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ON THE STABILITY OF HIGHER RING LEFT DERIVATIONS

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In this note, we investigate the Hyers-Ulam, the Isac and Rassias-type stability and the Bourgin-type superstability of a functional inequality corresponding to the following functional equation:

$$h_n(xy) = \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)h_j(y) + c_{ij}h_i(y)h_j(x)],$$

where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Key words : Higher left ring derivation; approximately higher left ring derivation; stability

1. Introduction and Preliminaries

Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$. An additive mapping $h : \mathcal{A} \to \mathcal{A}$ is said to be a *ring homomorphism* if the functional equation $h(xy) = h(x)h(y)$ holds for all $x, y \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *ring left derivation* (resp. *ring derivation*) if the functional equation $d(xy) = xd(y) + yd(x)$ (resp. $d(xy) = xd(y) + d(x)y$) holds for all $x, y \in \mathcal{A}$. Brešar and Vukman [5, Proposition 1.6] showed that every ring left derivation on a semiprime ring is a ring derivation which maps into its center.

Let $\mathbb{N}$ be the set of natural numbers. From $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \cdots, h_m\}$ (resp. $H = \{h_0, h_1, \cdots, h_n, \cdots\}$) of additive mappings on $\mathcal{A}$ is called a *higher ring left derivation* of rank m (resp. infinite rank) on $\mathcal{A}$ if the functional equation

$$h_n(xy) = \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)h_j(y) + c_{ij}h_i(y)h_j(x)], \tag{1.1}$$

holds for each $n = 0, 1, \cdots, m$ (resp. $n = 0, 1, \cdots$) and for all $x, y \in \mathcal{A}$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

A higher ring left derivation H on $\mathcal{A}$, particularly, is called *strong* if h_0 is an identity mapping. If, in addition, each h_n in H satisfies the functional equation $h_n(\lambda x) = \lambda h_n(x)$ for all $\lambda \in \mathbb{F}$ and all $x \in \mathcal{A}$, then we say that H is just a *higher left derivation*.

Of course, a higher ring left derivation of rank 0 from $\mathcal{A}$ into $\mathcal{B}$ (resp. a strong higher ring left derivation of rank 1 on $\mathcal{A}$) is a ring homomorphism (resp. a ring left derivation), that is, a higher ring left derivation is a generalization of both a ring homomorphism and a ring left derivation.

For example, given any ring left derivation d on an algebra $\mathcal{A}$ with unit and an invertible element $c \in \mathcal{A}$, let $\delta : \mathcal{A} \to \mathcal{A}$ be the mapping defined by $\delta(x) = cd(x)$ for all $x \in \mathcal{A}$. Then $\delta(xy) = \theta(x)\delta(y) + \theta(y)\delta(x)$ for all $x, y \in \mathcal{A}$, where the relation $\theta(x) = cxc^{-1}, x \in \mathcal{A}$ defines an inner automorphism of $\mathcal{A}$. Let $h_0 = \theta$, $h_n = 0$, $1 \leq n \leq m-1$ and $h_m = \delta$. Then we see that the sequence $H = \{h_0, h_1, \cdots, h_m\}$ satisfies the relation (1.1).

Here, it is of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation. The study of stability problems originated from a famous talk given by Ulam [20] in 1940 : "Under what condition does there exists a homomorphism near an approximate homomorphism ?" In the next year 1941, Hyers [8] was answered affirmatively the question of Ulam and the result can be formulated as follows: *if* $\varepsilon > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a map with* $\mathcal{X}$ *a normed space,* $\mathcal{Y}$ *a Banach space such that*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon$$

for all $x, y \in \mathcal{X}$*, then there exists a unique additive map* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$||f(x) - T(x)|| \leq \varepsilon$$

for all $x \in \mathcal{X}$*. Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* x *in* $\mathcal{X}$*, where* $\mathbb{R}$ *denotes the real field, then* T *is linear.*

A generalized version of the theorem of Hyers for approximately additive mappings was first given by Aoki [1] in 1950. In 1978, Rassias [16] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces: *if there exist a* $\theta \geq 0$ *and* $0 \leq p < 1$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{X}$*, then there exists a unique additive map* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in \mathcal{X}$.

