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COMPUTING EIGENELEMENTS OF STURM-LIOUVILLE PROBLEMS BY USING DAUBECHIES WAVELETS

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This work is our first step to get multiresolution approximation of eigenelements of Sturm-Liouville problems within bounded domain of varied nature. The formula for obtaining elements of representation of Sturm-Liouville operator involving polynomial coefficients in wavelet basis of Daubechies family have been derived in a form which can be readily used for their computations by a simple computer program. Estimates of errors for both the eigenvalues and eigenfunctions are also presented here. The proposed wavelet-Galerkin scheme based on scale functions and wavelets of Daubechies family having three or four vanishing moments of their wavelets has been applied to get approximate eigenelements of regular and singular Sturm-Liouville problems within bounded domain and compared with the exact or approximate results whenever available. From our study it appears that the proposed method is efficient and rapidly convergent in comparison to other approximation schemes based on variational method in Haar basis or finite difference methods studied by Bujurke *et al*. [39].

Key words : Daubechies scale functions and wavelets; multiresolution approximation of eigenelements; Sturm-Liouville problems; quantum mechanical bound state problems

1. INTRODUCTION

The development of efficient methods for the solution of ordinary/partial differential equations is an important task. There are many different methods for the numerical solution of differential equations which include finite-difference techniques, finite element method, Galerkin (variational) methods, collocation and spectral methods etc. In every method, the main interest is to develop numerical schemes that yield accurate solutions with a minimum of computational effort. The effort expended can be assessed in terms of the ease of implementation of the method and computer resources required to achieve a specified accuracy.

Sturmian theory is one of the most extensively developing fields in the domain of differential equations [1, 2, 3]. There has been increasing interest in the spectral analysis of boundary value problems with eigenvalue dependent boundary conditions [4], impulse effects (also known as interface conditions, transmission conditions, discontinuity conditions) [5], multi-interval problems [6], equations involving higher order even fractional derivatives [7, 8], quantum mechanical bound state problems at nano-scale etc. which appear in diverse fields of physical processes. Naturally, availability of efficient method for obtaining eigenelements of Sturm-Liouville problems (SLP) will be of highly beneficial for the mathematical study of the problems mentioned above.

In recent years, a lot of interest has been devoted to explore the possibilities of the development of wavelet based techniques to investigate various physical problems [9-12]. Since the inception of multiresolution analysis (MRA) of $L^2(\mathbb{R})$ [13] there is an intense activity on the development of MRA of L^2-space of functions over a bounded interval $[a, b] \subset \mathbb{R}$. Such efforts enriched the theory through the invention of different families of wavelets e.g. spline wavelets belonging to biorthogonal family [14, 15], interpolating wavelets contained in the multiwavelet family [16-18], second generation wavelets [19, 20] etc. But wavelet basis of each of these families has some advantages as well as limitations too.

Although the scale function and wavelets based on spline in biorthogonal family (involving separate L^2-spaces $V, \tilde{V}$) are symmetric, elements of these bases at resolution j are not orthonormal in the same space V_j or $\tilde{V}_j$. Thus four sequences of filters (cf. section 2) are necessary during the exercise of MRA of subtle (local) behaviour of signal or function f. Moreover, the regularity of the function cannot be predicted in a straightforward manner ($f \in C^s(\mathbb{R})$ iff $|\langle f|\psi_{j,k}\rangle| \leq \frac{c}{2^{j(s+\frac{1}{2})}}$) as in the case of wavelets in Daubechies family). The wavelets in (Legendre) multiwavelet family are not continuous at some points within its support at every resolution j. As a consequence the values of functions obtained from their representation in Legendre multiwavelet basis show distorted behaviour in the neighborhood of those singularities.

Vasilyev *et al.* [21] observed that the wavelet basis constructed using interpolating scaling functions does not provide Riesz basis for L^2, as the wavelet itself has nonzero mean, and the dual wavelets are delta functions which do not belong to L^2. In addition, the wavelet transform derived from interpolation introduces considerable aliasing, since the scales are not well separated by the interpolating wavelets. The later property of the interpolating wavelets can lead to either unstable or inaccurate results. In addition wavelet coefficients cannot be used for analysis or/and prediction of small scale phenomena, since severe aliasing completely distorts their values.

The underlying machinery of second generation wavelets is the lifting scheme. The basic idea of lifting is to start with simple MRA and gradually build it with specific, a-priori defined properties. Lifting scheme allows one to custom design the filters, needed in the transform algorithm, to the situation at hand. But the question is whether these filters actually generate functions which form a stable basis, or have smoothness, remain to be checked in each particular case [19, 20].

The methods based on wavelet for solving differential equation have many salient features due to the multiresolution properties of space of functions spanned by wavelets [22-27]. In spite of the limited success in finding eigen spectrum of quantum mechanical problems in the basis of multiresolution generator and wavelets having compact support [28, 29], Dai *et al*. [30] observed that "wavelets constructed through the usual method cannot be readily applied to operators with unbounded coefficients such as Schrödinger equation in quantum mechanics .....as well as operators with singular coefficients". In our recent studies [31] it is observed that the method based on scale functions in Daubechies family for solving singular integral equation have many salient features due to the multiresolution properties of space of functions. Of equal practical significance is the fact that the method's implementation requires no modification in the presence of singularities. The discrete system approximating a problem depends only on parameters involved in the equation regardless of whether it is singular or nonsingular. The error of the method converges to zero like $o(c/2^j)$, where j is the resolution of the detail space used, and c is a positive constant independent of the resolution j. Recently, Bulut and Polyzou [32], Brennen *et al*. [33] observed that orthonormality of the scale functions and wavelets in Daubechies family have advantages in applications to the problems in quantum mechanics and quantum field theory. These observations encouraged us to explore whether the numerical scheme based on wavelets in Daubechies family can be used to obtain eigenspectrum of SLP readily when the underlying domain is bounded.

In this paper formulae for multiscale represention of differential operators of both nonsingular or singular nature involved in SLP in a bounded domain in the basis comprising truncated scale function and wavelet of Daubechies family have been derived. These are applied to obtain eigenelements of the corresponding eigenvalue problem in a straightforward manner.

The paper is organized as follows. The multiresolution properties of scale function and wavelet of Daubechies family within a bounded domain, their moments and moments of their derivatives etc. required for transforming the differential equation to a system of linear simultaneous equation are presented in section 2. Section 3 deals with the construction of algebraic equations and their solutions by using existing mathematical tools executable by the solver in MATHEMATICA. The error of the wavelet-Galerkin method based on Daubechies wavelets has been discussed in section 4. A comparison of the efficiency of the method with existing methods is presented in section 5 with two illustrative examples while a short conclusion is given in section 6.

2. Multiresolution Properties of Daubechies Wavelets within Finite Domain

A MRA of the space $L^2(\mathbb{R})$, (set) of square integrable ($\int |f|^2 dx$ is finite) functions in the Lebesgue sense, is defined as a set of closed subspaces $V_j, j \in \mathbb{Z}$, with properties [34]:

1. Nestedness: $\{0\}.. \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots\ldots, V_j \subset V_{j+1} \subset \ldots ...\subset L^2(\mathbb{R})$,

2. Admissibility: $\bigcap\limits_{j\in\mathbb{Z}} V_j \ = \ \{0\}$,

3. Closure: $\overline{\bigcup\limits_{j\in\mathbb{Z}} V_j} \ = \ L^2(\mathbb{R})$,

4. Refinement: $f(\cdot) \in V_j$ iff $f(2\cdot) \in V_{j+1}\ \forall j \in \mathbb{Z}$,

5. Existence of generator and basis: $\exists\ \varphi(\cdot) \in V_0$ such that $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ is an unconditional basis of V_0,

6. Pseudo-distribution: $\int \varphi dx = 1$.

The function φ appearing in conditions 5 and 6 of MRA is known as ***multiresolution generator*** (MRG). In our discussion it is of finite support $[0, 2K-1]$ and satisfies the refinement equation or two-scale relation

$$\varphi_k(\cdot) = \varphi(\cdot - k) = \sum_{l=0}^{2K-1} h_l\ \varphi_1(\cdot - 2k - l) = \sqrt{2}\sum_{l=0}^{2K-1} h_l\ \varphi(2\cdot - 2k - l) = \boldsymbol{\Phi}_{1,2k}\boldsymbol{H} \tag{2.1}$$

with

$$\boldsymbol{H} = (h_0, h_1, \ldots h_{2K-1})_{2K\times 1}\,, \tag{2.2a}$$

$$\boldsymbol{\Phi}_{1,2k} = \sqrt{2}(\varphi(2\cdot -2k), \phi(2\cdot -2k-1)\ \ , ..., \varphi(2\cdot -2k-2K+1))_{1\times 2K}. \tag{2.2b}$$

The symbol K is a natural number, known as the genus of the wavelet in Daubechies family, representing the number of vanishing moments, *viz*.

$$\int x^n \psi_k(x)dx = 0 \;\; \forall \;\; n = 0, 1, ..., K-1 \text{ and } \; k \in \mathbb{Z}. \tag{2.3}$$

The function $\psi_k(\cdot) = \psi(\cdot - k)$, the translate of the wavelet $\psi(\cdot)$ are orthogonal to all $\varphi_k(\cdot), k \in \mathbb{Z}$ and can be expressed by the relation

$$\psi_k(\cdot) = \psi(\cdot - k) = \sum_{l=0}^{2K-1} g_l \, \varphi_1(\cdot - 2k - l) = \sqrt{2} \sum_{l=0}^{2K-1} (-1)^l h_{2K-1-l} \, \varphi(2 \cdot -2k - l) = \boldsymbol{\Phi}_{1,2k}\boldsymbol{G} \tag{2.4}$$

where the column vector

$$\boldsymbol{G} = (g_0, g_1, \ldots g_{2K-1})_{2K \times 1} = (h_{2K-1}, -h_{2K-2}, h_{2K-3}, \ldots, -h_0)_{2K \times 1} \, .$$

The matrices ***H*** and ***G*** of Eqs. (2.1) and (2.4) are known as low-pass and high-pass filters respectively for the MRA generated by the MRG φ. Wavelet $\psi(\cdot)$ and its translates $\psi(\cdot - k)$ are all orthonormal, span the closed subspace $W_0 \subset V_1$ such that

$$V_1 = V_0 \bigoplus W_0.$$

This formula for splitting the approximation space V_{j+1} into the approximation subspace V_j and detail subspace W_j have been followed for all $j \in \mathbb{Z}$. The basis for the subspaces V_j and W_j are, respectively

$$\varphi_{jk}(\cdot) = 2^{\frac{j}{2}} \varphi(2^j \cdot -k) \tag{2.5a}$$

and

$$\psi_{jk}(\cdot) = 2^{\frac{j}{2}} \psi(2^j \cdot -k), \;\; k \in \mathbb{Z}. \tag{2.5b}$$

But whenever the domain of interest becomes a finite subset, say $\Omega = [a, b] \subset \mathbb{R}$, the translation invariance and orthonormality of $\{\varphi_{j_0 k} \;\; k \in \mathbb{Z}, \; \psi_{jk'} \;\; k^{'} \in \mathbb{Z}, \; j \geq j_0\}$ have been lost. In such a situation, scale functions and wavelets are classified into three classes [35, 36], *viz*.,

$$\wedge_j^R = \begin{cases} \varphi_{jl}^R \chi_{[a,b]}, & l = a2^j - 2K + 2, \cdots\cdots, a2^j - 1, \\ \psi_{jl}^R \chi_{[a,b]}, & l = a2^j - 2K + 2, \cdots\cdots, a2^j - 1, \end{cases} \tag{2.6a}$$

$$\wedge_j^I = \begin{cases} \varphi_{jl}, & l = a2^j, \cdots\cdots, b2^j - 2K + 1, \\ \psi_{jl}, & l = a2^j, \cdots\cdots, b2^j - 2K + 1, \end{cases} \tag{2.6b}$$

$$\wedge_j^L = \begin{cases} \phi_{jl}^L \, \chi_{[a,b]}, & l = b2^j - 2K + 2, \cdots\cdots, b2^j - 1, \\ \psi_{jl}^L \, \chi_{[a,b]}, & l = b2^j - 2K + 2, \cdots\cdots, b2^j - 1. \end{cases} \tag{2.6c}$$

