi \ge 0$ for the 1-simple wave such that $p_r/p_\ell = \mathrm{e}^{-\xi}$, we rewrite the simple wave solution for ρ_r and u_r in terms of ξ, ρ_ℓ and u_ℓ as follows.

$$\frac{p_r}{p_\ell} = \mathrm{e}^{-\xi}, \quad \xi \ge 0 \tag{3.6a}$$

$$\frac{\rho_r}{\rho_\ell} = \frac{\mathrm{e}^{-\xi/\gamma}}{1 - b\rho_\ell\left(1 - \mathrm{e}^{-\xi/\gamma}\right)}, \tag{3.6b}$$

$$u_r = u_\ell + \frac{2F_\ell(1 - b\rho_\ell)}{\gamma - 1}\left(1 - \mathrm{e}^{-(\gamma-1)\xi/2\gamma}\right). \tag{3.6c}$$

Similarly, introducing $\xi \ge 0$ for the 3-simple wave such that $p_\ell/p_r = \mathrm{e}^{-\xi}$, we rewrite the simple wave solution for ρ_r and u_r in terms of ξ, ρ_ℓ and u_ℓ as follows

$$\frac{p_\ell}{p_r} = \mathrm{e}^{-\xi}, \tag{3.7a}$$

$$\frac{\rho_\ell}{\rho_r} = \frac{\mathrm{e}^{-\xi/\gamma}}{1 - b\rho_r\left(1 - \mathrm{e}^{-\xi/\gamma}\right)}, \tag{3.7b}$$

$$u_\ell = u_r + \frac{2F_r(1 - b\rho_r)}{\gamma - 1}\left(\mathrm{e}^{-(\gamma-1)\xi/2\gamma} - 1\right). \tag{3.7c}$$

Observe that as the van der Waals parameter b increases, for the 1-simple wave, the density ρ_r increases and the velocity u_r decreases, where as for the 3-simple wave, the density ρ_ℓ and the velocity u_ℓ increase as compared to the corresponding density and velocity in an ideal gas (i.e., $b = 0$).

4. Riemann Problem

Consider the initial data at $t = 0$ for the system (1.1) as

$$(\rho, u, p)|_{t=0} = \begin{cases} (\rho_b, u_b, p_b), & \text{if } x < 0, \\ (\rho_a, u_a, p_a), & \text{if } x \ge 0. \end{cases} \tag{4.1}$$

The initial discontinuity in the flow variables (ρ, u, p) originating at $x = 0$ propagates into the medium for $t > 0$. As a result, there exist four constant regions for any time $t > 0$ be connected by either shocks or simple waves or contact discontinuity. Let these four regions from left to right be named as region-1, region-2, region-3 and region-4 in which the flow variables are constant. Let the flow variables be given by ρ_1, u_1, p_1 in region-1, by ρ_2, u_2,

p_2 in region-2, by ρ_3, u_3, p_3 in region-3 and by ρ_4, u_4, p_4 in region-4 respectively. Observe that the left most trailing characteristic wave front originated at $x = 0$ propagates into the medium given by ρ_b, u_b, p_b, i.e., $\rho_1 = \rho_b$, $u_1 = u_b$ and $p_1 = p_b$. The region-2, next to region-1, is connected by either a 1-shock (solution given in equations (2.7)) or a 1-simple wave (solution given in equations (3.6)). Hence, there exists $\xi = \xi_1$ such that

$$\rho_2 = \rho_1 f(\xi_1, \rho_1), \tag{4.2a}$$

$$u_2 = u_1 + F_1 g(\xi_1, \rho_1), \tag{4.2b}$$

$$p_2 = p_1 \mathrm{e}^{-\xi_1}, \tag{4.2c}$$

where $F_1 = \sqrt{\dfrac{\gamma p_1}{\rho_1(1-b\rho_1)}}$,

$$f(\xi_1, \rho_1) = \begin{cases} \dfrac{(\beta \mathrm{e}^{-\xi_1} + 1)}{\beta + \mathrm{e}^{-\xi_1} - \frac{2b\rho_1}{\gamma-1}\left(1 - \mathrm{e}^{-\xi_1}\right)}, & \text{if } \xi_1 < 0 \\ \dfrac{\mathrm{e}^{-\xi_1/\gamma}}{1 - b\rho_1\left(1 - \mathrm{e}^{-\xi_1/\gamma}\right)}, & \text{if } \xi_1 \geq 0 \end{cases}$$

$$g(\xi_1, \rho_1) = \begin{cases} \sqrt{\dfrac{2}{\gamma(\gamma-1)}} \dfrac{(1-b\rho_1)\left(1-\mathrm{e}^{-\xi_1}\right)}{\sqrt{(\beta \mathrm{e}^{-\xi_1} + 1)}}, & \text{if } \xi_1 < 0 \\ \dfrac{2(1-b\rho_1)}{\gamma - 1}\left(1 - \mathrm{e}^{-(\gamma-1)\xi_1/2\gamma}\right), & \text{if } \xi_1 \geq 0 \end{cases}$$