In 1949, Bourgin [4] proved the following result, which is sometimes called the superstability of ring homomorphisms: *suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are Banach algebras with unit. If* $f : \mathcal{A} \to \mathcal{B}$ *is a surjective mapping such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$
$$\|f(xy) - f(x)f(y)\| \leq \delta$$

for some $\varepsilon \geq 0$, $\delta \geq 0$ *and all* $x, y \in \mathcal{A}$*, then* f *is a ring homomorphism.*

Badora [2] gave a generalization of the above Bourgin's result. He [3] also obtained the following result for the Hyers and Ulam-type stability and the Bourgin-type superstability of ring derivations: *let* $\mathcal{A}_1$ *be a subalgebra of a Banach algebra* $\mathcal{A}$*. Assume that* $f : \mathcal{A}_1 \to \mathcal{A}$ *is a mapping such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$
$$\|f(xy) - xf(y) - f(x)y\| \leq \delta$$

for some $\varepsilon \geq 0$, $\delta \geq 0$ *and all* $x, y \in \mathcal{A}_1$*. Then there exists a unique ring derivation* $d : \mathcal{A}_1 \to \mathcal{A}$ *such that*

$$\|f(x) - d(x)\| \leq \varepsilon$$

for all $x \in \mathcal{A}_1$*. Moreover,*

$$x\{f(y) - d(y)\} = 0$$

for all $x, y \in \mathcal{A}_1$*. In addition, if* $\mathcal{A}_1$ *and* $\mathcal{A}$ *have the unit element, then* f *is a ring derivation.*

Based on these facts, the author [15] obtained the results for the stability of module left derivations.

The main purpose of the present paper is to investigate the stability problem of higher ring left derivations. We first consider approximate higher ring left derivations of Hyers and Ulam-type: *a sequence* $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ *of mappings on* $\mathcal{A}$ *such that for some* $\delta \geq 0$, $\varepsilon \geq 0$ *and each* $n = 0, 1, \cdots$,

$$\|f_n(x + y) - f_n(x) - f_n(y)\| \leq \varepsilon \tag{1.2}$$

and

$$\left\| f_n(xy) - \sum_{\substack{i+j=n \\ i\leq j}} [f_i(x)f_j(y) + c_{ij}f_i(y)f_j(x)] \right\| \leq \delta \tag{1.3}$$

for all $x, y \in \mathcal{A}$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Furthermore, we show the Isac and Rassias-type stability [9] and the Bourgin-type superstability [4] for higher ring left derivations.

2. Main Results

By the similar way as in [2], we obtain the next theorem.

***Theorem* 2.1** — *Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings on a Banach algebra $\mathcal{A}$ satisfying (1.2) and (1.3). Then there exists a unique higher ring left derivation $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ of any rank on $\mathcal{A}$ such that for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$,*

$$\|f_n(x) - h_n(x)\| \leq \varepsilon.$$

Moreover,

$$\sum_{\substack{i+j=n \\ i\leq j}} h_i(x)[f_j(y) - h_j(y)] + \sum_{\substack{i+j=n \\ i\leq j}} c_{ij}[f_i(y) - h_i(y)]h_j(x) = 0 \tag{2.1}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$.

Proof : By (1.2), it follows from the Hyers theorem [8] that for each $n = 0, 1, \cdots$, there exists a unique additive mapping $h_n : \mathcal{A} \to \mathcal{A}$ given by

$$h_n(x) = \lim_{k\to\infty} \frac{1}{2^k} f_n(2^k x) \tag{2.2}$$

for all $x \in \mathcal{A}$ such that

$$\|f_n(x) - h_n(x)\| \leq \varepsilon$$

for all $x \in \mathcal{A}$.