Here $\chi_{[a,b]}$ represents the characteristic function for the interval Ω. Since the supports of scale functions and wavelets in the classes Λ_j^R and Λ_j^L overlaps partially on Ω, their two-scale relations do not follow Eqs. (2.1) and (2.4). Instead those are given by

$$\boldsymbol{\varphi}^R = \left(\boldsymbol{\varphi}_1^R(\cdot), \boldsymbol{\varphi}_1^{RI}(\cdot)\right) \begin{pmatrix} \mathbf{H^R} \\ \\ \mathbf{H^{RI}} \end{pmatrix}, \quad \boldsymbol{\varphi}^L = \left(\boldsymbol{\varphi}_1^L(\cdot), \boldsymbol{\varphi}_1^{LI}(\cdot)\right) \begin{pmatrix} \mathbf{H^L} \\ \\ \mathbf{H^{LI}} \end{pmatrix}, \tag{2.7a,b}$$

$$\boldsymbol{\psi}^R = \left(\boldsymbol{\varphi}_1^R(\cdot), \boldsymbol{\varphi}_1^{RI}(\cdot)\right) \begin{pmatrix} \mathbf{G^R} \\ \\ \mathbf{G^{RI}} \end{pmatrix}, \quad \boldsymbol{\psi}^L = \left(\boldsymbol{\varphi}_1^L(\cdot), \boldsymbol{\varphi}_1^{LI}(\cdot)\right) \begin{pmatrix} \mathbf{G^L} \\ \\ \mathbf{G^{LI}} \end{pmatrix}. \tag{2.8a,b}$$

The elements of the matrices $\mathbf{H^{R,L}}$, $\mathbf{H^{RI,LI}}$ involved in the two-scale relations and moments of $\boldsymbol{\varphi}^{R,L}$ or $\boldsymbol{\psi}^{R,L}$ are discussed in details in [31]. In Galerkin wavelet approximation to the solution of SLP with variable coefficients within finite domain, the moments of product of scale functions/wavelets and one of their derivatives are necessary.

2.1 *Values of MRG and its derivatives within its support* [37, 38]

The numerical values of the scale function φ or its derivatives $\varphi^{(p)}$ at any dyadic point within its support are obtained recursively whenever its values at any integer within its support are known. By construction, the eigenvector corresponding to the eigenvalue $\frac{1}{2^{p+\frac{1}{2}}}$ of the recursion matrix

$$L = [l_{mn}] = [h_{2m-n}]_{2K-2\times 2K-2}, \text{for given } K \in \mathbb{N}$$

provides values $c\left[\varphi^{(p)}(1), \varphi^{(p)}(2), \cdots\cdots, \varphi^{(p)}(2K-2)\right]$. However, it can be obtained directly as the solution of overdetermined system of linear simultaneous equations

$$\varphi^{(p)}(i) = 2^{p+\frac{1}{2}} \sum_{k=0}^{2K-1} h_k\, \varphi^{(p)}(2i-k), \;\; i = 0, 1, 2, \ldots, 2K-1 \tag{2.9a}$$

and

$$\sum_{k=-2K+1}^{0} \langle x^p \rangle_{\varphi_k}\, \varphi^{(p)}(-k) = p!. \tag{2.9b}$$

Once the values of $\varphi^{(p)}$ at integers are known, formula (2.9a) will provide values of $\varphi^{(p)}(q)$ at any dyadic q within its support for $p = 0, 1, \ldots, K-1$.

2.2 *Moments of product of Daubechies scale functions and Wavelets*

a. Moment of product of scale functions/wavelets on $\mathbb{R}$

If we denote

$$\Gamma V_{l\,k}^{I}(m) = \int_{-\infty}^{\infty} \varphi_l\,(x)\,x^m \varphi_k\,(x)\,dx,$$

and use (2.1) in its RHS, then the recursion relation for $\Gamma V_k^I(m) = \Gamma V_{0\,k}^I(m)$ takes the form

$$\Gamma V_k^I\,(m) = \frac{1}{2^m} \sum_{l_1=0}^{2K-1} \sum_{k_1=0}^{2K-1} h_{l_1} h_{k_1} \sum_{n=0}^{m} \begin{pmatrix} m \\ n \end{pmatrix} l_1^{m-n} \Gamma V_{2k+k_1-l_1}^I\,(n)\,. \tag{2.10}$$

Since support $[k, k+2K-1]$ of $\varphi_k(x)$ is finite,

$$\Gamma V_k^I\,(m) = \begin{cases} 0 & \text{for } \ |k| > 2\text{K}-2, \\ \text{Formula } \ (2.10) & \text{for } \ |k| \ \leq \ 2\text{K}-2. \end{cases}$$

Once $\Gamma V_k^I(m)$,$(-2K+2 \leq k \leq 2K-2)$ are known, $\Gamma V_{l\,k}^I(m)$ can be obtained by using the formula

$$\Gamma V_{l\,k}^I\,(m) = \begin{cases} 0 & \text{if } \ |k-l| > 2K-2, \\ \sum_{n=0}^{m} \begin{pmatrix} m \\ n \end{pmatrix} l^{m-n} \Gamma V_{k-l}^I\,(m) & \text{if } \ |k-l| \leq 2K-2. \end{cases} \tag{2.11}$$

Formula for the moments when both the scale functions in the integrand are at resolution $j_0 \neq 0$ is given by

$$\rho V_{j_0 lk}^I\,(m) = \Gamma V_{j_0 l\;k}^I\,(m) = \int_{-\infty}^{\infty} \varphi_{j_0 l}\,(x)\,x^m \varphi_{j_0 k}\,(x)\,dx \ = \begin{cases} 0 & \text{if } \ |k-l| > 2K-2, \\ \frac{1}{2^{j_0 m}} \Gamma V_{l\,k}^I\,(m) & \text{if } \ |k-l| \leq 2K-2. \end{cases} \tag{2.12}$$

Note that $\rho V_{j_0 lk}^I(0) = \delta_{lk}$, the Kronecker delta symbol, is independent of j_0. In case of moments involving product of two scale functions at two different scales say, $\varphi_{jl}(x)$ and $\varphi_{0k}(x)$,

$$\rho V_{jl;0k}^I\,(m) = \int_{-\infty}^{\infty} \varphi_{jl}\,(x)\,x^m \varphi_k\,(x)\,dx$$

$$= \begin{cases} \frac{1}{2^m} \sum_{k_1=0}^{2K-1} h_{k_1} \rho V_{j-1l;0\;2k+k_1}^I\,(m) & \text{for } \ -2K+2 \leq l - 2^j k \leq 2^j\,(2K-1) - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Moments involving product of two scale functions at two different non-zero scales say, $\varphi_{j\;l}(x)$ and $\varphi_{j_0 k}(x)$ are given by

$$\rho V_{ji;j_0 k}^I(m) = \int_{-\infty}^{\infty} \varphi_{jl}(x) x^m \varphi_{j_0 k}(x) dx$$

$$= \begin{cases} \frac{1}{2^{j_0 m}}\rho V^I_{j-j_0\ l;0\ k}(m) & \text{for} \quad j > j_0, -2K+2 \leqslant l - 2^{j-j_0}k \leqslant 2^{j-j_0}(2K-1)-1 \\ \frac{1}{2^{j_0 m}}\rho V^I_{0\ l;j_0-j\ k}(m) & \text{for} \quad j < j_0, -2K+2 \leqslant 2^{j-j_0}l - k \leqslant 2^{j-j_0}(2K-1)-1 \\ 0 & \qquad \text{otherwise.} \end{cases}, \quad \text{(2.14a)}$$

Following similar steps, the moments of product of scale function-wavelet and wavelet-wavelet are given by

$$\alpha V^I_{jl;j_0k}(m) = \int_{-\infty}^{\infty} \psi_{jl}(x) x^m \varphi_{j_0k}(x) dx$$

$$= \begin{cases} \frac{1}{2^{j_0 m}}\alpha V^I_{j-j_0 l;0k}(m) & \text{for} \quad j > j_0, -2K+2 \leqslant l - 2^{j-j_0}k \leqslant 2^{j-j_0}(2K-1)-1 \\ \frac{1}{2^{j m}}\alpha V^I{}_{0l;j_0-jk}(m) & \text{for} \quad j < j_0, -2K+2 \leqslant 2^{j-j_0}l - k \leqslant 2^{j-j_0}(2K-1)-1 \\ 0 & \qquad \text{otherwise.} \end{cases}, \quad \text{(2.14b)}$$

$$\beta V^I_{jl;j_0k}(m) = \int_{-\infty}^{\infty} \varphi_{jl}(x) x^m \psi_{j_0k}(x) dx$$

$$= \begin{cases} \frac{1}{2^{j_0 m}}\beta V^I_{j-j_0 l;0k}(m) & \text{for} \quad j > j_0, -2K+2 \leqslant l - 2^{j-j_0}k \leqslant 2^{j-j_0}(2K-1)-1 \\ \frac{1}{2^{j m}}\beta V^I{}_{0l;j_0-jk}(m) & \text{for} \quad j < j_0, -2K+2 \leqslant 2^{j-j_0}l - k \leqslant 2^{j-j_0}(2K-1)-1 \\ 0 & \qquad \text{otherwise.} \end{cases}, \quad \text{(2.14c)}$$

$$\gamma V^I_{jl;j_0k}(m) = \int_{-\infty}^{\infty} \psi_{jl}(x) x^m \psi_{j_0k}(x) dx$$

$$= \begin{cases} \frac{1}{2^{j_0 m}}\gamma V^I_{j-j_0 l;0k}(m) & \text{for} \quad j > j_0, -2K+2 \leqslant l - 2^{j-j_0}k \leqslant 2^{j-j_0}(2K-1)-1 \\ \frac{1}{2^{j m}}\gamma V^I{}_{0l;j_0-jk}(m) & \text{for} \quad j < j_0, -2K+2 \leqslant 2^{j-j_0}l - k \leqslant 2^{j-j_0}(2K-1)-1 \\ 0 & \qquad \text{otherwise.} \end{cases}, \quad \text{(2.14d)}$$

b. Moment of product of scale functions/wavelets on $\mathbb{R}^+$ and $\mathbb{R}^-$

Let us consider integrals on $\mathbb{R}^+ = [0, \infty)$ involving product of scale functions in Λ_0^R of (2.6) whose supports contain the origin 0 *viz*,

$$\Gamma V_{lk}^{R}(m) := \int_{0}^{\infty} \varphi_{l}^{R}(x) x^{m} \varphi_{k}^{R}(x) dx, \qquad -2K+1 < k, l < 0. \tag{2.15a}$$

