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COMPLETENESS THEOREM FOR THE DISSIPATIVE STURM-LIOUVILLE OPERATOR ON BOUNDED TIME SCALES

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In this paper we consider a second-order Sturm-Liouville operator of the form

$$l(y) := -\left[p(t)\, y^{\Delta}(t)\right]^{\nabla} + q(t)\, y(t)$$

on bounded time scales. In this study, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of the dissipative Sturm-Liouville operators in terms of boundary conditions. Using Krein's theorem, we proved a theorem on completeness of the system of eigenvectors and associated vectors of the dissipative Sturm-Liouville operators on bounded time scales.

Key words : Time scales; Sturm-Liouville operator; Δ-differentiable; dissipative operator; completeness of the system of eigenvectors and associated vectors; Krein theorem; boundary value space.

1. INTRODUCTION

The study of dynamic equations on time scales is a new area of theoretical exploration in mathematics. The first fundamental results in this area were obtained by Hilger [15]. Time scale calculus unites the study of differential and difference equations. The study of time scales has led to several important applications, e.g., in the study of neural networks, heat transfer, and insect population models, epidemic models [1]. We refer the reader to consult the reference [2, 3, 6, 7, 14, 19] for some basic definitions.

The completeness theorems are important for solving various problems in mathematical physics by the Fourier method, and also for the spectral theory itself. Dissipative operator is important part of non self adjoint operators. In the spectral analysis of a dissipative operator, we should answer the question that whether all eigenvectors and associated vectors of a dissipative operator span the whole space or not.

The organization of this document is as follows: In Section 2, some time scale essentials are included for the convenience of the reader. In Section 3, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of the dissipative Sturm-Liouville operators in terms of boundary conditions. Finally, we proved a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators under consideration. A similar way was employed earlier in the differential and difference operators case in [4, 5, 11, 12, 13]. In this paper, we unite the study of differential and difference operators.

2. Preliminaries

Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, t \in \mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$. We introduce the sets $\mathbb{T}^k,\ \mathbb{T}_k,\ \mathbb{T}^*$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Some function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show [14]

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

Let $f:\mathbb{T}\rightarrow\mathbb{R}$ be a function, and $a,b\in\mathbb{T}$. If there exists a function $F:\mathbb{T}\rightarrow\mathbb{R}$, such that $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{k}$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_{a}^{b}f(t)\,\Delta t=F(b)-F(a)\ \text{ for } a,b\in\mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.

Let $L_{\Delta}^{2}(\mathbb{T}^{*})$ be the space of all functions defined on $\mathbb{T}^{*}$ such that

$$\|f\|:=\left(\int_{a}^{b}|f(t)|^{2}\,\Delta t\right)^{1/2}<\infty.$$

The space $L_{\Delta}^{2}(\mathbb{T}^{*})$ is a Hilbert space with the inner product [21]

$$(f,g):=\int_{a}^{b}f(t)\,\overline{g(t)}\Delta t,\ \ f,g\in L_{\Delta}^{2}(\mathbb{T}^{*}).$$

Let $a\leq b$ be fixed points in $\mathbb{T}$ and $a\in\mathbb{T}_{k}, b\in\mathbb{T}^{k}$. We will consider the Sturm-Liouville equation

$$l(y):=-\left[p(t)\,y^{\Delta}(t)\right]^{\nabla}+q(t)\,y(t)\,,\ t\in[a,b], \tag{2.1}$$

where $q:\mathbb{T}\rightarrow\mathbb{C}$ is continuous function, $p:\mathbb{T}\rightarrow\mathbb{R}$ is ∇-differentiable on $\mathbb{T}^{k}$, $p(t)\neq 0$ for all $t\in\mathbb{T}$, and $p^{\nabla}:\mathbb{T}_{k}\rightarrow\mathbb{R}$ is continuous. The Wronskian of y,z is defined to be [14]

$$W(y,z)(t):=p(t)\left[y(t)\,z^{\Delta}(t)-y^{\Delta}(t)\,z(t)\right],\ \ t\in\mathbb{T}^{*}.$$

Let L_{0} denote the closure of the minimal operator generated by (2.1) and by D_{0} its domain. Besides, we denote by D the set of all functions $y(t)$ from $L_{\Delta}^{2}(\mathbb{T}^{*})$ such that $l(y)\in L_{\Delta}^{2}(\mathbb{T}^{*});$ D is the domain of the maximal operator L. Furthermore $L=L_{0}^{*}$ [20]. Suppose that the operator L_{0} has defect index (2,2).

For every $y,z\in D$ we have Lagrange's identity [14]

$$(Ly,z)-(y,Lz)=[y,z]_{b}-[y,z]_{a}$$

where $[y,z]_{t}:=p(t)\left[y(t)\,\overline{z^{\Delta}(t)}-y^{\Delta}(t)\,\overline{z(t)}\right]$.