Observe that, region-1 is left to either a 1-shock wave or a 1-simple wave and region-2 is right to either a 1-shock wave or a 1-simple wave. Thus to obtain the above solution, equations (2.7) or equations (3.7) are used in which $(\rho_\ell, u_\ell, p_\ell)$ is replaced by (ρ_1, u_1, p_1) and (ρ_r, u_r, p_r) is replaced by (ρ_2, u_2, p_2). The region-2 and region-3 are connected by a contact discontinuity (solution given in equations (2.9)) as follows.

$$\rho_3 = \rho_2 \mathrm{e}^{\xi_2}, \tag{4.3a}$$

$$u_3 = u_2, \tag{4.3b}$$

$$p_3 = p_2. \tag{4.3c}$$

Observe that, here also, region-2 is to the left to the contact discontinuity and region-3 is to the right to the contact discontinuity. Thus, to obtain the solution (4.3), equations (2.9) are used in which $(\rho_\ell, u_\ell, p_\ell)$ is replaced by (ρ_2, u_2, p_2) and (ρ_r, u_r, p_r) is replaced by (ρ_3, u_3, p_3). Similarly, region-3 and region-4 are connected by either a 3-shock (solution given in equations (2.11)) or a 3-simple wave (solution given in equations (3.7)). Hence there exists $\xi = \xi_3$ such that

$$\rho_3 = \rho_4 \hat{f}(\xi_3, \rho_4), \tag{4.4a}$$

$$u_3 = u_4 + F_4 \hat{g}(\xi_3, \rho_4), \tag{4.4b}$$

$$p_3 = p_4 \mathrm{e}^{-\xi_3}, \tag{4.4c}$$

where $F_4 = \sqrt{\dfrac{\gamma p_4}{\rho_4(1-b\rho_4)}}$,

$$\hat{f}(\xi_3, \rho_4) = \begin{cases} \dfrac{(\beta + \mathrm{e}^{\xi_3})}{(1+\beta \mathrm{e}^{\xi_3}) + \frac{2b\rho_4}{\gamma-1}(1-\mathrm{e}^{\xi_3})}, & \text{if } \xi_3 < 0 \\ \dfrac{\mathrm{e}^{-\xi_3/\gamma}}{1 - b\rho_4\left(1-\mathrm{e}^{-\xi_3/\gamma}\right)}, & \text{if } \xi_3 \geq 0 \end{cases}$$

$$\hat{g}(\xi_3, \rho_4) = \begin{cases} \sqrt{\dfrac{2}{\gamma(\gamma-1)}}\dfrac{(1-b\rho_4)(1-\mathrm{e}^{\xi_3})}{\sqrt{(\beta+\mathrm{e}^{\xi_3})\mathrm{e}^{\xi_3}}}, & \text{if } \xi_3 < 0 \\ \dfrac{2(1-b\rho_4)}{\gamma-1}\left(\mathrm{e}^{-(\gamma-1)\xi_3/2\gamma} - 1\right), & \text{if } \xi_3 \geq 0 \end{cases}$$

Observe that, here also, region-3 is to the left to either a 3-shock wave or a 3-simple wave and region-3 is to the right to either a 3-shock wave or a 3-simple wave. Thus, to obtain the solution (4.4), equations (2.11) are used in which $(\rho_\ell, u_\ell, p_\ell)$ is replaced by (ρ_3, u_3, p_3) and (ρ_r, u_r, p_r) is replaced by (ρ_4, u_4, p_4). The leading characteristic of the system is propagating in the constant state $\rho = \rho_a$, $u = u_a$ and $p = p_a$ which is the 4-state, i.e., $\rho_4 = \rho_a$, $u_4 = u_a$ and $p_4 = p_a$. Thus in view of equations (4.2), (4.3) and (4.4), the following equations hold.

$$\rho_4 \hat{f}(\xi_3, \rho_4) = \rho_1 f(\xi_1, \rho_1)\mathrm{e}^{\xi_2}, \tag{4.5a}$$

$$u_4 + F_4 \hat{g}(\xi_3, \rho_4) = u_1 + F_1 g(\xi_1, \rho_1), \tag{4.5b}$$

$$p_4 \mathrm{e}^{-\xi_3} = p_1 \mathrm{e}^{-\xi_1}. \tag{4.5c}$$

Since $(\rho_1, u_1, p_1) = (\rho_b, u_b, p_b)$ and $(\rho_4, u_4, p_4) = (\rho_a, u_a, p_a)$, the above equations (4.5) are rewritten as

$$\frac{\rho_a}{\rho_b} = \frac{f(\xi_1, \rho_b)\mathrm{e}^{\xi_2}}{\hat{f}(\xi_3, \rho_a)}, \tag{4.6a}$$

$$\frac{p_a}{p_b} = \mathrm{e}^{-\xi_1+\xi_3}, \tag{4.6b}$$

$$(u_a - u_b) = F_b g(\xi_1, \rho_b) - F_a \hat{g}(\xi_3, \rho_a), \tag{4.6c}$$

where $F_a = \sqrt{\dfrac{\gamma p_a}{\rho_a(1 - b\rho_a)}}$ and $F_b = \sqrt{\dfrac{\gamma p_b}{\rho_b(1 - b\rho_b)}}$.