Note that

$$\Gamma V_{l\,k}^{R}(m) = \begin{cases} \Gamma V_{l\,k}^{I}(m) & \text{if} \quad l \text{ or } k \ \geq 0, \\ 0 & \text{if} \quad l \text{ or } k \ \leq -2K+1 \end{cases} \tag{2.15b}$$

due to finite support of $\varphi(x)$. Using relations (2.7a) and (2.15b) in (2.15a) one gets system of linear algebraic equations

$$\mathbf{\Gamma}\boldsymbol{V}^{\mathrm{R}}(m) - \frac{1}{2^{m}} \mathbf{H}^{\mathrm{R}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{R}}(m) (\mathbf{H}^{\mathrm{R}})^{\mathrm{T}} = \mathbf{H}^{\mathrm{R}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{RI}}(m) (\mathbf{H}^{\mathrm{RI}})^{\mathrm{T}} + \mathbf{H}^{\mathrm{RI}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{IR}}(m) (\mathbf{H}^{\mathrm{R}})^{\mathrm{T}} + \boldsymbol{H}^{\mathrm{RI}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{RII}}(m) (\mathbf{H}^{\mathrm{RI}})^{\mathrm{T}} \tag{2.16}$$

for $(2K-2)\times(2K-2)$ elements $\Gamma V_{lk}^{R}(m)$ of $\mathbf{\Gamma}\boldsymbol{V}^{R}(m)$. Here $\mathbf{\Gamma}\boldsymbol{V}^{RI}(m)$, $\mathbf{\Gamma}\boldsymbol{V}^{IR}(m)$ and $\mathbf{\Gamma}\boldsymbol{V}^{RII}(m)$ are all $(2K-2)\times(2K-2)$ matrices given by

$$\mathbf{\Gamma V}^{RI}(m) = \left(\rho V_{1;l\ k}^{I}(m)\right)_{(2K-2)\times(2K-2)}; l = -2K+2, -2K+3, \cdots, -1, \quad k = 0, 1, \cdots, 2K-3 \tag{2.17a}$$

$$\mathbf{\Gamma V}^{IR}(m) = \left(\rho V_{1;l\ k}^{I}(m)\right)_{(2K-2)\times(2K-2)}; l = 0, 1, \cdots, 2K-3, \quad k = -2K+2, -2K+3, \cdots, -1, \tag{2.17b}$$

$$\mathbf{\Gamma V}^{RII}(m) = \left(\rho V_{1;l\ k}^{I}(m)\right)_{(2K-2)\times(2K-2)}; \ l, k = 0, 1, \cdots, 2K-3. \tag{2.17c}$$

The linear simultaneous equations for the integrals

$$\Gamma V_{l\,k}^{L}(m) = \int_{-\infty}^{0} \varphi_{l}^{L}(x)\, x^{m} \varphi_{k}^{L}(x)\, dx, \ -2K+1 < k, l < 0 \tag{2.18}$$

on $\mathbb{R}^{-} = (-\infty, 0]$, involving product of scale functions whose supports contain the origin 0 are given by

$$\mathbf{\Gamma}\boldsymbol{V}^{L}(m) - \frac{1}{2^{m}} \mathbf{H}^{\mathrm{L}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{L}}(m) (\mathbf{H}^{\mathrm{L}})^{\mathrm{T}} = \mathbf{H}^{\mathrm{L}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{LI}}(m) (\mathbf{H}^{\mathrm{LI}})^{\mathrm{T}} + \mathbf{H}^{\mathrm{LI}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{IL}}(m) (\mathbf{H}^{\mathrm{L}})^{\mathrm{T}} + \mathbf{H}^{\mathrm{LI}} \mathbf{\Gamma}\boldsymbol{V}^{\mathrm{LII}}(m) (\mathbf{H}^{\mathrm{LI}})^{\mathrm{T}} \tag{2.19}$$

with

$$\mathbf{\Gamma V}^{LI}(m) = \left(\rho V_{1;l\ k}^{I}(m)\right)_{(2K-2)\times(2K-2)}; l = -2K+2, , \cdots, -1, \quad k = -4K+4, , \cdots, -2K+1; \tag{2.20a}$$

$$\mathbf{\Gamma V}^{IL}(m) = \left(\rho V^{I}_{1;l\ k}(m)\right)_{(2K-2)\times(2K-2)}; l=-4K+4,\cdots,-2K+1,\ k=-2K+2,\cdots,-1 \tag{2.20b}$$

$$\mathbf{\Gamma V}^{IL}(m) = \left(\rho V^{I}_{1;l\ k}(m)\right)_{(2\mathrm{K}-2)\times(2\mathrm{K}-2)}; l,k=-4K+4,-4K+3,\cdots,-2K+1. \tag{2.20c}$$

Note that

$$\Gamma \mathrm{V}^{\mathrm{L}}_{\mathrm{l\ k}}(m) = \begin{cases} 0 & \text{if} \quad l \text{ or k } \geq 0\,, \\ \\ \Gamma \mathrm{V}^{\mathrm{I}}_{\mathrm{l\ k}}(m) & \text{if} \quad l \text{ or k } \leq -2\mathrm{K}+1. \end{cases} \tag{2.21}$$

The values of above integrals involving boundary scale functions $\varphi^{R,L}_{k}(x)$, $\varphi^{R,L}_{jl}(x)$, one at 0 -resolution and the other at resolution-j are given by

$$\boldsymbol{\rho V}^{R,L}_{j;0}(m) = \frac{1}{2^m}\left[\boldsymbol{\rho V}^{R,L}_{j-1;0}(m)\left(\mathbf{H}^{R,L}\right)^T + \boldsymbol{\rho V}^{RI,LI}_{j-1;0}(m)\left(\mathbf{H}^{RI,LI}\right)^T\right]. \tag{2.22a}$$

The moment of the product of two boundary scale functions at two different non-zero resolution can be obtained recursively by using the results of (2.22a) into the formula

$$\boldsymbol{\rho V}^{R,L}_{j;j_0}(m) = \int_{0,-\infty}^{\infty,0} \varphi^{R,L}_{j\ l}(x)\, x^m \varphi^{R,L}_{j_0\ k}(x)\, dx = \begin{cases} \frac{1}{2^{j_0 m}}\boldsymbol{\rho V}^{R,L}_{j-j_0;0}(m) & \text{if} \ \ j > j_0, \\ \\ \frac{1}{2^{jm}}\boldsymbol{\rho V}^{R,L}_{j_0-j;0}(m) & \text{if} \ \ j < j_0. \end{cases} \tag{2.23a}$$

The moments

$$\alpha V^{R}_{l\ k}(m) = \int_{0,-\infty}^{\infty,0} \psi^{R}_{l}(x)\, x^m \varphi^{R}_{k}(x)\, dx,$$

$$\beta V^{R,L}_{l\ k}(m) = \int_{0,-\infty}^{\infty,0} \varphi^{R}_{l}(x)\, x^m \psi^{R}_{k}(x)\, dx,$$

and

$$\gamma V^{R,L}_{l\ k}(m) = \int_{0,-\infty}^{\infty,0} \psi^{R}_{l}(x)\, x^m \psi^{R}_{k}(x)\, dx$$

of product of boundary scale functions and wavelets or two boundary wavelets can be evaluated from the formulae

$$\begin{aligned} \boldsymbol{\alpha V}^{R,L}(m) &= \tfrac{1}{2^m}\mathbf{G}^{R,L}\boldsymbol{\Gamma V}^{R,L}(m)\left(\mathbf{H}^{R,L}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{R,L}}\boldsymbol{\Gamma V}^{\mathrm{LI}}(m)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \\ &+ \mathbf{G}^{RI,LI}\boldsymbol{\Gamma V}^{IL}(m)\left(\mathbf{H}^{R,L}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\Gamma V}^{\mathrm{II}}(m)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \end{aligned} \tag{2.24a,b}$$

$$\begin{aligned} \boldsymbol{\beta V}^{R,L}(m) &= \tfrac{1}{2^m}\mathbf{H}^{R,L}\boldsymbol{\Gamma V}^{R,L}(m)\left(\mathbf{G}^{R,L}\right)^{\mathrm{T}} + \mathbf{H}^{R,L}\boldsymbol{\Gamma V}^{\mathrm{LI}}(m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \\ &+ \mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\Gamma V}^{IL}(m)\left(\mathbf{G}^{R,L}\right)^{\mathrm{T}} + \mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\Gamma V}^{\mathrm{II}}(m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \end{aligned} \tag{2.25a,b}$$

and

$$\begin{aligned}\boldsymbol{\gamma V}^{R,L}(m) &= \tfrac{1}{2^m}\mathbf{G}^{R,L}\boldsymbol{\Gamma V}^{R,L}(m)\left(\mathbf{G}^{R,L}\right)^{\mathrm{T}}+\mathbf{G}^{\mathrm{R,L}}\boldsymbol{\Gamma V}^{\mathrm{LI}}(m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \\ &+ \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\Gamma V}^{\mathrm{IL}}(m)\left(\mathbf{G}^{R,L}\right)^{\mathrm{T}}+\mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\Gamma V}^{\mathrm{II}}(m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}}\end{aligned} \tag{2.26a,b}$$

respectively. Similar to eq. (2.22a), moment of products at two different resolutions (one of them at zero-resolution) are given by

$$\boldsymbol{\alpha V}^{R,L}_{j;0}(m)=\frac{1}{2^m}[\boldsymbol{\alpha V}^{R,L}_{j-1;0}(m)(\mathbf{H}^{R,L})^T+\boldsymbol{\alpha V}^{\mathrm{RI,LI}}_{j-1;0}(m)(\mathbf{H}^{\mathrm{RI,LI}})^T] \tag{2.22b}$$

$$\boldsymbol{\beta V}^{R,L}_{0;j}(m)=\frac{1}{2^m}[\mathbf{H}^{R,L}\boldsymbol{\beta V}^{R,L}_{0;j-1}(m)+\mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\beta V}^{RI,LI}_{0;j-1}(m)] \tag{2.22c}$$

and

$$\boldsymbol{\gamma}\mathbf{V}^{R,L}_{j;0}(m)=\frac{1}{2^m}[\boldsymbol{\alpha V}^{R,L}_{j-1;0}(m)(\mathbf{G}^{R,L})^T+\boldsymbol{\alpha V}^{RI,LI}_{j-1;0}(m)(\mathbf{G}^{RI,LI})^T] \tag{2.22d}$$

$$\boldsymbol{\gamma}\mathbf{V}^{R,L}_{0;j}(m)=\frac{1}{2^m}[\mathbf{G}^{\mathbf{R,L}}\boldsymbol{\alpha V}^{R,L}_{j-1;0}(m)+\mathbf{G}^{RI,LI}\boldsymbol{\alpha V}^{RI,LI}_{j-1;0}(m)] \tag{2.22d'}$$

c. Moment of product of scale functions/wavelets on $[a,b]\subset\mathbb{R}$

The same integrals within finite interval $[a,\ b]\subset\mathbb{R}$ involving boundary scale function $\varphi^{R,L}_{j_0\ l\ \mathrm{or\ k}}$ and wavelet $\psi^{R,L}_{j\ l\ \mathrm{or\ k}}$ or two wavelets at two different resolutions which are nonzero can be evaluated by using the results stated above into the formula

$$\boldsymbol{\rho V}^{R,L}_{j;j_0}(m;a,b)=\sum_{q=0}^{m}\binom{m}{q}\{a,b\}^{m-q}\boldsymbol{\rho V}^{R,L}_{j;j_0}(q), \tag{2.27a,b}$$

$$\boldsymbol{\alpha V}^{R,L}_{j;j_0}(m;a,b)=\sum_{q=0}^{m}\binom{m}{q}\{a,b\}^{m-q}\boldsymbol{\alpha V}^{R,L}_{j;j_0}(q), \tag{2.28a,b}$$

$$\boldsymbol{\beta V}^{R,L}_{j;j_0}(m;a,b)=\sum_{q=0}^{m}\binom{m}{q}\{a,b\}^{m-q}\boldsymbol{\beta V}^{R,L}_{j;j_0}(q) \tag{2.29a,b}$$

$$\boldsymbol{\gamma V}^{R,L}_{j;j_0}(m;a,b)=\sum_{q=0}^{m}\binom{m}{q}\{a,b\}^{m-q}\boldsymbol{\gamma V}^{R,L}_{j;j_0}(q), \tag{2.30a,b}$$

with

$$\boldsymbol{\alpha V}^{R,L}_{j;j_0}(m)=\frac{1}{2^{j_0 m}}\boldsymbol{\alpha V}^{R,L}_{j-j_0;0}(m),\quad \boldsymbol{\beta V}^{R,L}_{j;j_0}(m)=\frac{1}{2^{j_0 m}}\boldsymbol{\beta V}^{R,L}_{j-j_0;0}(m) \tag{2.23b,c}$$

and

$$\boldsymbol{\gamma V}^{R,L}_{j;j_0}(m)=\begin{cases}\frac{1}{2^{j_0 m}}\boldsymbol{\gamma V}^{R,L}_{j-j_0;0}(m) & if\quad j>j_0,\\ \\ \frac{1}{2^{jm}}\boldsymbol{\gamma V}^{R,L}_{0;j_0-j}(m) & if\quad j<j_0.\end{cases} \tag{2.23d}$$

In (2.27a,b)-(2.30a,b), $\{a,b\}^{m-q}$ assumes the value a^{m-q} or b^{m-q} when superscript assumes value R and L respectively.