Next, we want to show that the sequence $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ satisfies the identity

$$h_n(xy) = \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)h_j(y) + c_{ij}h_i(y)h_j(x)]$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. The inequality (1.3) implies that the function $\Delta_n : \mathcal{A}\times\mathcal{A} \to \mathcal{B}$ defined by

$$\Delta_n(x, y) = f_n(xy) - \sum_{\substack{i+j=n \\ i\leq j}} [f_i(x)f_j(y) + c_{ij}f_i(y)f_j(x)] \tag{2.3}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$, is bounded. Hence we see that

$$\lim_{k\to\infty} \frac{\Delta_n(2^k x, y)}{2^k} = 0 \tag{2.4}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Now, using (2.2), (2.3) and (2.4), we have

$$\begin{aligned}
h_n(xy) &= \lim_{k\to\infty} \frac{f_n(2^k(xy))}{2^k} = \lim_{k\to\infty} \frac{f_n((2^k x)y))}{2^k} \\
&= \lim_{k\to\infty} \frac{\displaystyle\sum_{\substack{i+j=n \\ i\leq j}} [f_i(2^k x)f_j(y) + c_{ij}f_i(y)f_j(2^k x)] + \Delta_n(2^k x, y)}{2^k} \\
&= \lim_{k\to\infty} \sum_{\substack{i+j=n \\ i\leq j}} \frac{1}{2^k}[f_i(2^k x)f_j(y) + c_{ij}f_i(y)f_j(2^k x)] + \lim_{k\to\infty} \frac{\Delta_n(2^k x, y)}{2^k} \\
&= \sum_{\substack{i+j=n \\ i\leq j}} \left\{ \lim_{k\to\infty} \frac{1}{2^k} f_i(2^k x)f_j(y) + c_{ij} \lim_{k\to\infty} \frac{1}{2^k} f_i(y)f_j(2^k x) \right\} \\
&= \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)f_j(y) + c_{ij}f_i(y)h_j(x)]
\end{aligned}$$

That is, we obtain that

$$h_n(xy) = \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)f_j(y) + c_{ij}f_i(y)h_j(x)] \tag{2.5}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Let $k \in \mathbb{N}$ be fixed. Then, applying (2.5) and the additivity of each h_n, $n = 0, 1, \cdots$, we get

$$\begin{aligned}
&\sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)f_j(2^k y) + c_{ij}f_i(2^k y)h_j(x)] \\
&= h_n(x(2^k y)) = h_n((2^k x)y) \\
&= \sum_{\substack{i+j=n \\ i\leq j}} [h_i(2^k x)f_j(y) + c_{ij}f_i(y)h_j(2^k x)] \\
&= 2^k \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)f_j(y) + c_{ij}f_i(y)h_j(x)]
\end{aligned}$$

Hence we get

$$\sum_{\substack{i+j=n\\ i\leq j}} [h_i(x)f_j(y)+c_{ij}f_i(y)h_j(x)] = \sum_{\substack{i+j=n\\ i\leq j}} \left[h_i(x)\frac{1}{2^k}f_j(2^k y)+c_{ij}\frac{1}{2^k}f_i(2^k y)h_j(x)\right] \tag{2.6}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Taking $k \to \infty$ in (2.6), we see that

$$\sum_{\substack{i+j=n\\ i\leq j}} [h_i(x)f_j(y)+c_{ij}f_i(y)h_j(x)] = \sum_{\substack{i+j=n\\ i\leq j}} [h_i(x)h_j(y)+c_{ij}h_i(y)h_j(x)] \tag{2.7}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$ which means (2.1). Combining (2.7) with (2.5), it follows that $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ satisfies the relation

$$h_n(xy) = \sum_{\substack{i+j=n\\ i\leq j}} [h_i(x)h_j(y)+c_{ij}h_i(y)h_j(x)].$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. This completes the proof of the theorem. □

Let $\mathbb{R}^+$ be the set of positive real numbers. Isac and Rassias [9] generalized the Hyers theorem introducing a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \tag{2.8}$$

$$\psi(ts) \leq \psi(t)\psi(s) \quad \text{for all } t,\ s \in \mathbb{R}^+, \tag{2.9}$$

$$\psi(t) < t \quad \text{for all } t > 1. \tag{2.10}$$

Here we obtain the Isac and Rassias-type stability result for higher ring left derivations which is a generalization of Theorem 2.1.