For arbitrary $y,z\in D$, one has the equality

$$[y,z]_{t}=[y,u]_{t}[z,v]_{t}-[y,v]_{t}[z,u]_{t}. \tag{2.2}$$

Denote $u(t,\lambda)$, $v(t,\lambda)$ the solutions of the equation $l(y)=\lambda y$ satisfying the initial conditions

$$\begin{aligned} u(a,\lambda) &= \cos\alpha,\ \ p(a)\,u^{\Delta}(a,\lambda)=\sin\alpha, \\ v(a,\lambda) &= -\sin\alpha\ \ p(a)\,v^{\Delta}(a,\lambda)=\cos\alpha, \end{aligned}$$

where $\alpha\in\mathbb{R}$. The solutions $u(t,\lambda)$ and $v(t,\lambda)$ form a fundamental system of solutions of $l(y)=\lambda y$ and they are entire functions of λ [8]. Let $u(t)=u(t,0)$ and $v(t)=v(t,0)$ the solutions of the equation $l(y)=0$ satisfying the initial conditions

$$\begin{aligned} u(a) &= \cos\alpha,\ \ p(a)\,u^{\Delta}(a)=\sin\alpha, \\ v(a) &= -\sin\alpha\ \ p(a)\,v^{\Delta}(a)=\cos\alpha. \end{aligned}$$

Let's define by Γ_1,Γ_2 the linear maps from D to $\mathbb{C}^2$ by the formula

$$\Gamma_1 y=\begin{pmatrix} -y(a) \\ [y,u]_b \end{pmatrix},\ \Gamma_2 y=\begin{pmatrix} p(a)\,y^{\Delta}(a) \\ [y,v]_b \end{pmatrix},\ y\in D. \tag{2.3}$$

We recall that a triple $(\mathbb{H},\Gamma_1,\Gamma_2)$ is called a space of boundary values of a closed symmetric operator A on a Hilbert space H if Γ_1,Γ_2 are linear maps from $D(A^*)$ to H with equal deficiency numbers and such that:

(i) for every $f,g\in D(A^*)$

$$(A^*f,g)_H-(f,A^*g)_H=(\Gamma_1 f,\Gamma_2 g)_{\mathbb{H}}-(\Gamma_2 f,\Gamma_1 g)_{\mathbb{H}};$$

(ii) any $F_1,F_2\in H$ there is a vector $f\in D(A^*)$ such that $\Gamma_1 f=F_1,\ \Gamma_2 f=F_2$ [10].

3. Main Results

***Theorem* 1** — *The triple $(\mathbb{C}^2,\Gamma_1,\Gamma_2)$ defined by (2.3) is a boundary spaces of the operator L_0.*

PROOF : For any $y,z\in D$, we have

$$\begin{aligned} (\Gamma_1 y,\Gamma_2 z)_{\mathbb{C}^2}-(\Gamma_2 y,\Gamma_1 z)_{\mathbb{C}^2} &= -y(a)\,\overline{p(a)\,z^{\Delta}(a)}+p(a)\,y^{\Delta}(a)\,\overline{z(a)} \\ &\quad +[y,u]_b[z,v]_b-[z,u]_b[y,v]_b \\ &= [y,z]_b-[y,z]_a=(Ly,z)-(y,Lz), \end{aligned}$$

i.e., the first condition in the definition of the space of boundary values holds. It is easy to see that the second condition of the boundary value space holds. □

Recall that a linear operator T (with dense domain $D(T)$) acting on some Hilbert space H is called dissipative (accretive) if $\operatorname{Im}(Tf, f) \geq 0$ ($\operatorname{Im}(Tf, f) \leq 0$) for all $f \in D(T)$ and maximal dissipative (maximal accretive) if it does not have a proper dissipative (accretive) extension.

From [10, 17], following theorem is obtained.

***Theorem* 2** — *For any contraction K in $\mathbb{C}^2$ the restriction of the operator L to the set of functions $y \in D$ satisfying either*

$$(K - I)\,\Gamma_1 y + i\,(K + I)\,\Gamma_2 y = 0 \tag{3.1}$$

or

$$(K - I)\,\Gamma_1 y - i\,(K + I)\,\Gamma_2 y = 0 \tag{3.2}$$

is respectively the maximal dissipative and accretive extension of the operator L_0. Conversely, every maximal dissipative (accretive) extension of the operator L_0 is the restriction of L to the set of functions $y \in D$ satisfying (3.1) (3.2), and the extension uniquely determines the contraction K. Conditions (3.1) (3.2), in which K is an isometry describe the maximal symmetric extensions of L_0 in $\mathbb{C}^2$. If K is unitary, these conditions define self-adjoint extensions. In particular, the boundary conditions

$$\cos\alpha\; y(a) + \sin\alpha\; p(a)\, y^{\Delta}(a) = 0 \tag{3.3}$$

$$[y, u]_b + h[y, v]_b = 0 \tag{3.4}$$

with $\operatorname{Im}\ h \geq 0$, describe the maximal dissipative extensions of L_0 with separated boundary conditions.