2.3 *Moments of product of Daubechies scale functions/wavelets and their derivatives*

a. *On* $\mathbb{R}$

The mth-moment of product of scale function and its pth-derivative

$$\rho_l^I(p,m) = \int_{-\infty}^{\infty} \varphi_l(x)\, x^m \frac{d^p\varphi}{dx^p} dx$$

are the solution of a overdetermined system of linear simultaneous equations obtained by using (2.1) for both φ's in the integrand

$$\rho_l^I\ (p,m) = \frac{2^p}{2^m} \sum_{l_1=0}^{2K-1} \sum_{k_1=0}^{2K-1} h_{l_1} h_{k_1} \sum_{s=0}^{m} \begin{pmatrix} m \\ s \end{pmatrix} l_1^{m-s} \rho^I_{2l+l_1-k_1}(p,s) \tag{2.31}$$

and algebraic version

$$\sum_{k=-2K+1}^{2K-1} \left\{ \sum_{r=0}^{p-1} \begin{pmatrix} p \\ r \end{pmatrix} \{ \prod_{i=0}^{p-r-1} (m-i)\} \sum_{s=0}^{m-p+r} \begin{pmatrix} m-p+r \\ s \end{pmatrix} k^{m-p+r-s} \rho^I_{-k}\ (r,s) \right.$$
$$\left. + \sum_{s=0}^{m} \begin{pmatrix} m \\ s \end{pmatrix} k^{m-s} \rho^I_{-k}\ (p,s) \right\} =< x^{m-p} >_\varphi \tag{2.32}$$

of the completeness condition

$$\sum_k \varphi(x) \frac{d^p\left(x^m \varphi_k(x)\right)}{dx^p} = \{\prod_{i=0}^{p-1} (m-i)\}\, x^{m-p} \varphi(x)\,. \tag{2.33}$$

Due to finite support $[k,\ k+2K-1]$ of $\varphi_k(x)$ and its p^{th} -order derivative $\varphi_k^{(p)}(x)$ for all $k \in \mathbb{Z}$,

$$\rho_l^I\ (p,m) = \begin{cases} 0 & \text{for} \quad |l| > 2K-2\,, \\ \\ \Gamma V_l^I\ (m) & \text{for} \quad |l|\ \le\ 2K-2 \text{ and } p=0\,. \end{cases}$$

Values of $\rho_l^I\ (p,m)$ can be used to evaluate moments involving product of different translates of ϕ and its derivatives from the formula

$$\rho_{l\ k}^I(p,m) = \int_{-\infty}^{\infty} \varphi_l\ x^m\ \frac{d^p\varphi_k}{dx^p} dx = \begin{cases} 0 & \text{if} \quad |k-l| > 2K-2, \\ \sum_{r=0}^{m} \begin{pmatrix} m \\ r \end{pmatrix} k^{m-r} \rho^I_{l-k}(p,r) & \text{if} \quad |k-l| \le 2K-2. \end{cases} \tag{2.34}$$

Moments when both φ and $\varphi^{(p)}$ in the integrand are in non-zero finite resolution $j,\ j_0$ respectively, are given by

$$\rho^I_{jlj_0k}(p,m) = \int_{-\infty}^{\infty} \varphi_{jl}(x)\, x^m \frac{d^p\varphi_{j_0k}}{dx^p} dx =$$

$$\begin{cases} 2^{j_0(p-m)}\rho^I_{0\ l;0\ k}(p,m) & \text{for} \quad j = j_0 \neq 0,\ \ |l-k| \le 2K-2 \\ 2^{(p-m)}\sum_{k_1=0}^{2K-1} h_{k_1}\rho^I_{j-1\ l;0\ 2k+k_1}(p,m) & \text{for} \quad j > j_0 = 0, -2K+2 \le l-2^j k \le 2^j(2K-1)-1 \\ 2^{(p-m)}\sum_{l_1=0}^{2K-1} h_{l_1}\rho^I_{0\ 2l+l_1;j_0-1\ k}(p,m) & \text{for} \quad j_0 > j = 0, -2K+2 \le\ 2^{j_0}l-k\ \le 2^{j_0}(2K-1)-1 \\ 2^{j_0(p-m)}\rho^I_{j-j_0\ l;0\ k}(p,m) & \text{for} \quad j > j_0, -2K+2 \le l-2^{j-j_0}k \le 2^{j-j_0}(2K-1)-1 \\ 2^{j(p-m)}\rho^I_{0\ l;j_0-j\ k}(p,m) & \text{for} \quad j_0 > j, -2K+2 \le 2^{j_0-j}l-k \le 2^{j_0-j}(2K-1)-1 \\ 0 & \quad\ \ \text{otherwise.} \end{cases} \tag{2.35a}$$

Following the same artifice, the integrals for moments involving product of wavelet and derivatives of scale function or vice-versa as well as wavelets and its derivatives can be obtained by using the formulae

$$\alpha^I_{jlj_0k}(p,m) = \int_{-\infty}^{\infty} \psi_{jl}(x)\, x^m \frac{d^p\varphi_{j_0k}}{dx^p} dx$$

$$= \begin{cases} 2^{j_0(p-m)}\sum_{l_1,k_1=0}^{2K-1} g_{l_1}h_{k_1}\rho^I_{0\ 2l+l_1;0\ 2k+k_1}(p,m) & \text{for} \quad j = j_0 \ge 0, \quad |l-k| \le 2K-2 \\ 2^{(p-m)}\sum_{k_1=0}^{2K-1} h_{k_1}\alpha^I_{j-1\ l;0\ 2k+k_1}(p,m) & \text{for} \quad j > j_0 = 0, -2K+2 \le l-2^j k \le 2^j(2K-1)-1 \\ 2^{(p-m)}\sum_{l=0}^{2K-1} g_{l_1}\rho^I_{0\ 2l+l_1;j_0-1\ k}(p,m) & \text{for} \quad j_0 > j = 0, -2K+2 \le\ 2^{j_0}l-k\ \le 2^{j_0}(2K-1)-1 \\ 2^{j_0(p-m)}\alpha^I_{j-j_0\ l;0\ k}(p,m) & \text{for} \quad j > j_0, -2K+2 \le l-2^{j-j_0}k \le 2^{j-j_0}(2K-1)-1 \\ 2^{j(p-m)}\alpha^I_{0\ l;j_0-j\ k}(p,m) & \text{for} \quad j_0 > j, -2K+2 \le 2^{j_0-j}l-k \le 2^{j_0-j}(2K-1)-1 \\ 0 & \quad\ \ \text{otherwise,} \end{cases} . \tag{2.35b}$$

$$\beta^I_{jlj_0k}(p,m) = \int_{-\infty}^{\infty} \varphi_{jl}(x)\, x^m \frac{d^p\psi_{j_0k}}{dx^p} dx$$

$$= \begin{cases} 2^{j_0(p-m)}\sum_{l_1,k_1=0}^{2K-1} g_{k_1}h_{l_1}\rho^I_{0\ 2l+l_1;0\ 2k+k_1}(p,m) & \text{for} \quad j = j_0 \ge 0,\ |l-k| \le 2K-2 \\ 2^{(p-m)}\sum_{k_1=0}^{2K-1} g_{k_1}\rho^I_{j-1\ l;0\ 2k+k_1}(p,m) & \text{for} \quad j > j_0 = 0, -2K+2 \le l-2^j k \le 2^j(2K-1)-1 \\ 2^{(p-m)}\sum_{l=0}^{2K-1} h_{l_1}\beta^I_{0\ 2l+l_1;j_0-1\ k}(p,m) & \text{for} \quad j_0 > j = 0, -2K+2 \le\ 2^{j_0}l-k\ \le 2^{j_0}(2K-1)-1 \\ 2^{j_0(p-m)}\beta^I_{j-j_0\ l;0\ k}(p,m) & \text{for} \quad j > j_0, -2K+2 \le l-2^{j-j_0}k \le 2^{j-j_0}(2K-1)-1 \\ 2^{j(p-m)}\beta^I_{0\ l;j_0-j\ k}(p,m) & \text{for} \quad j_0 > j, -2K+2 \le 2^{j_0-j}l-k \le 2^{j_0-j}(2K-1)-1 \\ 0 & \quad\ \ \text{otherwise,} \end{cases} \tag{2.35c}$$

and,

$$\gamma^I_{jlj_0k}(p,m) = \int_{-\infty}^{\infty} \psi_{jl}(x)\, x^m \frac{d^p\psi_{j_0k}}{dx^p} dx$$

$$= \begin{cases} 2^{j_0(p-m)}\sum_{l_1,k_1=0}^{2K-1} g_{k_1}g_{l_1}\rho^I_{0\ 2l+l_1;0\ 2k+k_1}(p,m) & \text{for} \quad j = j_0 \ge 0, |l-k| \le 2K-2 \\ 2^{(p-m)}\sum_{k_1=0}^{2K-1} g_{k_1}\alpha^I_{j-1\ l;0\ 2k+k_1}(p,m) & \text{for} \quad j > j_0 = 0, -2K+2 \le l-2^j k \le 2^j(2K-1)-1 \\ 2^{(p-m)}\sum_{l=0}^{2K-1} g_{l_1}\beta^I_{0\ 2l+l_1;j_0-1\ k}(p,m) & \text{for} \quad j_0 > j = 0, -2K+2 \le\ 2^{j_0}l-k\ \le 2^{j_0}(2K-1)-1 \\ 2^{j_0(p-m)}\gamma^I_{j-j_0\ l;0\ k}(p,m) & \text{for} \quad j > j_0, -2K+2 \le l-2^{j-j_0}k \le 2^{j-j_0}(2K-1)-1 \\ 2^{j(p-m)}\gamma^I_{0\ l;j_0-j\ k}(p,m) & \text{for} \quad j_0 > j, -2K+2 \le 2^{j_0-j}l-k \le 2^{j_0-j}(2K-1)-1 \\ 0 & \quad\ \ \text{otherwise.} \end{cases} . \tag{2.35d}$$

b. *On* $\mathbb{R}^+$ *and* $\mathbb{R}$

As in the case of section $2.2.b$, the integrals

$$\boldsymbol{\rho}^{R,L}\left(p,\ m\right)=\ \int_{0,-\infty}^{\infty,0}\boldsymbol{\varphi}^{R,L}\left(x\right)x^{m}\frac{d^{p}}{dx^{p}}\boldsymbol{\varphi}^{R,L}\left(x\right)\ dx$$