The Riemann problem for the system (1.1) subject to (4.1) has been determined in terms ξ_1, ξ_2 and ξ_3 satisfying the conditions (4.6). Since $g(\xi_1, \rho_b)$ (respectively, $\hat{g}(\xi_3, \rho_a)$) is an increasing (respectively, decreasing) function with respect to ξ_1 (respectively, ξ_3) for a fixed ρ_b (respectively, ρ_a) and it is bounded above (respectively, below) by $2(1 - b\rho_b)/(\gamma - 1)$ (respectively, $-2(1 - b\rho_a)/(\gamma - 1)$), the condition for the Riemann problem to the system (1.1) subject to (4.1) is given in terms ρ_a, ρ_b, u_a, u_b, p_a and p_b as follows

$$u_a - u_b < \frac{2}{\gamma - 1}\left(F_b(1 - b\rho_b) + F_a(1 - b\rho_a)\right). \tag{4.7}$$

Hence, we proved the following thoerem.

***Theorem* 1** — *The solution for a Riemann problem to the equations of gas dynamics with the given initial data (1.1) and (4.1) for a polytropic van der Waals gas is given by (4.2), (4.3) and (4.4) if the initial data is satisfied by*

$$u_a - u_b < \frac{2}{\gamma - 1}\left(F_b(1 - b\rho_b) + F_a(1 - b\rho_a)\right).$$

Now, we give a simple corollary which suggests the possibilities for shocks or simple waves to occur in a 1-family and a 3-family.

Corollary — Consider the solution (4.2), (4.3) and (4.4) of the Riemann problem for the system (1.1) satisfying in the initial data (4.1) subject to the inequality (4.7). Then the following are true:

(I) The 1-component of the solution is a simple wave if and only if

$$\begin{aligned} -F_a \;\; \hat{g}\left(\log\frac{p_a}{p_b}, \rho_a\right) &< u_a - u_b \\ &< \frac{2}{\gamma - 1}\left(F_b(1 - b\rho_b) + F_a(1 - b\rho_a)\right). \end{aligned}$$

Otherwise, it is a shock.

(II) The 3-component of the solution is a simple wave if and only if

$$\begin{aligned} F_b \;\; g\left(-\log\frac{p_a}{p_b}, \rho_b\right) &< u_a - u_b \\ &< \frac{2}{\gamma - 1}\left(F_b(1 - b\rho_b) + F_a(1 - b\rho_a)\right). \end{aligned}$$

Otherwise, it is a shock.

Proof : From the Theorem 1, we can see that the inequalities on the right must hold in both (I) and (II).

Now we consider a 1-family. In view of (4.6b), we rewrite the equation (4.6c) in terms of ξ_1 as

$$(u_a - u_b) = F_b\, g(\xi_1, \rho_b) - F_a\, \hat{g}(\xi_1 + \log\frac{p_a}{p_b}, \rho_a).$$

To prove the left inequality of (I), we define

$$\phi(x) = F_b\, g(x, \rho_b) - F_a\, \hat{g}(x + \log\frac{p_a}{p_b}, \rho_a)$$

and observe that $\phi(0) = -F_a\, \hat{g}\left(\log\frac{p_a}{p_b}, \rho_a\right)$ and $\phi'(x) > 0$. Thus, replacing x by ξ_1, in view of $u_a - u_b = \phi(\xi_1)$ being an increasing function for ξ_1, where $\xi_1 \geq 0$ (respectively; $\xi_1 < 0$) represents a 1-simple wave (respectively; a 1-shock wave), we conclude that $u_a - u_b > \phi(0)$ (respectively; $u_a - u_b < \phi(0)$) for a 1-simple wave (respectively; a 1- shock wave). Hence, a 1-wave is the simple wave if and only if

$$u_a - u_b > -F_a\, \hat{g}\left(\log\frac{p_a}{p_b}, \rho_a\right).$$

Otherwise it is a shock.

