Indian J. Pure Appl. Math., **47**(4): 617-639, December 2016

DOI: 10.1007/s13226-016-0206-3

DYNAMICAL ANALYSIS AND CHAOS CONTROL IN A HETEROGENEOUS KOPEL DUOPOLY GAME

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(*Received* 6 *November* 2013; *after final revision* 25 *June* 2014;
accepted 29 *February* 2016)

A dynamic of a nonlinear Kopel duopoly game with heterogeneous players is presented. By assuming two heterogeneous players where one player use naive expectation whereas the other employs a technique of adaptive. The stability conditions of equilibrium points are analyzed. Numerical simulations are used to show bifurcation diagrams, phase portraits and sensitive dependence on initial conditions. The chaotic behavior of the game has been controlled by using feedback control method.

Key words : Kopel duopoly game; heterogeneous players; Nash equilibrium point; bifurcation; chaos; feedback control method.

1. INTRODUCTION

The classic model of oligopolies was proposed by the French mathematician, Cournot [7] introduced the first well-known model, which gives a mathematical description of competition in a duopolistic market consisting of two quantity-setting firms which produce homogeneous goods and make the optimal output choice through assuming the last value taken by competitors in every step. In the literature on oligopoly games most of papers focus on games with homogeneous players, that is players who adopt the same expectation rule see in [20, 21]. Expectations also play an important role in modeling of economic phenomena a producer can choose his expectations rules from many available techniques to adjust his production outputs. Cournot [7] investigated the case that each firm is provided with naive expectation in duopoly, that is in every step each firm assumes that the output of his competitor at his period is the same as the previous period. Bounded rationality has been considered by some scholars [2, 3, 29]. Recently, several works have shown that the Cournot model may lead to complex behavior such as cyclic and chaotic see for example [1, 9, 15]. Multistability and complex dynamics in a simple discrete economic model have been studied in [23].

Another branch of literature is considered more real mechanisms through which game with heterogeneous players form their expectations about decisions of competitors. The player can use different strategies called heterogeneous expectations as naive or adaptive and bounded rationality. The output duopoly game with heterogeneous players is analyzed in order to study the influence of player's different behavior on the dynamics of game, so the dynamic behavior of a duopoly game representing two firms using heterogeneous expectations rules have been studied in [4, 5, 30]. In actual economies, players sometimes would offer an upper limiter to their output due to capacity constraints, financial constraints and cautious response to uncertainty in the world, or offer a lower limiter to their output due to economies of scale or break-even consideration, so the dynamic duopoly game with heterogeneous players by assuming that one of them imposes an upper limiter on output, and the other one imposes a lower limiter has been discussed in [12]. Compared with the dynamic Cournot model, the dynamic Bertrand model is seldom investigated. However, Bertrand competition is more likely to occur than Cournot competition in reality. The chaotic dynamics of a differentiated Bertrand duopoly model with heterogeneous players and nonlinear cost function have been studied in [27]. The effects of micro-founded differentiated products demand on stability has been studied in [20].

A duopoly is a specific type of oligopoly where only two producers exist in one market. In reality, this definition is generally used where only two firms have dominant control over a market. In the field of industrial organization, the most popular example of duopoly is between Visa and Mastercard which exercise a major control over the electronic payment processing market in the world. Pepsi and Coca-cola are the two major shareholders in the soft drinks market. Airbus and Boeing are duopolies in the commercial jet aircraft market. Applications are given for the market such as the competition between cellular firms [14]. Neural networks as models of the firm are able to converge to the unique Nash equilibrium of a Cournot game when facing a linear demand function with stochastic parameters [11].

In this paper, we investigate the over all dynamical analysis and chaos control of a nonlinear Kopel duopoly game with heterogeneous firms. We assume the existence of two firms, firm X and firm Y, each firm thinks with a different strategy to maximize his output. One firm (firm X) is endowed with naive expectations and the other firm (firm Y) is assumed adaptive method.

The paper is organized as follows. Section 2, introducing the model of heterogenous Cournot-Kopel duopoly game. Our model is constructed, the Nash-equilibria and the stability analysis are provided in Section 3. Section 4, present various types of changes in behavior that can occur at bifurcation values In Section 5, there is numerical simulation. In Section 6, chaos control of the model is considered with the feedback control method. Finally, the results are summarized.