involving product of scale functions within $\Lambda^{R,L}$ and their derivatives can be obtained by solving the system of linear equations

$$\boldsymbol{\rho}^{\mathrm{R,L}}(p,m)-2^{p-m}\mathbf{H}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{R,L}}(p,m)(\mathbf{H}^{\mathrm{R,L}})^{\mathrm{T}}=$$

$$\mathbf{H}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{RI,LI}}\left(p,m\right)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}}+\mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{IR,IL}}\left(p,m\right)\left(\mathbf{H}^{\mathrm{R,L}}\right)^{\mathrm{T}}+\boldsymbol{\rho}^{\mathrm{RII,LII}}\left(p,m\right)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \quad \text{(2.36a,b)}$$

in conjunction with equations extracted from the completeness condition-

$$\sum_{k=-2K+2}^{-1}\rho_{l\ k}^{R}(p,m)+\sum_{k=0}^{l+2K-2}\rho_{l\ k}^{I}(p,m)=\left(\prod_{i=0}^{p-1}(m-i)\right)\left\langle x^{m-p}\right\rangle_{\varphi_l^R}\times$$

$$-\sum_{r=0}^{p-1}\left(\begin{array}{c}p\\ r\end{array}\right)\left(\prod_{i=0}^{p-r-1}(m-i)\right)\left\{\sum_{k=-2K+2}^{-1}\rho_{l\ k}^{R}\left(r,m-p+r\right)+\left\{\sum_{k=0}^{l+2K-2}\rho_{l\ k}^{I}\left(r,m-p+r\right)\right\}\right\} \quad \text{(2.36c)}$$

and

$$\sum_{k=-2K+2}^{-1}\rho_{l\ k}^{L}(p,m)+\sum_{k=l-2K+2}^{-2K+1}\rho_{l\ k}^{I}(p,m)=\left(\prod_{i=0}^{p-1}(m-i)\right)\left\langle x^{m-p}\right\rangle_{\varphi_l^R}$$

$$\times\sum_{r=0}^{p-1}\left(\begin{array}{c}p\\ r\end{array}\right)\left(\prod_{i=0}^{p-r-1}(m-i)\right)\left\{\sum_{k=-2K+2}^{-1}\rho_{l\ k}^{L}\left(r,m-p+r\right)+\left\{\sum_{k=l-2K+2}^{-2K+1}\boldsymbol{\rho}_{l\ k}^{I}\left(r,m-p+r\right)\right\}\right\}. \quad \text{(2.36d)}$$

As in the case of section $2.2.b$, matrices $\boldsymbol{\rho}^{\mathrm{RI,LI}}\left(p,m\right)$, $\boldsymbol{\rho}^{\mathrm{IR,IL}}\left(p,m\right)$ and $\boldsymbol{\rho}^{\mathbf{RII,LII}}\left(p,m\right)$ are all ($2K$-2)×($2K$-2) matrices given by

$$\boldsymbol{\rho}^{RI}\left(p,m\right)=\ \left(\boldsymbol{\rho}_{1l\ 1k}^{I}(p,m)\right)_{(2K-2)\times(2K-2)};\ l=-2K+2,\cdots,-1,\ k=0,\cdots,2K-3 \quad \text{(2.37a)}$$

$$\boldsymbol{\rho}^{IR}\left(p,m\right)=\ \left(\boldsymbol{\rho}_{1l\ 1k}^{I}(p,m)\right)_{(2K-2)\times(2K-2)};\ l=0,1,\cdots,2K-3,\ k=-2K+2,\cdots,-1 \quad \text{(2.37b)}$$

$$\boldsymbol{\rho}^{RII}\left(p,m\right)=\ \left(\boldsymbol{\rho}_{1l\ 1k}^{I}(p,m)\right)_{(2K-2)\times(2K-2)};\ l,k=0,1,\cdots,2K-3. \quad \text{(2.37c)}$$

and

$$\boldsymbol{\rho}^{LI}(p,m) = \left(\boldsymbol{\rho}^{I}_{1l\;1k}(p,m)\right)_{(2K-2)\times(2K-2)};\ l=-2K+2,\cdots,-1,\ k=-4K+4,\cdots,-2K+1, \tag{2.38a}$$

$$\boldsymbol{\rho}^{IL}(p,m) = \left(\boldsymbol{\rho}^{I}_{1l\;1k}(p,m)\right)_{(2K-2)\times(2K-2)};\ l=-4K+4,\cdots,-2K+1,\ \ k=-2K+2\,,\cdots,-1, \tag{2.38b}$$

$$\boldsymbol{\rho}^{LII}(m) = \left(\boldsymbol{\rho}^{I}_{1;l\;k}(m)\right)_{(2K-2)\times(2K-2)};\ l,k=-4K+4,-4K+3,\cdots,-2K+1. \tag{2.38c}$$

Note that

$$\boldsymbol{\rho}^{R}_{l\;k}(p,m) = \begin{cases} 0 & \text{if} \quad l \text{ or } k \le\ -2K+1 \\ \boldsymbol{\rho}^{I}_{l\;k}(p,m) & \text{if } l \text{ or } k \ge 0 \end{cases} \tag{2.39a}$$

and

$$\boldsymbol{\rho}^{L}_{l\;k}(p,m) = \begin{cases} 0 & \text{if} \qquad l \text{ or } k\ \ge 0 \\ \boldsymbol{\rho}^{I}_{l\;k}\ (p,m) & \text{if} \quad l \text{ or } k\ \le -2K+1. \end{cases} \tag{2.39b}$$

The value of above integrals involving boundary scale functions, one at 0-resolution and other at resolution-j are given by

$$\boldsymbol{\rho}\boldsymbol{V}^{R,L}_{j;0}(p,m) = 2^{p-m}\left[\boldsymbol{\rho}\boldsymbol{V}^{R,L}_{j-1;0}(p,m)\left(\mathbf{H}^{\mathrm{R,L}}\right)^{T} + \boldsymbol{\rho}\boldsymbol{V}^{\mathrm{RI,LI}}_{j-1;0}(p,m)\,(\mathbf{H}^{\mathrm{RI,LI}})^{T}\right], \tag{2.40a,b}$$

$$\boldsymbol{\rho}\boldsymbol{V}^{R,L}_{0;j}(p,m) = 2^{p-m}[\mathbf{H}^{\mathrm{R,L}}\ \boldsymbol{\rho}\boldsymbol{V}^{R,L}_{0;j-1}(p,m) + \mathbf{H}^{\mathrm{RI,LI}}\ \boldsymbol{\rho}\boldsymbol{V}^{\mathrm{RI,LI}}_{0;j-1}\ (p,m)]. \tag{2.41a,b}$$

When they are in two different non-zero resolutions say j and j_0, values for

$$\boldsymbol{\rho}^{R,L}_{j;j_0}(p,m) = \int_{0,-\infty}^{\infty,0} \varphi^{R,L}_{j\;l}(x)\,x^m \frac{d^p}{dx^p}\varphi^{R,L}_{j_0\;k}(x)\,dx$$

can be conveniently calculated by using the formula

$$\boldsymbol{\rho}^{R,L}_{j;j_0}(p,m) = \begin{cases} 2^{j_0(p-m)}\boldsymbol{\rho}^{R,L}_{j-j_0;0}(p,m) & \text{if } \ j > j_0, \\ & . \\ 2^{j(p-m)}\boldsymbol{\rho}^{R,L}_{j_0-j;0}(p,m) & \text{if } \ j < j_0 \end{cases} \tag{2.42}$$

The moments of product of boundary scale functions/wavelets and their derivatives

$$\boldsymbol{\alpha}^{R,L}_{l\;k}(p,m) = \int_{0,-\infty}^{\infty,0} \psi^{R}_{l}(x)\,x^m \frac{d^p}{dx^p}\phi^{R}_{k}(x)\,dx,$$

$$\boldsymbol{\beta}^{R,L}_{l\;k}(p,m) = \int_{0,-\infty}^{\infty,0} \phi^{R}_{l}(x)\,x^m \frac{d^p}{dx^p}\psi^{R}_{k}(x)\,dx$$

and

$$\boldsymbol{\gamma}^{R,L}_{l\;k}(p,m) = \int_{0,-\infty}^{\infty,0} \psi^{R}_{l}(x)\,x^m \frac{d^p}{dx^p}\psi^{R}_{k}(x)\,dx$$

can be evaluated from the formulae

$$\boldsymbol{\alpha}^{\mathrm{R,L}}(p,m) = 2^{p-m}\mathbf{G}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{R,L}}(p,m)\ \left(\mathbf{H}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{RI,LI}}(p,m)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{RI,LI}}\ \boldsymbol{\rho}^{\mathrm{IR,IL}}(p,m)\left(\mathbf{H}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{RII,LII}}(p,m)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}}, \tag{2.43a}$$

$$\boldsymbol{\beta}^{\mathrm{R,L}}(p,m) = 2^{p-m}\mathbf{H}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{R,L}}(p,m)\ \left(\mathbf{G}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{H}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{RI,LI}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} + \mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{IR,IL}}(p,m)\left(\mathbf{G}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{RII,LII}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \tag{2.43b}$$

and

$$\boldsymbol{\gamma}^{\mathrm{R,L}}(p,m) = 2^{p-m}\mathbf{G}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{R,L}}(p,m)\ \left(\mathbf{G}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{R,L}}\boldsymbol{\rho}^{\mathrm{RI,LI}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{IR,IL}}(p,m)\left(\mathbf{G}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{RII,LII}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}} \tag{2.43c}$$

respectively. Similar to eq. (2.40a,b) and (2.41a,b), moment of products scale functions and wavelets having two different resolutions (one of them at zero-resolution) one of their derivatives are given by

$$\boldsymbol{\alpha}^{R,L}_{j;0}(p,m) = 2^{p-m}\left[\boldsymbol{\alpha}^{R,L}_{\mathrm{j-1;0}}(p,m)\left(\mathbf{H}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \boldsymbol{\alpha}^{\mathrm{RI,LI}}_{\mathrm{j-1;0}}(p,m)\left(\mathbf{H}^{\mathrm{RI,LI}}\right)^{\mathrm{T}}\right], \tag{2.44a,b}$$

$$\boldsymbol{\alpha}^{R,L}_{0;j}(p,m) = 2^{p-m}\left[\mathbf{G}^{R,L}\boldsymbol{\rho}^{\mathrm{R,L}}_{\mathrm{0;j-1}}(p,m) + \mathbf{G}^{\mathrm{RI,LI}}\boldsymbol{\rho}^{\mathrm{RI,LI}}_{\mathrm{0;j-1}}(p,m)\right], \tag{2.44c,d}$$

$$\boldsymbol{\beta}^{R,L}_{0;j}(p,m) = 2^{p-m}\left[\mathbf{H}^{R,L}\boldsymbol{\beta}^{\mathrm{R,L}}_{\mathrm{0;j-1}}(p,m) + \mathbf{H}^{\mathrm{RI,LI}}\boldsymbol{\beta}^{\mathrm{RI,LI}}_{\mathrm{0;j-1}}(p,m)\right], \tag{2.45a,b}$$

$$\boldsymbol{\beta}^{R,L}_{j;0}(p,m) = 2^{p-m}\left[\boldsymbol{\rho}^{R,L}_{0;\,j-1}(p,m)\left(\mathbf{G}^{\mathrm{R,L}}\right)^{\mathrm{T}} + \boldsymbol{\rho}^{\mathrm{RI,LI}}_{\mathrm{0;j-1}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{\mathrm{T}}\right], \tag{2.45c,d}$$

$$\boldsymbol{\gamma}^{R,L}_{j;0}(p,m) = 2^{p-m}\left[\boldsymbol{\alpha}V^{R,L}_{\mathrm{j-1;0}}(p,m)\left(\mathbf{G}^{\mathrm{R,L}}\right)^{T} + \boldsymbol{\alpha}V^{\mathrm{RI,LI}}_{\mathrm{j-1;0}}(p,m)\left(\mathbf{G}^{\mathrm{RI,LI}}\right)^{T}\right], \tag{2.46a,b}$$

$$\boldsymbol{\gamma}^{R,L}_{0;j}(p,m) = 2^{p-m}\left[\mathbf{G}^{\mathrm{R,L}}\ \boldsymbol{\alpha}V^{R,L}_{\mathrm{j-1;0}}(p,m) + \mathbf{G}^{\ \mathrm{RI,LI}}\ \boldsymbol{\alpha}V^{\mathrm{RI,LI}}_{\mathrm{j-1;0}}(p,m)\right]. \tag{2.46c,d}$$