In view of (4.6b), we rewrite the equation (4.6c) in terms of ξ_3 as

$$(u_a - u_b) = F_b\, g(\xi_3 - \log\frac{p_a}{p_b}, \rho_b) - F_a\, \hat{g}(\xi_3, \rho_a).$$

To prove the left inequality of (II), we define

$$\psi(x) = F_b\, g(x - \log\frac{p_a}{p_b}, \rho_b) - F_a \hat{g}(x, \rho_a)$$

and observe that $\psi(0) = F_b\, g\left(-\log\frac{p_a}{p_b}, \rho_b\right)$ and $\psi'(x) > 0$. Thus, replacing x by ξ_3, in view of $u_a - u_b = \phi(\xi_3)$ being an increasing function for ξ_3 where $\xi_3 \geq 0$ (respectively; $\xi_3 < 0$) represents a 3- simple wave (respectively; a 3-shock wave), we conclude that $u_a - u_b > \psi(0)$ ($u_a - u_b < \psi(0)$) for a 3- simple wave (respectively; a 3- shock wave). Hence, a 3-wave is a simple wave if and only if

$$u_a - u_b > F_b\, g\left(-\log\frac{p_a}{p_b}, \rho_b\right).$$

Otherwise, it is a shock. □

Table 1: Initial data for Riemann problem

Test	ρ_b	u_b	p_b	ρ_a	u_a	p_a
data-1	1.0	0.0	1.0	0.125	0.0	0.1
data-2	0.96	1.0833	2.8333	1.7741	1.1187	4.0
data-3	1.0	-2.0	0.4	1.0	2.0	0.4
data-4	5.99924	19.5975	460.894	5.9924	-6.19633	46.095

Table 2: Solution to the Riemann problem for $b = 0$ and $\gamma = 1.4$, for data in Table 1

Test	ρ_2	u_2	p_2	ρ_3	u_3	p_3
data-1	0.42632	0.92745	0.30313	0.26557	0.92745	0.30313
data-2	1.06518	0.86951	3.27765	1.53884	0.86951	3.27764
data-3	0.02185	0.00000	0.00189	0.02185	0.00000	0.00189
data-4	14.2823	8.68977	1691.65	31.0426	8.68977	1691.65

5. Interaction of Shock Waves

Theorem 2 — *The interaction of the two weak 3-shocks (respectively, 1-shocks) leads to a 1-simple wave (respectively, a 3-simple wave) if* $\gamma < 5/3$ *and to a 1-shock (respectively, a 3-shock) if* $\gamma > 5/3$.

Proof : Let us consider the three regions (i.e., region-1, region-2 and region-3) which are separated by the states (ρ_b, u_b, p_b) , (ρ_m, u_m, p_m) and (ρ_a, u_a, p_a). Assume that the regions-1 and 2 and regions-2 and 3 are separated by 3-shocks. Now we need to solve the Riemann problem (1.1) with the data (ρ_b, u_b, p_b) and (ρ_a, u_a, p_a) at the time of interaction.

Since, the region-1 is connected to the region-2 with a 3-shock, there exists a parameter $\eta(\leq 0)$ such that ρ_b, u_b and p_b are given by equations (2.11) with $\xi = \eta$, $\rho_r = \rho_m$, $u_r = u_m$

Table 3: Solution to the Riemann problem for $b = 0.05$ and $\gamma = 1.4$, for data in Table 1

Test	ρ_2	u_2	p_2	ρ_3	u_3	p_3
data-1	0.43506	0.91271	0.29934	0.26172	0.91271	0.29934
data-2	1.05795	0.87807	3.27002	1.5548	0.87807	3.27002
data-3	0.01975	0.00000	0.00153	0.01975	0.00000	0.00153
data-4	10.8403	8.16593	2216.43	13.9208	8.16593	2216.43

Table 4: Solution to the Riemann problem for $b = 0.1$ and $\gamma = 1.4$, for data in Table 1

Test	ρ_2	u_2	p_2	ρ_3	u_3	p_3
data-1	0.44433	0.89737	0.29537	0.25786	0.89737	0.29537
data-2	1.05074	0.88709	3.26127	1.57117	0.88709	3.26127
data-3	0.01759	0.00000	0.00121	0.01759	0.00000	0.00121
data-4	8.37217	7.59328	3511.02	8.92873	7.59328	3511.02

and $p_r = p_m$, i.e.,

$$\begin{aligned} \rho_b &= \rho_m \hat{f}(\eta, \rho_m), \\ u_b &= u_m + F_m \hat{g}(\eta, \rho_m), \\ p_b &= p_m e^{-\eta}, \end{aligned} \tag{5.1}$$

where $F_m = \sqrt{\dfrac{\gamma p_m}{\rho_m(1 - b\rho_m)}}$.