***Theorem* 2.2** — *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with properties (2.8), (2.9) and (2.10), and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a function satisfying the condition*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{2.11}$$

Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings on a Banach algebra $\mathcal{A}$ such that for some $\delta \geq 0$ and each $n = 0, 1, \cdots$,

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \leq \delta(\psi(\|x\|) + \psi(\|y\|)) \tag{2.12}$$

and

$$\left\| f_n(xy) - \sum_{\substack{i+j=n\\ i\leq j}} [f_i(x)f_j(y)+c_{ij}f_i(y)f_j(x)] \right\| \leq \varphi(\|x\|\,\|y\|) \tag{2.13}$$

for all $x, y \in \mathcal{A}$. *Then there exists a unique higher ring left derivation* $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ *of any rank on* $\mathcal{A}$ *and a constant* $c \in \mathbb{R}$ *such that for each* $n = 0, 1, \cdots$ *and all* $x, y \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \le c\,\delta\psi(\|x\|).$$

Moreover, the relation (2.1) is fulfilled.

PROOF : By the Isac-Rassias theorem [9], for each $n = 0, 1, \cdots$, there exist a unique additive mapping $h_n : \mathcal{A} \to \mathcal{A}$ given by

$$h_n(x) = \lim_{k\to\infty} \frac{1}{2^k} f_n(2^k x) \tag{2.14}$$

for all $x \in \mathcal{A}$ and a constant $c \in \mathbb{R}$ such that

$$\|f_n(x) - h_n(x)\| \le c\,\delta\psi(\|x\|)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to that of Theorem 2.1. Hence we obtain the desired conclusion. □

Remark 2.3 : The typical example of the function ψ fulfilling (2.8), (2.9) and (2.10) is given by $\psi(t) = t^p$, where $p < 1$. The example of the mapping φ satisfying (2.11) is $\varphi(t) = t^q$, where $q < 1$. If we intend to consider the case of $p, q > 1$, then we adopt the method given by Gajda in [6] to obtain the Isac and Rassias-type stability result for the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ fulfilling the conditions

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \tag{2.15}$$

$$\psi(ts) \le \psi(t)\psi(s) \quad \text{for all } t,\ s \in \mathbb{R}^+, \tag{2.16}$$

$$\psi(t) < t \quad \text{for all } t \in (0,1) \tag{2.17}$$

In the proof of Theorem 2.1, if we replace (2.2) by

$$h_n(x) = \lim_{k\to\infty} 2^k f_n\Big(\frac{1}{2^k}x\Big)$$

and (2.4) by

$$\lim_{k\to\infty} 2^k \Delta_n\Big(\frac{1}{2^k}x, y\Big) = 0,$$

then under the conditions (2.15), (2.16) and (2.17), Theorem 2.2 is still true.

As consequences of Theorem 2.1, we get the following Bourgin-type superstability for higher ring left derivations H which are *onto*, i.e., if $h_0 \in H$ maps $\mathcal{A}$ onto $\mathcal{B}$.

Corollary 2.4 — Let $\mathcal{A}$ be a Banach algebra with unit. Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings on a Banach algebra $\mathcal{A}$ satisfying (1.2) and (1.3), where f_0 is onto. Then $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a higher ring left derivation of any rank on $\mathcal{A}$.

PROOF : By induction, we lead the conclusion. By the Bourgin's theorem [4] and the relation (2.2) in the proof of Theorem 2.1, we see that f_0 is a ring homomorphism on a Banach algebra $\mathcal{A}$ and $f_0 = h_0$. If $n = 1$, then it follows from (2.1) and the definition of c_{ij} that $f_1(x) = h_1(x)$ holds for all $x \in \mathcal{A}$ since h_0 is onto. Let us assume that $f_m(x) = h_m(x)$ is valid for all $x \in \mathcal{A}$ and all $m < n$. Then (2.1) implies that $h_0(x)\{f_n(y) - h_n(y)\} = 0$ for all $x, y \in \mathcal{A}$. Since h_0 is onto, we have $f_n(y) = h_n(y)$ for all $y \in \mathcal{A}$. Hence we conclude that $f_n(x) = h_n(x)$ holds for all $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$. Now Theorem 2.1 tells us that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a higher ring left derivation of any rank on $\mathcal{A}$. The proof of the theorem is complete. □

Corollary 2.5 — Let $\mathcal{A}$ be a Banach algebra with unit. Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings on $\mathcal{A}$ satisfying (1.2) and (1.3) on $\mathcal{A}$. Then $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a strong higher ring left derivation of any rank on $\mathcal{A}$.