Their values in case of scale functions and wavelets in different non-zero resolutions follow the same as in the case of $\boldsymbol{\rho}^{R,L}_{j;j_0}(p,m)$ given in formula (2.42).

c. Moments of product of scale functions/wavelets and their derivatives on $[\boldsymbol{a},\boldsymbol{b}] \subset \mathbb{R}$

The same integrals within finite interval $\Omega = [a,\ b] \subset \mathbb{R}$ involving boundary scale function $\varphi^{R,L}_{j_0\ l\ \mathrm{or\ k}}$ and wavelet $\psi^{R,L}_{j\ l\ \mathrm{or\ k}}$ or two wavelets at two different resolutions which are nonzero can be evaluated by using the results stated above into the formula

$$\boldsymbol{\rho},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}^{R}_{j;j_0}(p,m;a,b) = \sum_{q=0}^{m}\binom{m}{q}a^{m-q}\boldsymbol{\rho},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}^{R}_{j;j_0}(p,q), \tag{2.47a,b,c,d}$$

$$\boldsymbol{\rho},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}^{L}_{j;j_0}(p,m;a,b) = \sum_{q=0}^{m}\binom{m}{q}b^{m-q}\boldsymbol{\rho},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}^{L}_{j;j_0}(p,q). \tag{2.48a,b,c,d}$$

with

$$\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}_{j;j_0}^{R,L}(p,m)=\begin{cases}2^{j_0(p-m)}\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}_{j-j_0;0}^{R,L}(p,m) & \text{with} \quad j>j_0,\\ 2^{j_0(p-m)}\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}_{0;j_0-j}^{R,L}(p,m) & \text{with} \quad j<j_0\,.\end{cases} \tag{2.49a,b,c,d}$$

3. WAVELET-GALERKIN APPROXIMATION OF SOLUTION AND MATRIX EIGENVALUE PROBLEM

Let us now consider a SLP for $u(x)$ in one-dimension in their general form

$$\widehat{\mathcal{O}}[u](x) \equiv p(x)\frac{d^2u}{dx^2}+\ q(x)\frac{du}{dx}+r(x)u(x)=\lambda\ f(x)\ u(x) \tag{3.1}$$

on $\Omega = [a,b] \subset \mathbb{R}$ with Dirichlet boundary conditions *u(a) = u(b) = 0.* It is assumed here that all the coefficients $p(x)$, $q(x)$, $r(x)$ and $f(x)$ are polynomials in *x*. Using the Galerkin expansion

$$u(\cdot)=\sum_{k=a2^{j_0}-2K+2}^{b2^{j_0}-1}u_{j_0k}\phi_{j_0k}(\cdot)\ +\sum_{k=a2^{j_0}}^{b2^{j_0}-2K+1}d_{j_0k}\psi_{j_0k}(\cdot)=(\boldsymbol{\varphi}_{j_0}(\cdot),\boldsymbol{\psi}_{j_0}(\cdot))\begin{pmatrix}\boldsymbol{u}_{j_0}\\ \boldsymbol{d}_{j_0}\end{pmatrix} \tag{3.2}$$

for the unknown solution *u(x)* in the basis of $\Lambda_{j_0}^R$, $\Lambda_{j_0}^I$ and $\Lambda_{j_0}^L$ followed by the inner product with all the elements of the basis involved in the expansion over Ω leads to linear simultaneous equation

$$\mathcal{A}_{j_0}\begin{pmatrix}\boldsymbol{u}_{j_0}\\ \boldsymbol{d}_{j_0}\end{pmatrix}=\lambda_{j_0}\ \ \mathcal{F}_{j_0}\begin{pmatrix}\boldsymbol{u}_{j_0}\\ \boldsymbol{d}_{j_0}\end{pmatrix}. \tag{3.3}$$

Here

$$\mathcal{A}_{j_0}=\begin{pmatrix}\rho\equiv<\boldsymbol{\varphi}_{j_0},\widehat{\mathcal{O}}[\varphi_{j0}]> & \beta\equiv<\boldsymbol{\varphi}_{j_0},\widehat{\mathcal{O}}[\psi_{j0}]>\\ \alpha\equiv<\boldsymbol{\psi}_{j_0},\widehat{\mathcal{O}}[\varphi_{j0}]> & \gamma\equiv<\boldsymbol{\psi}_{j_0},\widehat{\mathcal{O}}[\psi_{j0}]>\end{pmatrix}$$

being the stiffness matrix of dimension equal to the dimension of union of approximation space V_{j_0} and detail space W_{j_0} and

$$\mathcal{F}_{\boldsymbol{j_0}}=\begin{pmatrix}<\boldsymbol{\varphi},f\boldsymbol{\varphi}> & <\boldsymbol{\varphi},f\boldsymbol{\psi}>\\ <\boldsymbol{\psi},f\boldsymbol{\varphi}> & <\boldsymbol{\psi},f\boldsymbol{\psi}>\end{pmatrix}$$

is also a matrix having the same dimension as of $\mathcal{A}_{j_0}$. The numerical values of elements of above two matrices have been calculated by using formulae (2.35), (2.47), (2.48) derived in section 2, $\boldsymbol{u}_{j_0}$ is the column vector of coefficients of the expansion of multiresolution approximation of unknown solution $u(x)$ in the basis of $\varphi_{j_0\,l}$, $\boldsymbol{d}_{j_0}$ is the column vector of coefficients of multiresolution approximation of function $u(x)$ in the basis of $\psi_{j_0\,l}$ whose supports ***overlaps fully*** on Ω. In the process of formation of generalized matrix eigenvalue problem (3.3) through inner product with elements of $\Lambda_{j_0}^R, \Lambda_{j_0}^I$ and $\Lambda_{j_0}^L$, we have replaced two equations corresponding to the inner product with the elements $\varphi_{j_0\,2^{j_0}a-1}^R(\cdot) \in \Lambda_{j_0}^R$ and $\varphi_{j_0\,2^{j_0}b-1}^L(\cdot) \in \Lambda_{j_0}^L$ by two boundary conditions

$$\sum_k u_{j_0k}\boldsymbol{\varphi}_{j_0k}(a)\ =\sum_{k=a2^{j_0}-2K+2}^{a2^{j_0}-1}u_{j_0k}\boldsymbol{\varphi}_{j_0k}(a)\ =0$$

$$\sum_k u_{j_0k}\boldsymbol{\varphi}_{j_0k}(b)=\sum_{k=b2^{j_0}-2K+2}^{b2^{j_0}-1}u_{j_0k}\boldsymbol{\varphi}_{j_0k}(b)\ =0. \tag{3.4a,b}$$

Values of $\varphi_{j_0k}(a)$ and $\varphi_{j_0k}(b)$ at the boundaries have been calculated by using values of $\varphi(x)$ at integers obtained from the eigenvectors of recursion matrix L of §2.1 with condition (2.9) and relations in (2.5).

4. Estimation of Error

a. Eigensolution

The advantage for choosing MRG and wavelets in Daubechies family is that their orthonormality properties (boundary scale functions and wavelets deserves additional treatment in special cases) will help to estimate errors via norm equivalence relation

$$\left\| u(x) \ - \sum_{k} u_{j_0k}\boldsymbol{\varphi}_{j_0k}(x) \ - \ \sum_{k'} d_{j_0k'}\,\boldsymbol{\psi}_{j_0k'}(x)\right\|_{L^2} = \sqrt{\sum_{j'\geq j_0+1}\sum_{k} d^2_{j'k}} \approx C\sqrt{\sum_{k} d^2_{j_0+1\ k}}. \tag{4.1}$$

Thus at every time, the wavelet expansion $\sum_{k'} d_{jk'}\,\psi_{jk'}(x)$ of the (unknown) solution would be treated as an error estimator capable of assessing the error in multiresolution approximation (3.2) of solution to the problem.

b. Eigenvalue

Let us write the eigenvalue λ and the eigensolutions y as

$$\lambda = \ \lambda_0 + \delta\lambda,\ y = \ y_0 + \delta y \tag{4.2a,b}$$

where $\lambda_0,\ y_0$ are the approximate eigenvalues and eigensolutions with errors $\delta\lambda$ and δy respectively. Substituting these decompositions into the given equation.

$$\widehat{\mathcal{O}}\,[y]\,(x) = \lambda\ f\,(x)\ \ y(x) \tag{4.3}$$

expressed in compact form we get

$$\widehat{\mathcal{O}}\,[y_0+\delta y]\,(x) = (\lambda_0+\delta\lambda)\ f\,(x)\ \ (y_0+\delta y).$$

Since $\widehat{\mathcal{O}}$ is an linear operator, this equation can be split into

$$\widehat{\mathcal{O}}\,[y_0]\,(x) + \widehat{\mathcal{O}}\,[\delta y]\,(x) \ = \lambda_0 f\,(x)\,y_0 \ + \delta\lambda\ f\,(x)\ \ y_0 + \lambda_0\ f\,(x)\ \ \delta y + \delta\lambda\ f\,(x)\ \ \delta y. \tag{4.4}$$

Assume that the eigenvalue problem

$$\widehat{\mathcal{O}}\,[y_0]\,(x) \ = \lambda_0\ f\,(x)\,y_0(x) \tag{4.5}$$

is well defined and has the eigenvalue λ_0 with the eigensolution y_0. Then discarding quantities involving product of two or more correction terms, eq. (4.4) can be recast into

$$\delta\lambda\ f\,(x)\ \ y_0(x) = \widehat{\mathcal{O}}[\delta y]\,(x) - \ \lambda_0\ f\,(x)\ \ \delta y(x). \tag{4.6}$$

Taking inner product with $y_0(x)$ on both sides we get

$$\delta\lambda = \frac{\left\langle y_0(x) \;\middle|\; \widehat{\mathcal{O}} \;\middle|\; \delta y(x)\right\rangle \lambda_0 \; \langle y_0(x) \mid f(x) \mid \delta y(x)\rangle}{\langle y_0(x) \mid f(x) \mid y_0(x)\rangle}. \tag{4.7}$$

Using the expansions

$$y_0\,(x) = \sum_{k=a2^j-2K+2}^{b2^j-1} y_{jk}\;\varphi_{jk}(x)$$

and

$$\delta y(x) \cong \sum_{k=a2^j}^{b2^j-2K+1} d_{j\;k}\;\psi_{jk}(x)$$

into the RHS of the (4.7) one gets

$$\delta\lambda = \frac{\begin{array}{c}\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j}^{b2^j-2K+1} y_{jk_2}\;d_{jk_1}\left\langle \varphi_{jk_2} \;\middle|\; \widehat{\mathcal{O}} \;\middle|\; \psi_{jk_1}\right\rangle \\ -\lambda_0\;\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j}^{b2^j-2K+1} y_{jk_2}d_{jk_1}\,\langle \varphi_{jk_2} \mid f(x) \mid \psi_{jk_1}\rangle\end{array}}{\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j-2K+2}^{b2^j-1} y_{jk_2}y_{jk_1}\,\langle \varphi_{jk_2} \mid f(x) \mid \varphi_{jk_1}\rangle}. \tag{4.8}$$