Similarly, as the region-2 is connected to the region-3 with a 3-shock, there exists a parameter $\xi(\leq 0)$ such that ρ_m, u_m and p_m are given by equations (2.11) with $\rho_b = \rho_m$, $u_b = u_m$ and $p_b = p_m$, i.e.,

$$\begin{aligned} \rho_m &= \rho_a \hat{f}(\xi, \rho_a), \\ u_m &= u_a + F_a \hat{g}(\xi, \rho_a), \\ p_m &= p_a e^{-\xi}, \end{aligned} \tag{5.2}$$

where $F_a = \sqrt{\dfrac{\gamma p_a}{\rho_a(1 - b\rho_a)}}$.

Using ρ_m, u_m and p_m from the equations (5.1) in (5.2), we get

$$\begin{aligned}
\rho_b &= \rho_a \frac{(\beta+e^{\xi})(\beta+e^{\eta}-\frac{2b\rho_b}{\gamma-1}(1-e^{\eta}))}{(1+\beta e^{\eta})(1+\beta e^{\xi}+\frac{2b\rho_a}{\gamma-1}(1-e^{\xi}))},\\
p_b &= p_a e^{-(\xi+\eta)},\\
u_b &= u_a + \left(\frac{2}{\gamma(\gamma-1)}\right)^{1/2} F_a\Big(\frac{(1-b\rho_a)(1-e^{\xi})}{\sqrt{e^{\xi}(\beta+e^{\xi})}}+\\
&\qquad \sqrt{\frac{(1-b\rho_a)(1-b\rho_a\hat{f}(\xi,\rho_a))}{e^{\xi+\eta}(\beta+e^{\eta})\hat{f}(\xi,\rho_a)}}(1-e^{\eta})\Big)
\end{aligned} \tag{5.3}$$

As a result of interaction of these two shocks, the resulting wave could be a 1-simple wave if $u_a - u_b > -F_a\hat{g}\left(log\dfrac{p_a}{p_b}, \rho_a\right)$, otherwise it is the 1-shock.

In view of the equation (5.3), the inequality reduces to

$$H(\xi) + H(\eta)K(\xi) < H(\xi+\eta), \tag{5.4}$$

Where

$$H(x) = \frac{1-e^{x}}{\sqrt{e^{x}(\beta+e^{x})}}$$

and

$$K(\xi) = \sqrt{\frac{1-b\rho_a\hat{f}(\xi,\rho_a)}{e^{\xi}(1-b\rho_a)\hat{f}(\xi,\rho_a)}}$$

The above inequality can be written as a Taylor series expansion for small values ξ and η. Thus we get

$$\frac{1}{2}\xi^2\eta(H'(0)K''(0) - H^{'''}(0)) + \frac{1}{2}\xi\eta^2(H''(0)K'(0) - H^{'''}(0)) + O(\xi^n\eta^m) < 0,$$

which gives

$$\xi\eta(\xi+\eta)\left(\frac{\beta(\beta-4)}{8(1+\beta)^{5/2}}\right) + O(\xi^n\eta^m) < 0.$$

Where $n+m \geq 4$. Since $\xi < 0$ and $\eta < 0$ and for very small values of ξ and η, the above inequality holds when $\beta(\beta-4) > 0$, i.e., $\beta > 4$ which implies $\gamma < 5/3$.