PROOF : For all $x \in \mathcal{A}$, we have, by (2.2),

$$h_0(x) = \lim_{k\to\infty} \frac{1}{k} f_0(kx) = x$$

and so $h_0(= f_0)$ is an identity mapping on $\mathcal{A}$. Following the same method as in the proof of Corollary 2.4 using the induction and the relation (2.1), we get

$$x\{f_n(y) - h_n(y)\} = 0$$

for all $n \in \mathbb{N}$ and all $x, y \in \mathcal{A}$. Since $\mathcal{A}$ contains the unit, it follows that $f_n(y) = h_n(y)$ for all $n \in \mathbb{N}$ and all $y \in \mathcal{A}$. So, by Theorem 2.1, we see that $F = \{f_1, f_2, \cdots, f_n, \cdots\}$ is a strong higher ring left derivation of any rank on $\mathcal{A}$. This completes the proof. □

Singer and Wermer [18] obtained a fundamental result which started investigation into the ranges of linear derivations on Banach algebras. The result, which is called the Singer-Wermer theorem, states that every continuous linear derivation (or linear left derivation) on a commutative Banach algebra maps into the Jacobson radical. In the same paper they conjectured that the assumption of continuity is not necessary. Thomas [19] proved the conjecture. Using the Thomas' result, we see that every linear derivation (or linear left derivation) on a commutative semisimple Banach algebra is identically zero which is the result of Johnson [11]. Roy [17] and Jewell [10] showed that the continuity of linear derivations on semisimple Banach algebras [12] can be extended to higher derivations. Jun and Lee [13] generalized the Singer-Wermer theorem to strong higher derivations.

The following is a modified result of Johnson [11] for approximately strong higher left derivations.

***Theorem* 2.6** — *Let $\mathcal{A}$ be a semisimple Banach algebra with unit. Suppose that $F = \{f_1, f_2, \cdots, f_n \cdots\}$ is a sequence of mappings on $\mathcal{A}$ such that for some $\varepsilon \geq 0$, $\delta \geq 0$ and each $n \in \mathbb{N}$,*

$$\|f_n(\alpha x + \beta y) - \alpha f_n(x) - \beta f_n(y)\| \leq \varepsilon \tag{2.18}$$

and

$$\left\| f_n(xy) - \sum_{\substack{i+j=n \\ i \leq j}} [f_i(x) f_j(y) + c_{ij} f_i(y) f_j(x)] \right\| \leq \delta$$

hold for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, where f_0 is an identity mapping on $\mathcal{A}$. Then we have $F = \{0\}_{n \in \mathbb{N}}$ on $\mathcal{A}$.

PROOF : Put $\alpha = \beta = 1 \in \mathbb{U}$ in (2.18). Then it follows from the Hyers theorem that for each $n \in \mathbb{N}$, there exists a unique additive mapping h_n on $\mathcal{A}$ such that

$$\|f_n(x) - h_n(x)\| \leq \varepsilon$$

for all $x \in \mathcal{A}$. For each $n \in \mathbb{N}$, the additive mapping $h_n : \mathcal{A} \to \mathcal{A}$ is given by

$$h_n(x) = \lim_{k \to \infty} \frac{1}{2^k} f_n(2^k x)$$

for all $x \in \mathcal{A}$. Setting $y = x$ in (2.18) yields

$$\|f_n((\alpha + \beta)x) - (\alpha + \beta) f_n(x)\| \leq \varepsilon$$

for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$. Thus we see that

$$2^{-k} \|f_n(2^k(\alpha + \beta)x) - (\alpha + \beta) f_n(2^k x)\| \to 0$$

as $k \to \infty$ which implies that for each $n \in \mathbb{N}$,

$$h_n((\alpha + \beta)x) = \lim_{k \to \infty} \frac{f_n(2^k(\alpha + \beta)x)}{2^k} = \lim_{k \to \infty} \frac{(\alpha + \beta) f_n(2^k x)}{2^k} = (\alpha + \beta) h_n(x)$$

for all $x \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$.