It follows from Theorem 2, all the eigenvalues of L lie in the closed upper half plane $\operatorname{Im}\lambda \geq 0$.

***Theorem* 3** — *The operator L has not any real eigenvalue.*

PROOF : Suppose that the operator L has a real eigenvalue λ_0. Let $\eta_0(x) = \eta(x, \lambda_0)$ be the corresponding eigenfunction. Since

$$\operatorname{Im}(L\eta_0, \eta_0) = \operatorname{Im}\left(\lambda_0 \|\eta_0\|^2\right) = \operatorname{Im}h\,([\eta_0, v]_b)^2\,,$$

we get $[\eta_0, v]_b = 0$. By the boundary condition (3.4), we have $[\eta_0, u]_b = 0$. Thus

$$[\eta_0(t, \lambda_0), u]_b = [\eta_0(t, \lambda_0), v]_b = 0. \tag{3.5}$$

Let $\xi_0(t) = v(t, \lambda_0)$. Then

$$1 = [\eta_0, \xi_0]_b = [\eta_0, u]_b[\xi_0, v]_b - [\eta_0, v]_b[\xi_0, u]_b.$$

By the equality (3.5), the right -hand side is equal to 0. This contradiction proves the theorem. □

Lemma 1 — Zero is not an eigenvalue L.

PROOF : Let $y \in D(L)$ and $Ly = 0$. Then

$$-\left[p(t)\, y^{\Delta}(t)\right]^{\nabla} + q(t)\, y(t) = 0,$$

and $y(t) = c_1 u(t) + c_2 v(t)$. Substituting this in the boundary conditions (3.3)-(3.4) we find that $c_1 = c_2 = 0;\ i.e., y = 0.$ □

Let's remind the Krein's theorem.

Definition 1 — Let f be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that

$$|f(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C} \tag{3.6}$$

then f is called an entire function of order ≤ 1 of growth and minimal type [9].

Let A denote the linear non-selfadjoint operator in the Hilbert space with domain $D(A)$. A complex number λ_0 is called an eigenvalue of the operator A if there exists a non-zero element $y_0 \in D(A)$ such that $Ay_0 = \lambda_0 y_0$; in this case, y_0 is called the eigenvector of A for λ_0. The eigenvectors for λ_0 span a subspace of $D(A)$, called the eigenspace for λ_0.

The element $y \in D(A),\ y \neq 0$ is called a root vector of A corresponding to the eigenvalue λ_0 if $(T - \lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. The root vectors for λ_0 span a linear subspace of $D(A)$, is called the root lineal for λ_0. The algebraic multiplicity of λ_0 is the dimension of its root lineal. A root vector is called an associated vector if it is not an eigenvector. The completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

***Theorem* 4** [9] — *The system of root vectors of a compact dissipative operator B with nuclear imaginary component is complete in the Hilbert space H so long as at least one of the following two conditions is fulfilled*:

$$\lim_{\rho\to\infty} \frac{n_+(\rho, B_R)}{\rho} = 0, \text{ or } \lim_{\rho\to\infty} \frac{n_-(\rho, B_R)}{\rho} = 0,$$

where $n_+(\rho, B_R)$ *and* $n_-(\rho, B_R)$ *denote the numbers of the characteristic values of the real component* B_R *of the operator* B *in the intervals* $[0,\rho]$ *and* $[-\rho, 0]$*, respectively.*

***Theorem* 5** [18] — *If the entire function* f *satisfies the condition (3.6), then*

$$\lim_{\rho\to\infty}\frac{n_+(\rho, f)}{\rho}\ \lim_{\rho\to\infty}\frac{n_-(\rho, f)}{\rho}=0$$

where $n_+(\rho, f)$ *and* $n_-(\rho, f)$ *denote the numbers of the zeros of the function* f *in the intervals* $[0,\rho]$ *and* $[-\rho, 0]$*, respectively.*

From Lemma 1, there exist the inverse operator L^{-1}. In order to describe the operator L^{-1} we use the Green's function method. We consider the functions $v(x)$ and $\theta(x) = u(x) + hv(x)$. These functions belong to the space $L^2_\Delta(\mathbb{T}^*)$. Their Wronskian $W(v,\theta) = -1$.