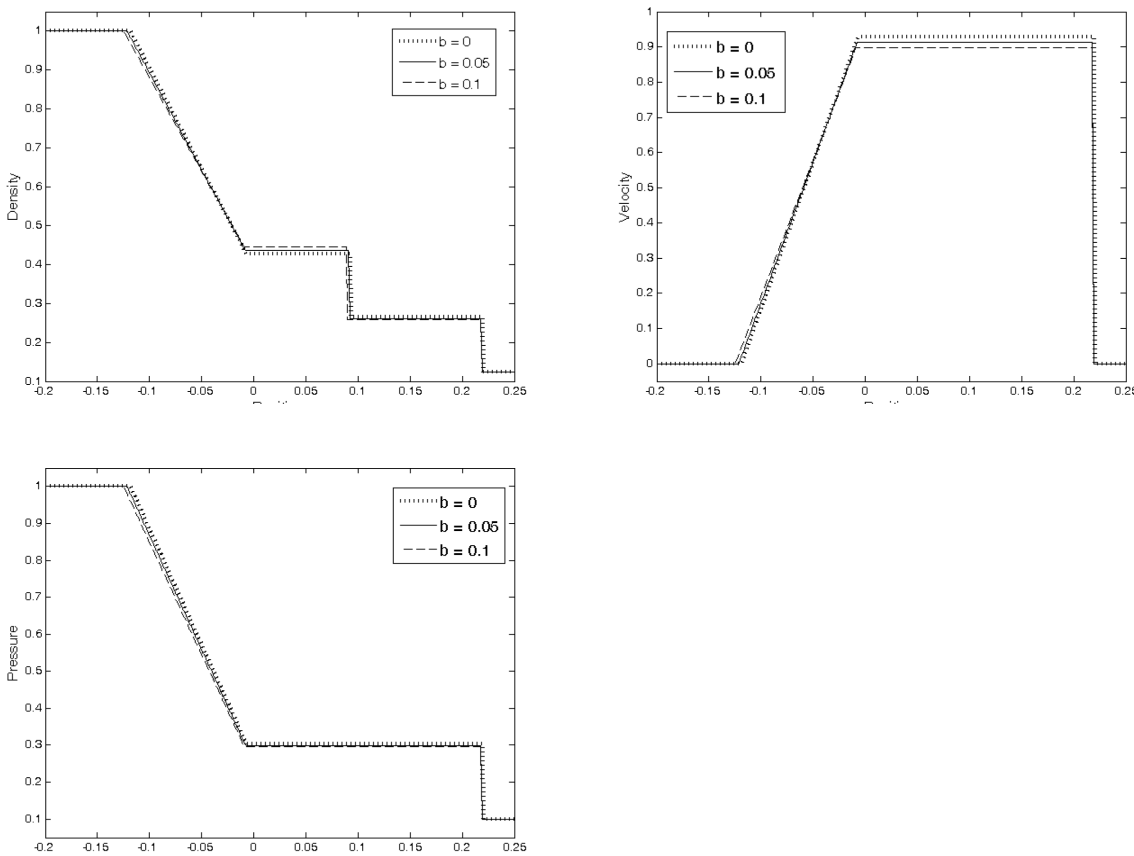

Fig. 1 The solution for the density, velocity and pressure at time t = 0.1 for data-1.

Hence if two incident 3-shocks are weak then the resultant 1- component wave is a 1-simple wave if $\gamma < 5/3$, and is a 1-shock wave if $\gamma > 5/3$. Similarly, we can observe that the interaction of two weak 1-shocks gives a resultant 3-simple wave if $\gamma < 5/3$ and a 3-shock wave if $\gamma > 5/3$. Hence the theorem. □

6. Numerical Solution

For a given initial data, the solution to the Riemann problem is known by determining the values ξ_1 , ξ_2 and ξ_3 from the equations (4.6a), (4.6b) and (4.6c). Consequently, the density, the velocity and the pressure in the unknown region are found from the equations (4.2), (4.3) and (4.4).

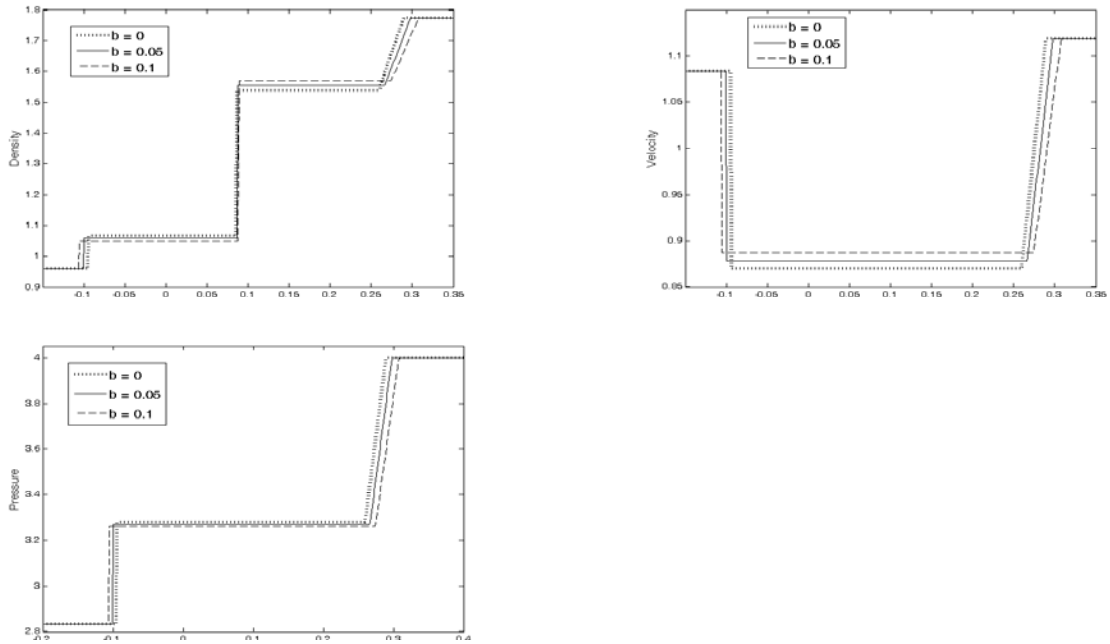

Fig. 2 The solution for the density, velocity and pressure at time t = 0.1 for data-2