Let us assume the linear differential operator $\widehat{\mathcal{O}}$ in the form.

$$\widehat{\mathcal{O}} \equiv \left(\sum_{i=0}^{m_p} p_i x^i\right)\frac{d^2}{dx^2} + \left(\sum_{i=0}^{m_q} q_i x^i\right)\frac{d}{dx} + \left(\sum_{i=0}^{m_r} r_i x^i\right)\,\mathbb{I} \tag{4.9a}$$

with polynomial coefficients so that $(\sum_{i=0}^{m_p} p_i x^i) = 0$ may have zero with multiplicity at most two within two boundaries and

$$f(x) \;\equiv \sum_{i=0}^{m_f} f_i x^i\;. \tag{4.9b}$$

is nonzero within Ω. Using (4.9b) into RHS of eq. (4.8) we get an estimate of error for the eigenvalue as

$$\delta\lambda = \frac{\begin{array}{c}\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j}^{b2^j-2K+1}\,(\sum_{i=0}^{m_p} p_i\;\beta\,(2,i,j,k_2,j,k_1) + \sum_{i=0}^{m_q} q_i\;\beta\,(1,i,j,k_2,j,k_1) \\ +\sum_{i=0}^{m_r} r_i\;\beta\,(i,j,k_2,j,k_1)) - \lambda_0\;\;\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j}^{b2^j-2K+1} y_{jk_2}d_{jk_1}(\sum_{i=0}^{m_f} f_i\;\beta\,(i,j,k_2,j,k_1))\end{array}}{\sum_{k_2=a2^j-2K+2}^{b2^j-1}\sum_{k_1=a2^j-2K+2}^{b2^j-1} y_{jk_2}y_{jk_1}(\sum_{i=0}^{m_f} f_i\;\rho\,(i,j,k_2,j,k_1))}. \tag{4.10}$$

Table 1: Few eigenvalues of Eq.(5.1) for cases i) and ii) with $n = 0$ and $n = 2$.

Res. j	$m(x) = \frac{\pi^2}{2(1+\pi^2 x^2)^2}$ $V(x) = 0$					$m(x) = 1$ $V(x) = x^{2n}, n = 0$				$m(x) = 1$ $V(x) = x^{2n}, n = 2$			
	Dau-3 λ_1	Haar λ_1[25]	FDM λ_1	Dau-3 λ_2	Dau-3 λ_3	Dau-3 λ_1	Haar λ_1[25]	Dau-3 λ_2	Dau-3 λ_3	Dau-3 λ_1	Haar λ_1[25]	Dau-3 λ_2	Dau-3 λ_3
3	5.389	6.0401	5.2208	29.364	88.354	10.84	11.13	40.30	91.54	9.95		39.48	9.071
4	5.214	5.4233	5.0998	24.434	59.892	10.865	10.93	40.42	89.66	9.97	10.34	39.60	88.86
5	5.194			23.845	55.356	10.869	FDM[39]	40.469	89.786	9.983		39.64	88.98
6	5.191			23.772	54.782	10.8695		40.4771	89.8201	9.9831		39.652	88.9831
	Exact not available					*10.8696*		*40.4784*	*89.8264*	*9.95067*	FDM[39]	Not available	

5. Illustrative Examples

The applicability of wavelet-Galerkin approximation scheme based on Daubechies wavelets developed above has been tested for two types of Sturm-Liouville problems, viz. equations without singularity and with singularity at one end, in the following two examples.

Example 1 : Sturm-Liouville problem with no singularity:

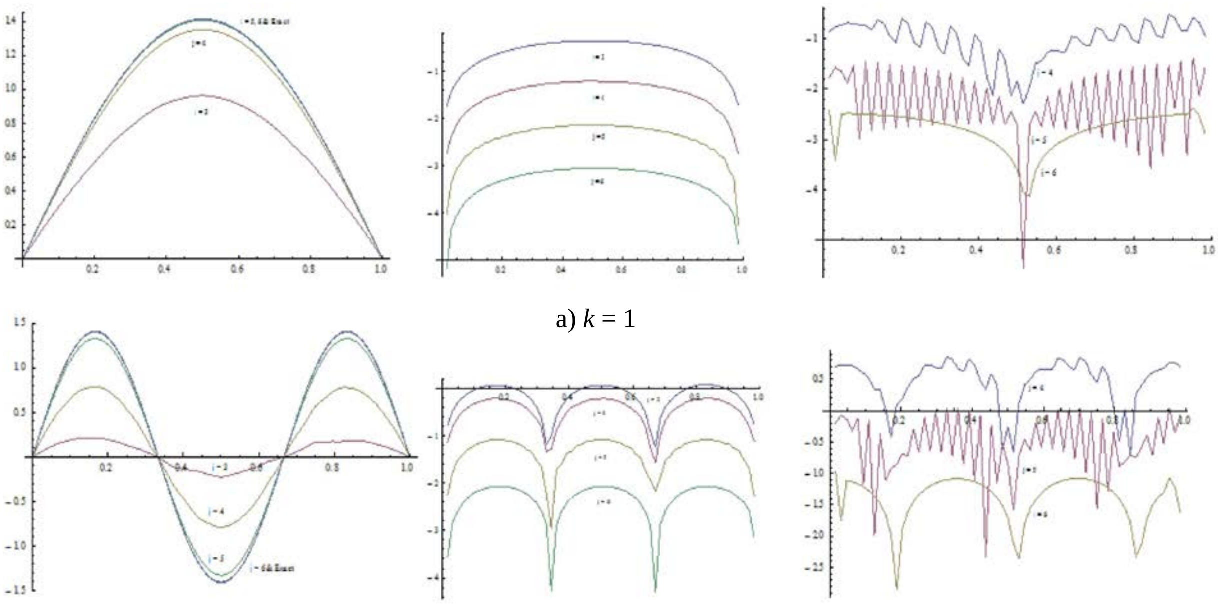
Here we consider the SLP

$$-\frac{1}{2\,m(x)}\frac{d^2u}{dx^2} + \; V(x)\; u = \; E\; u \tag{5.1}$$

in the finite domain $[0,\ 1]$ with the homogeneous Dirichlet boundary condition $u(0) \;=\; 0$ and $u(1) = 0$. These problems have been studied recently by Bujurke *et al.* [39] with the help of variational method in Haar (Daubechies family with *K=1*) bases for i) $m(x) \;=\; \frac{\pi^2}{2\;(1+\pi^2 x^2)^2},\; V(x) \;=\; 0$; and ii) $m\,(x) = \;\frac{1}{2},\;\; V\,(x) = x^{2n},\; n = 0,\; 2$.

While the case i) fits with the Schrödinger equation for free particle with variable mass/space dependent kinetic energy term, the case ii) may be regarded as the same equation describing the motion of a particle with fixed mass moving freely for $n \;=\; 0$ case and under the influence of quartic potential in case of $n \;=\; 2$. In all cases, the particle is confined within the region $[0,\ 1]$. Although the method based on variational principle employed by Bujurke *et al.* [39] produced a rough estimate for the least eigenvalue and the corresponding eigenfunction for equation (5.1) with *m(x)* given by (i) and (ii), the scheme developed here provides not only more accurate estimates for the least eigenvalue but also other eigenvalues as is obvious from the Table 1.

For case ii) with $n = 0$, the problem has exact analytical eigensolution $u_k(x) = \sqrt{2}\sin k\pi\, x$ with eigenvalue $E_k \;=\; 1 + (k\;\pi)^2$. The eigenvalues presented in Table 1 and deviation of corresponding approximate eigenfunctions or first order derivatives from their respective exact values exhibited in Figures 1 indicate the superiority of the wavelet-Galerkin approximation in Daubechies wavelet basis compared to the variational method involving the Haar basis as well as the finite difference method (FDM) for evaluating eigenelements of regular Sturm-Liouville problems.



a) k = 1

b) k = 3

Fig. 1 Exact and approximate eigenfunctions of eq. (5.1) for free particle i.e. ($n = 0$ for the case ii) and their deviations (2nd Fig.), deviations of first order derivatives (3rd Fig.) from their exact values respectively in log_{10}-scale for $a)$ $k = 1$ and $b)$ $k = 3$. Here k stands for the level of the eigenspectrum.

Example 2 : Sturm-Liouville problem with singularity at one end:

Getting eigenelements of Sturm-Liouville problems having singularity at the origin has a long history since the inception of quantum theory. Apart from the problems appearing in the classical or quantum mechanics, the interactions among the constituents of physical processes in other areas too are dominated by Coulomb or harmonic oscillator type interactions in addition to the centrifugal term when the system is isotropic. In particular, we consider the differential equation

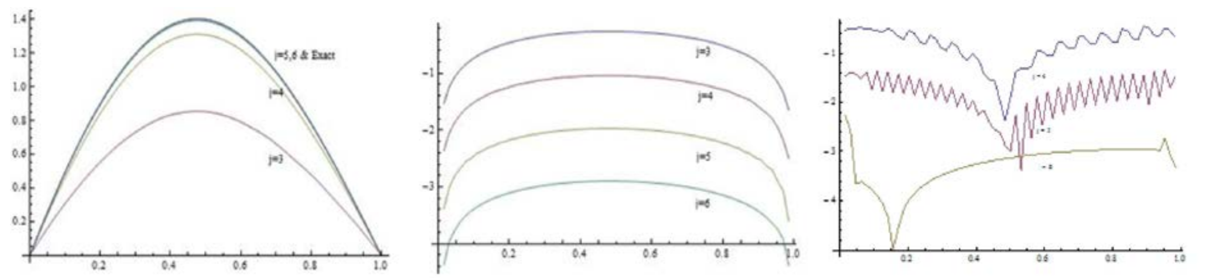
$$-u^{''} + \frac{l(l+1)}{x^2}u + V\,(x)\;u = \;\lambda\;\,u \tag{5.2}$$

with Dirichlets boundary condition $u\,(0) \;=\; 0,\;\;\; u\,(1) \;=\; 0$ where i) $V\,(x) \;=\; -\frac{1}{x}$ (Coulomb) and ii) $V\,(x) = x^2$ (Isotropic oscillator). The approximate values of λ for these two functional forms of $V(x)$ for first few levels for $l\,=\,0,\;1,\;2$ are calculated by using Daubechies scale functions and wavelets within bounded domain for $K = 3$ and $K = 4,$ and the results are presented in Table 2 along with corresponding exact values. As in the case of infinite domain, eq. (5.2) has exact analytic eigenfunction. However, instead of a simple algebraic function of number of zeros of eigenfunction in case of infinite domain [40], the eigenvalues of (5.2) in a bounded domain become the solution of a transcendental equation. Their explicit forms are given by