2. Heterogeneous Kopel Duopoly Game

Let us consider two firms X and Y, which produce the quantities x_t and y_t in period t . To determine the quantity of period $t+1$, the firms form expectations on the quantity of the other firm, which might for example depend on their own quantity and the quantity of the other firm. Assuming the linearity of the inverse demand curve unchanged. let

$$p(x,y) = a - b(x+y),$$

define the inverse demand curve, where $x+y$ is the industry output and $a, b > 0$. If we assume that the cost function of the form $c(x, y)$ and the profit function of firms X and Y are.

$$\begin{aligned} \Pi^x(x,y) &= [a - b(x+y)]\, x - c^x(x,y) \\ \Pi^y(x,y) &= [a - b(x+y)]\, y - c^y(x,y). \end{aligned}$$

We have the following result:

Proposition 1 — Let the cost function of firm X is given by:

$$c^x(x_t, x_{t+1}, y_t) = d^x + a\, x_{t+1} - b\, x_{t+1}\, y_t(1+2\mu_x) + 2b\, \mu_x\, x_{t+1}\, y_t^2$$

and the cost function of firm Y is given by:

$$c^y(y_t, y_{t+1}, x_t) = d^y + a\, y_{t+1} - b\, y_{t+1}\, x_t(1+2\mu_y) + 2b\, \mu_y\, y_{t+1}\, x_t^2$$

where d^x, d^y are nonnegative constants representing the fixed costs, $a, b > 0$, $\lambda \in [0,1]$ and $\mu_x, \mu_y \in [1,4]$. Then the reaction functions of the two firms are given by:

$$\begin{aligned} x_{t+1} &= \mu_x\, y_t\, (1 - y_t) \\ y_{t+1} &= \mu_y\, x_t\, (1 - x_t)\, . \end{aligned}$$

PROOF : The expectation profit of firm X is given by:

$$\Pi^x = [a - b(x_{t+1} + y_{t+1})]\, x_{t+1} - d^x - a\, x_{t+1} + b\, x_{t+1}\, y_t(1+2\mu_x) - 2b\, \mu_x\, x_{t+1}\, y_t^2.$$

Maximization under the condition $x = x_{t+1}$ leads to the first-order condition (the second- order condition is $\frac{\partial \Pi^x}{\partial x_{t+1}} = -2b < 0$; so we indeed deal with a maximum):

$$\frac{\partial \Pi^x}{\partial x_{t+1}} = -2b\, x_{t+1} + 2b\, \mu_x\, y_t - 2b\, \mu_x\, y_t^2 = 0,$$

and hence we obtain the reaction curve for firm X ,

$$x_{t+1} = x^*_{t+1} = \mu_x\, y_t\, (1 - y_t)\, .$$

The expectation profit of firm Y is given by

$$\Pi^y = [a - b(x_{t+1} + y_{t+1})]\, y_{t+1} - d^y - a\, y_{t+1} + b\, y_{t+1}\, x_t(1 + 2\mu_y) - 2b\, \mu_y\, y_{t+1}\, x_t^2$$

the first-order condition

$$\frac{\partial \Pi^y}{\partial y_{t+1}} = -2b\, y_{t+1} + 2b\, \mu_y\, x_t - 2b\, \lambda\, \mu_y\, x_t^2 = 0,$$

leads to the result

$$y_{t+1} = y^*_{t+1} = \mu_y\, x_t\, (1 - x_t)\, ,$$

and this is the reaction curve for firm Y. The proof is completed now. □

3. The Map and the Local Stability Analysis of its Equilibria

The model that we use is a two-dimensional map described two firms heterogeneous with their expectations, firm X is endowed with naive expectations and the other firm (firm Y) is assumed adaptive method. Let x_t and y_t denote the outputs of the first firm and the second in discrete time period t , respectively, and decide their productions for the next period x_{t+1} and y_{t+1}. The time evolution of the model is determined by the two-dimensional map $T : ((x_t, y_t\,) \rightarrow (x_{t+1}, y_{t+1}))$ defined by:

$$T : \begin{cases} x_{t+1} = \mu_x\, y_t\, (1 - y_t) \\ y_{t+1} = (1-\lambda) y_t + \lambda\, \mu_y\, x_t\, (1 - x_t)\, . \end{cases} \tag{3.1}$$

Where the positive parameters μ_x ,$\mu_y \in [1, 4]$. λ is called the adjustment coefficient. The meaning of the model implies that the parameter $\lambda \in [0, 1]$.

We investigate the Nash-equilibria and equilibrium stability of a nonlinear Kopel duopoly game with heterogeneous firms for special case of the model, when $\mu_x = \mu_y = \mu$.