Clearly, $h_n(0x) = 0 = 0h_n(x)$ for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$. Now, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$), and let $M \in \mathbb{N}$ greater than $|\lambda|$. By appying a geometric argument, we see that there exist $\lambda_1, \lambda_2 \in \mathbb{U}$ such that $2\frac{\lambda}{M} = \lambda_1 + \lambda_2$. By the additivity of each h_n, $n \in \mathbb{N}$, we get $h_n(\frac{1}{2}x) = \frac{1}{2} h_n(x)$ for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$.

Therefore,

$$\begin{aligned} h_n(\lambda x) &= h_n\left(\frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} x\right) = M h_n\left(\frac{1}{2} \cdot 2 \cdot \frac{\lambda}{M} x\right) = \frac{M}{2} h_n((\lambda_1 + \lambda_2)x) \\ &= \frac{M}{2}(\lambda_1 + \lambda_2) h_n(x) = \frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} h_n(x) = \lambda h_n(x) \end{aligned}$$

for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$, so that h_n is $\mathbb{C}$-linear for each $n \in \mathbb{N}$.

Let Δ_n be the mapping defined by (2.3) for each $n \in \mathbb{N}$. Then it follows from (1.3) that the mapping Δ_n satisfies the condition (2.4). The remainder of the proof is similar to the one of Theorem 2.1. Now, setting $f_0 = h_0$, then we see that $H = \{h_0, h_1 \cdots, h_n, \cdots\}$ is a strong higher left derivation of any rank on $\mathcal{A}$ and that the relation

$$\sum_{\substack{i+j=n \\ i \leq j}} h_i(x)[f_j(y) - h_j(y)] + \sum_{\substack{i+j=n \\ i \leq j}} c_{ij}[f_i(y) - h_i(y)]h_j(x) = 0 \tag{2.19}$$

holds for each $n \in \mathbb{N}$ and all $x, y \in \mathcal{A}$.

Using the relation (2.19) and the similar process as in the proof of Corollary 2.5, we obtain that $f_n(x) = h_n(x)$ for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$. This means that that $H = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a strong higher left derivation of any rank on $\mathcal{A}$.

Since f_1 is a linear left derivation on $\mathcal{A}$, we have $f_1 = 0$ on $\mathcal{A}$ by [14, Corollary 3.7]. Assume that $n \geq 2$ and $f_m = 0$ for all $m < n$. Since we have

$$f_n(xy) = xf_n(y) + yf_n(x) + \sum_{\substack{i+j=n \\ i \leq j,\, i \neq 0,n}} [f_i(x)f_j(y) + c_{ij}f_i(y)f_j(x)],$$

it follows from the hypothesis that

$$f_n(xy) = xf_n(y) + yf_n(x)$$

for all $x, y \in \mathcal{A}$. This implies that f_n is a linear left derivation on $\mathcal{A}$. Therefore, [14, Corollary 3.7] again gives $f_n = 0$ on $\mathcal{A}$. By the induction, we have $F = \{0\}_{n \in \mathbb{N}}$ on $\mathcal{A}$ which completes the proof. □

Remark 2.7 : As in Theorem 2.2 and Remark 2.3, we can generalize our results by substituting another functions satisfying appropriate conditions (see, for instance, [7]) for the bounds ε and δ of the inequalities corresponding to the functional equations

$$f_n(x + y) = f_n(x) + f_n(y) \text{ and } f_n(xy) = \sum_{\substack{i+j=n \\ i \leq j}} [f_i(x)f_j(y) + c_{ij}f_i(y)f_j(x)].$$

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