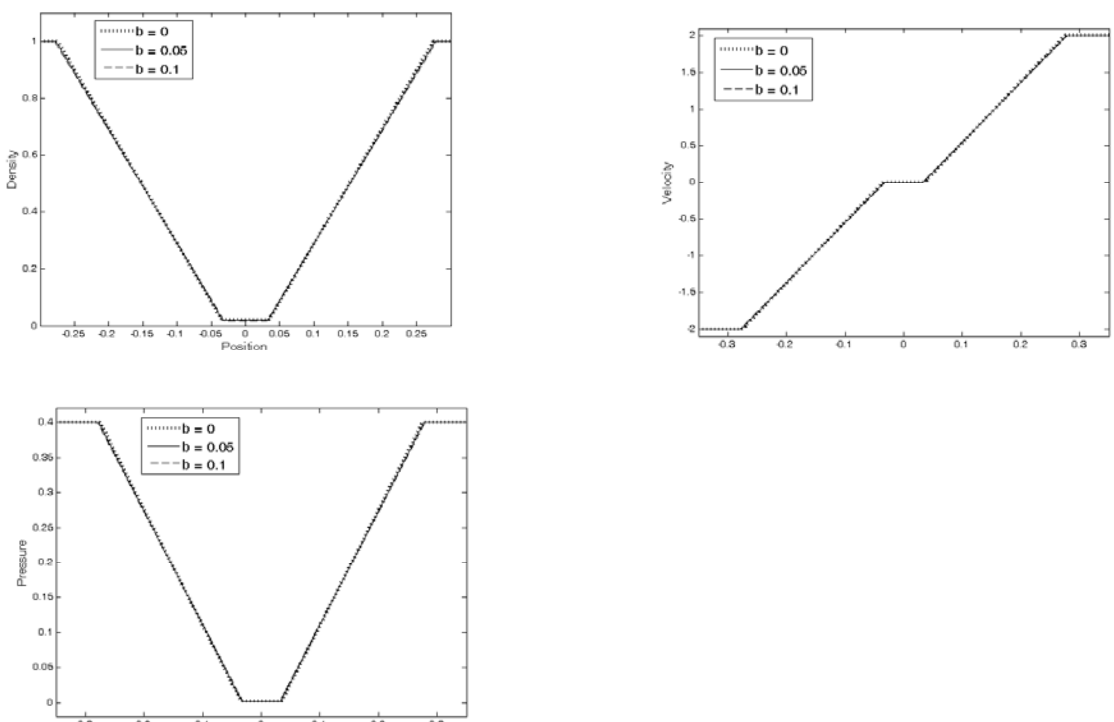

Fig. 3 The solution for the density, velocity and pressure at time t = 0.1 for data-3.

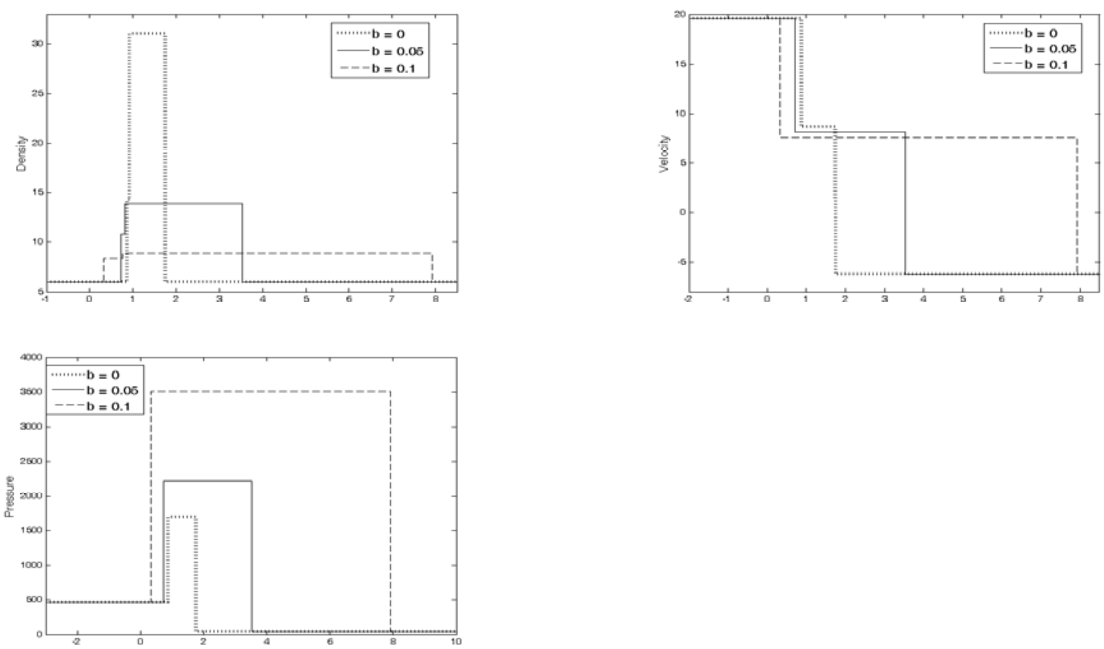

Fig. 4 The solution for the density, velocity and pressure at time t = 0.1, for the data-4.