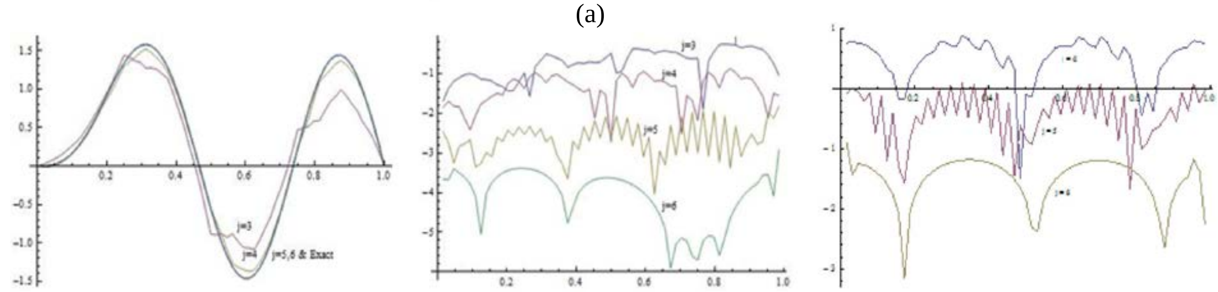
$$u\,(x) = \begin{cases} Ne^{-\sqrt{-\lambda}\;x}\;M(l+1-\frac{1}{2\lambda},2l+2,\;\;2\lambda\;x)\;\;\text{for}\;\;(i) \\ Ne^{-\frac{x^2}{2}}\;\;M(\frac{2l+3-\lambda}{4},l+\frac{3}{2},\;\;x^2)\;\;\text{for}\;\;(ii) \end{cases} \tag{5.3a,b}$$

with eigenvalues as the solution of

$$\lambda = \begin{cases} M\left(l+1-\frac{1}{2\lambda}, 2l+2,\ 2\lambda\right) = 0 & \text{for (i)}\,, \\ \\ M\left(\frac{2l+3-\lambda}{4}, l+\frac{3}{2},\ 1\right) = 0 & \text{for (ii)}\,. \end{cases} \tag{5.4a,b}$$



(a)



(b)

Fig. 2 Exact and approximate eigenfunctions of eq. (5.2) in case i) and deviations of approximate solutions or their first-order derivatives from exact values respectively for $a)\ l=0,\ n^r=0$ and $b)\ l=2, n_r=2$. Here n_r stands for the level of the eigenvalue.

Here $M(\cdot,\cdot,\cdot)$ denotes the confluent hypergeometric function ${}_1F_1$. The exact values of λ presented in Table 2 are the numerical solutions of eqs. (5.4a,b), which have been solved numerically by using library function “FindRoot” available in MATHEMATICA. The library function “FindRoot” desires a rough estimate of λ as input which is different for distinct levels. Many eigenelements $\lambda_n = \ 2(2n_r+l+\frac{3}{2})$ of equation in (5.2) for ISO in infinite domain will be missing whenever the domain is restricted to bounded domain like [0, 1]. So, guessing of appropriate input of “FindRoot” is a difficult task. We have supplied those inputs here as the approximate values of λ obtained by using wavelet Galerkin method developed here at the lowest resolution $j_0 = 3$ or 4. From a comparison of data presented in Table 2, it appears that the order of accuracy of eigenvalues improves with increase in the resolution levels j as well as regularity or smoothness [10, pp. 234] (*K*, number of vanishing moments) of wavelets. From the trend in the order of accuracy of approximate λ’s obtained by using *K=3* and *K=4* Daubechies basis in the proposed method, it appears that the order of accuracy improves substantially with increase in *K*. Thus, it is possible to get approximate value of the eigenvalue λ as accurate as desired by choosing smoothness parameter *K* and the resolution j appropriately. We have compared the eigenfunctions and its deviation from the exact values given by (5.3ab) in ”$\log_{10}$” scale for both the Coulomb and isotropic oscillator problems for few levels mentioned above in Figs.2 and 3 respectively. From the results illustrated here it appears that multiresolution approximation of eigensolution improves by the o(10^{-1}) with increase in each step of resolution j. These features have been maintained for other levels too.

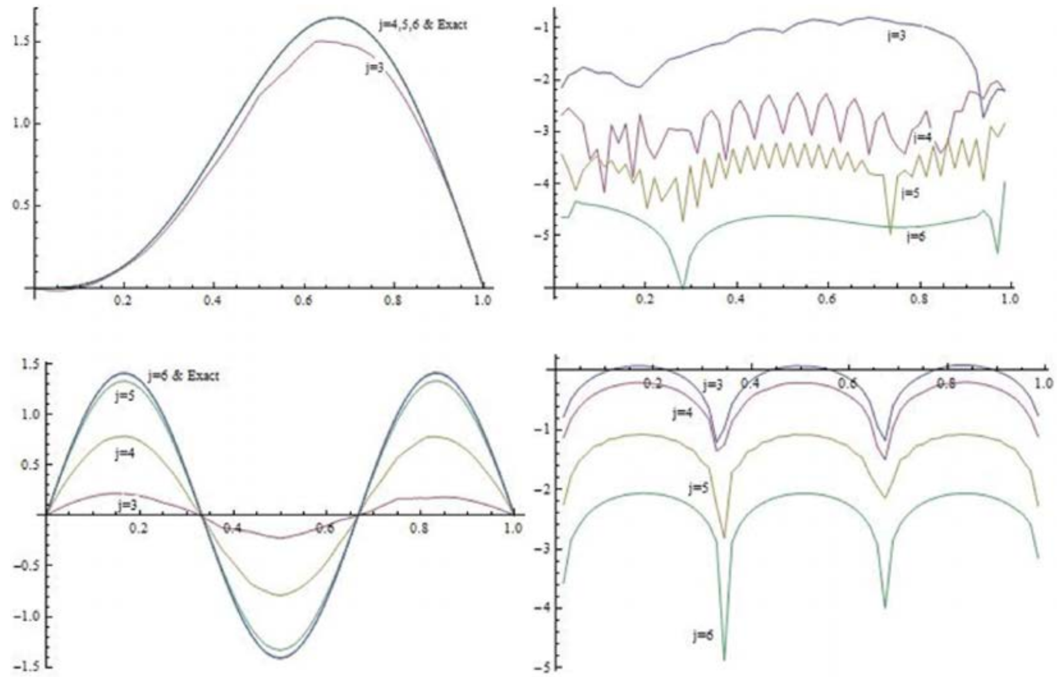


Fig. 3 Approximate eigenfunctions of SLP (5.2) in case of IHO and their deviation from exact solution (5.3b) respectively for $l = 2,\ n_r = 0$ and $l = 0,\ n_r = 2$.

6. CONCLUSION

This work is an initial step towards our studies on the possibilities of development of an efficient straightforward approximation method based on usual MRA of L^2-space for getting highly accurate eigenspectrum of eigenvalue problems of varied nature. This includes singular or discontinuous coefficients of differential equations or boundary conditions involved in the problem or having multiple domain etc. In this work, we have developed wavelet Galerkin approximation in the basis of Daubechies wavelet for eigensolution of regular and singular Sturm-Liouville problems having algebraic coefficients defined over a bounded domain $[0, 1]$. The efficiency of the method has been tested on a variety of problems of physical interests whose exact solutions or approximate solutions obtained by some other methods are available. Examples illustrated exhibit the fact that wavelet Galerkin method based on Daubechies wavelets basis (truncated in case of elements touching boundary) is quite efficient in evaluating eigenspectrum of regular or singular Sturm-Liouville problems with better accuracy in comparison to other approximation scheme e.g. versatile variational method based on simple Haar wavelets or FEM (cf. Bujurke *et al*. [39]). Extension of this method to other wavelet basis involving pair of filter sequences (instead of four) e.g. Coiflet, is straightforward. Since there is an efficient quadrature rule for evaluation of coefficients of multiscale expansion of smooth functions beyond polynomial types or singular

Table 2: Eigenvalues for singular SLPs (5.2) for first three levels and $l = 0, 1, 2$ in both cases. The entries in the first row against each j are eigenvalues calculated by using MRG and wavelets in Daubechies family with $K = 3$ and in second row corresponding to $K = 4$.

		Coulomb plus centrifugal term			Isotropic harmonic oscillator plus centrifugal term		
l		0	1	2	0	1	2
n_r	j						
0	3	7.	18.	31.	10.	20.	33.
	4	7.3	18.3	31.5	10.1	20.5	33.6
		7.372 7	**18.329 8**	**31.581**	**10.150**	**20.564 9**	**33.655 6**
	5	7.37	18.32	31.57	10.15	20.56	33.6
		7.373 91	**18.329 45**	**31.580 9**	**10.151 14**	**20.564 53**	**33.655 55**
	6	7.373	18.329	31.57	10.15	20.564	33.654 6
		7.373 981	**18.329 438**	**31.580 924**	**10.151 163 3**	**20.564 514**	**33.655 554 4**
Exact		***7.373 985 02***	***18.329 437 94***	*31.580 092 23*	*10.151 164 03*	*20.564 513 87*	*33.655 554 21*
1	3	35.	56.	79.	38.	59.	82.
	4	36.	57.	80.	39.	59.8	83.
		36.29	**57.33**	**80.68**	**39.76**	**60.03**	**83.10**
	5	36.	57.2	80.6	39.7	60.0	83.0
		36.334	**57.322**	**80.678 55**	**39.798**	**60.027**	**83.095**
	6	36.33	57.32	80.	39.79	60.02	83.09
		36.335 92	**57.321 789**	**80.678 505**	**39.799 35**	**60.026 98**	**83.094 946**
Exact		***36.336 019 60***	***57.321 776 26***	*80.678 503 30*	*39.799 393 00*	*60.026 975 18*	*83.094 944 33*
2	3	82.	113.	156.	86.	117.	159.
	4	84	115	148	88	118.7	151.6
		84.	**116.3**	**149.6**	**88**	**119.33**	**152.3**
	5	85.	116.	149.	88.9	119.	152.
		85.27	**116.216**	**149.533**	**89.140 1**	**119.243**	**152.212 1**
	6	85.2	116.2	149.5	89.1	119.2	152.18
		85.291 8	**116.214 07**	**149.532 37**	**89.153 8**	**119.240**	**152.211 36**
Exact		***85.292 582 09***	***116.213 980 03***	*149.532 357 70*	*89.154 342 45*	*119.243 459 6*	*152.211 343 12*
3	3	171.	208.	281.	175.	212.	284.
	4	151.	193.	237.	156.	196.8	240.
		152.	**195.**	**238.**	**156.**	**198.**	**241.6**
	5	153.	194.	238.	157.7	197.9	240.8
		154.008	**194.938**	**238.16**	**158.16**	**198.20**	**241.058 52**
	6	154.0	194.9	238.1	158.17	198.15	240.99
		154.095 09	**194.925 8**	**238.161 8**	**158.241**	**198.195 8**	**241.050 9**
Exact		***154.098 623 74***	***194.925 448 57***	*238.161 747 78*	*158.243 961 70*	*198.195 449 90*	*241.058 495*
4	3	334.	362.	433.	337.	365.	437.
	4	238.	291.	348.	243.	295.	351.
		237.	**294.**	**348.**	**242.**	**297.**	**351.**
	5	242.	293.	346.	245.9	296.	349.
		242.3	**293.47**	**346.59**	**246.8**	**296.93**	**349.66**
	6	242.5	293.3	346.5	246.9	296.8	349.5
		242.69	**293.426**	**346.560 1**	**247.061**	**296.892**	**349.623 7**
Exact		***242.705 559 36***	***293.424 433 38***	*346.559 842 87*	*247.071 500 2*	*296.896 221 5*	*349.623 486 36*

functions [41], the present scheme can be smoothly reformulated for SLP with transcendental or discontinuous coefficients within the domain, even for multi-interval problems. Due to higher regularity and computational simplicity of evaluation of connection coefficients for derivatives of scale functions and wavelets in Daubechies family, the proposed scheme can be easily applied to eigenspectrum of SLPs involving higher order derivatives and reformulated for SLPs involving fractional derivatives. Works in these directions are in progress and will be reported in due course.

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