Let

$$G(x,t) = \begin{cases} v(x)\,\theta(t), & a \le x \le t \le b \\ v(t)\,\theta(x), & a \le t \le x \le b \end{cases}. \tag{3.7}$$

The integral operator K defined by the formula

$$Kf = \int_a^b G(x,t)\,\overline{f(t)}\Delta t \quad \left(f \in L^2_\Delta(\mathbb{T}^*)\right) \tag{3.8}$$

is a compact linear operator in the space $L^2_\Delta(\mathbb{T}^*)$. K is a Hilbert Schmidth operator. It is evident that $K = L^{-1}$. Consequently the root lineals of the operator L and K coincide and, therefore, the completeness in $L^2_\Delta(\mathbb{T}^*)$ of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of L may have only a finite number of linear independent associated vectors.

Let $\varphi(x,\lambda)$ is the single linearly independent solution of the equation $l(y) = \lambda y$, and

$$\begin{aligned} \tau_1(\lambda) &:= [\varphi(x,\lambda), u(x)]_b, \\ \tau_2(\lambda) &:= [\varphi(x,\lambda), v(x)]_b, \\ \tau(\lambda) &:= \tau_1(\lambda) + h\tau_2(\lambda). \end{aligned}$$

It is clear that

$$\sigma_p(L) = \{\lambda : \lambda \in \mathbb{C},\ \tau(\lambda) = 0\}$$

where $\sigma_p(L)$ denotes the set of all eigenvalues of L. Since for arbitrary $c\ (a \leq c < b)$, the functions $\varphi(c,\lambda)$ and $\varphi^{\triangle}(c,\lambda)$ are entire functions of λ of order ≤ 1 (see [8]), consequently, $\tau(\lambda)$ have the same property. Then $\tau(\lambda)$ is entire functions of the order ≤ 1 of growth, and of minimal type.

***Theorem* 6** — *The system of all root vectors of the dissipative operator K is complete in $L^2_\Delta(\mathbb{T}^*)$.*

PROOF : It will be sufficient to prove that the system of all root vectors of the operator $K = L^{-1}$ in (3.8) is complete in $L^2_\Delta(\mathbb{T}^*)$. Since $\theta(x) = u(x) + hv(x)$, setting $h = h_1 + ih_2\ (h_1, h_2 \in \mathbb{R})$, we get from (3.8) in view of (3.7) that $K = K_1 + iK_2$, where

$$K_1 f = \left(G_1(x,t), \overline{f(t)}\right)_{L^2_\Delta(\mathbb{T}^*)},\ K_2 f = \left(G_2(x,t), \overline{f(t)}\right)_{L^2_\Delta(\mathbb{T}^*)}$$

and

$$G_1(x,t) = \begin{cases} v(x)\,[u(t) + h_1 v(t)], & a \leq t \leq x \leq b, \\ v(t)\,[u(x) + h_1 v(x)], & a \leq t \leq x \leq b, \end{cases}$$

$$G_2(x,t) = -h_2 v(x)\, v(t),\ h_2 = \mathrm{Im} h > 0.$$

The operator K_1 is the self-adjoint Hilbert–Schmidt operator in $L^2_\Delta(\mathbb{T}^*)$, and K_2 is the self-adjoint one dimensional operator in $L^2_\Delta(\mathbb{T}^*)$. Let L_1 denote the operator in $L^2_\Delta(\mathbb{T}^*)$ generated by the differential expression l and the boundary conditions

$$\begin{aligned} \cos\alpha\, y(a) + \sin\alpha\, p(a)\, y^{\Delta}(a) &= 0, \\ [y,u]_b + h_1[y,v]_b &= 0, \end{aligned}$$

where $h_1 = \mathrm{Re} h$. It is easy to verify that K_1 is the inverse L_1. Further

$$\sigma_p(L_1) = \{\lambda : \lambda \in \mathbb{C},\ \Psi(\lambda) = 0\} \tag{3.9}$$

where

$$\Psi(\lambda) := \tau_1(\lambda) + h_1 \tau_2(\lambda). \tag{3.10}$$

Then we find

$$|\Psi(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|},\ \ \forall \lambda \in \mathbb{C}. \tag{3.11}$$

Let $T = -K$ and $T = T_1 + iT_2$, where $T_1 = -K_1,\ T_2 = -K_2$. The characteristic values of the operator K_1 coincide with the eigenvalues of the operator L_1. From (3.9), (3.11) and Theorem 5, we have

$$\lim_{\rho\to\infty} \frac{m_+(\rho, T_1)}{\rho} = 0, \text{ or } \lim_{\rho\to\infty} \frac{m_-(\rho, T_1)}{\rho} = 0,$$

where $m_+(\rho, T_1)$ and $m_-(\rho, T_1)$ denote the numbers of the characteristic values of the real component $T_R = T_1$ in the intervals $[0, \rho]$ and $[-\rho, 0]$, respectively. Thus the dissipative operator T (also of K) carries out all the conditions of Krein's theorem on completeness. The theorem is proved. □

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