Three typical values are considered for van der Waals gas parameter $b = 0, 0.05$ and 0.1 for several initial data $(\rho_b, u_b, , p_b)$, (ρ_a, u_a, p_a) given in Table-1. For each initial data (see, Toro [16]) the numerical values for the flow variables in regions-2,3 are tabulated in Tables-2, 3 and 4 for $b = 0$, 0.05, 0.1. The solution for the pressure, the velocity and the density are calculated at time $t = 0.1$ and are depicted in Figs. 1, 2, 3 and 4.

The data-1 (respectively, data-2) corresponds to a solution of the Riemann problem consisting of a 1-simple wave (respectively, a 1-shock wave), a 2-wave as a contact discontinuity and a 3-shock wave (respectively, a 3-simple wave) (see, Fig. 1 (respectively, Fig. 2)); however, the data-3 (respectively, data-4) corresponds to a solution consisting of both a 1-wave and a 3-wave as simple waves (respectively, as shock waves) with 2-wave as a contact discontinuity (see, Fig. 3 (respectively, Fig. 4)).

7. Results and Discussions

The Riemann problem for gas dynamic equations governing the one dimensional flow of van der Waals gases is studied. The existence and uniqueness of shocks and simple wave solutions are discussed using R-H conditions and Lax conditions. The explicit form of solutions for shocks, contact discontinuities and simple waves are derived. It is observed that with an increase in b, both the shock speed and the density decrease and the velocity increases for a 1-shock, whereas both the density and the shock speed increase and the velocity decreases for a 3-shock. However, for a 1-simple wave, the density increases and the velocity decreases with an increase in b and for a 3-simple wave, both the density and the velocity increase as b increases. A necessary condition is derived on the initial data for the existence of a unique solution to the Riemann problem with an arbitrary initial data. Further, a relation is derived on the initial data to know whether the solution to the Riemann problem involves either shocks or simple waves or both a shock wave and a simple wave. For a given initial data (Table 1), the solution of the Riemann problem is determined numerically for van der Waals gas parameter $b = 0$, $b = 0.05$ and $b = 0.1$ and is given in Tables 2, 3 and 4. It is observed that the results obtained here for an ideal gas by putting $b = 0$, are in good agreement with results given by Smoller [15].

Acknowledgement

One of the authors (K.Ambika) is thankful to the Council of Scientific and Industrial Research for providing financial support.

References

1. P. Colella and H. M. Glaz, Efficient solution algorithms for the Riemann problem for real gases, *J. Comput. Phy.*, **59** (1985), 264-289.
2. R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience publishers, New York (1999).
3. J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure. Appl. Math.*, **18** (1963), 697-715.
4. S. K. Godunov, *Numerical solution of multidimensional problems in gasdynamics*, Nauka Press, Moscow (1976).
5. A. Guardone and L. Vigevano, Roe linearization for the van der Waals gas, *J. Comput. Phy.*, **175** (2002), 50-78.
6. H. Hattori, The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion - isothermal case, *Arch. Rat. Mech. and Anal.*, **92** (1986), 247-263.

7. P. D. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure. Appl. Math.*, **10** (1957), 537-566.

8. L. Quartapelle, L. Castelletti, A. Guardone and G. Quaranta, Solution of the Riemann problem of classical gasdynamics, *J. Comput. Phy.*, **190** (2003), 118-140.

9. T. Raja Sekhar and V. D. Sharma, Riemann problem and elementary wave interactions in isentropic magnetogasdynamics, *Nonlinear Analysis: Real World Applications*, **11** (2010), 619-636.

10. T. Raja Sekhar and V. D. Sharma, Solution to the Riemann problem in a one-dimensional magnetogasdynamic flow, *International Journal of Computer Mathematics*, **89** (2012), 200-216.

11. P. L. Roe, Approximate Riemann solvers, parameter vectors, and difference schemes, *J. Comput. Phy.*, **43** (1981), 357-372.

12. R. Saurel, Exact and approximate Riemann solvers for real gases, *J. Comput. Phy.*, **112** (1994), 126-137.

13. M. Shearer, The Riemann problem for a class of conseravtion laws of mixed type, *J. Diff. Eq.*, **46** (1982), 426-443.

14. C. Shen, The limits of Riemann solutions to the isentropic magnetogasdynamics, *Appl. Math. Lett.*, **24** (2011), 1124-1129.

15. J. Smoller, *Shock waves and reaction diffusion equations*, Springer Verlag (2nd ed.), New York (1994).

16. E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics*, 2nd ed., Springer, Berlin (1997).