This follows since fixed points of the generalized system have to fulfill the equations:

$$\begin{aligned} x &= \mu\, y\, (1 - y) \\ y &= \mu\, x\, (1 - x)\, . \end{aligned} \tag{3.2}$$

Similar with the work by Agiza [1], the solutions of equation (3.2) give four equilibria:

1. Trivial solution $E_1(x_1, y_1) = (0, 0)$.

2. A positive symmetric fixed point exists for $\mu > 1$, given by
   $E_2(x_2, y_2) = (\frac{\mu-1}{\mu}, \frac{\mu-1}{\mu})$.

3. Two further non-symmetric Nash equilibria, given by
   $E_3\left(x_3, y_3\right) = \left(\frac{(\mu+1)+\sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{(\mu+1)-\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right)$,
   and its $(x, y) \longmapsto (y, x)$ reflection E_4 for $\mu \geq 3$

4. $E_4\left(x_4, y_4\right) = \left(\frac{(\mu+1)-\sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{(\mu+1)+\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right)$.

In order to study the local stability of the fixed points which are based on calculated the eigenvalues θ_1 and θ_2 of the Jacobian matrix of equation (3.1) at any points (x, y) is given by the matrix:

$$J = \begin{bmatrix} 0 & \mu\left(1-2y\right) \\ \lambda\mu\left(1-2x\right) & 1-\lambda \end{bmatrix}, \tag{3.3}$$

the characteristic equation of J has the form

$$P\left(\theta\right) = \theta^2 - T\,\theta + D = 0, \tag{3.4}$$

where T is the trace and D is the determinate of the Jacobian matrix defined in equation (3.3),

$$\begin{aligned} T &= \left(1-\lambda\right), \\ D &= -\lambda\mu^2\left(1-2x\right)\left(1-2y\right). \end{aligned} \tag{3.5}$$

Then the corresponding eigenvalues of equation (3.4) are given by,

$$\theta_{1,2} = \frac{1}{2}\left(1-\lambda\right) \pm \frac{1}{2}\sqrt{\left(1-\lambda\right)^2 + 4\lambda\mu^2\left(1-2x\right)\left(1-2y\right)}. \tag{3.6}$$

The fixed points, E_1 upto E_4 of the discrete dynamical system equation (3.1) are stable if $|\theta_i| < 1, i = 1, 2$. In order to study the qualitative behavior of the solution of equation (3.1) we estimate equation (3.3) and equation (3.6) at the corresponding fixed point. For each fixed point, E_1

upto E_4 we calculate the eigenvalues θ_1 and θ_2 which can be judged by the following Jury criterion [13, 26]. Which are generically given by:

$$\begin{cases} i)\ F := 1 + T + D > 0 \\ ii)\ TC := 1 - T + D > 0 \ , \\ iii)\ H := 1 - D > 0 \end{cases} \tag{3.7}$$

where the trace and determinate of the Jacobian matrix are given respectively by:

$$\begin{aligned} T &: = J_{11} + J_{22} \\ D &: = J_{11}J_{22} - J_{12}J_{21} \end{aligned}$$

These conditions are satisfied in a region which is called the ”stability region” All these regions are sketched in Figure (1).

4. Bifurcations in Discrete-time Dynamical Systems

Bifurcation theory describes the way that topological features of a flow vary as one or more parameters are varied. A bifurcation occurs when the solutions of a nonlinear dynamical system change their qualitative character as a parameter changes. Bifurcation is usually associated with the existence of strange attractors.

In this section, we present various types of changes in behavior that can occur at bifurcation values. The types of bifurcations depend on how the dynamics of a map change as a single parameter is varied.

Consider a discrete-time dynamical system depending on a parameter

$$x \to H\left(\mu, x\right), \ \ x \in R^n, \ \mu \in R,$$

where the map H is smooth with respect to both x and μ. Let $x = x_0$ be a hyperbolic fixed point of the system for $\mu = \mu_0$. Let us monitor this fixed point and its eigenvalues while this parameter varies. It is clear that there are generically, only three ways in which the hyperbolicity condition can be violated. Either a simple positive eigenvalue approaches the unit circle and we have $\lambda_1 = 1$, or a simple negative eigenvalue approaches the unit circle and we have $\lambda_1 = -1$, or a pair of simple complex eigenvalues reach the unit circle and we have $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, for some value of parameter. Now, we give the following theorems.

***Theorem* 1** *(Fold bifurcation)* [18] — *Let the mapping* $x_{n=1} = H(x_n, \mu), x \in R, \mu \in R$. *Let* H *be* C^2 *and assume that there is a fixed point* $(x^*, \mu_0) = (0, 0)$. *If:*

$$\frac{\partial H(0,0)}{\partial x} = \lambda = 1, \tag{4.1}$$

$$\frac{\partial H^2(0,0)}{\partial x^2} \neq 0, \tag{4.2}$$

$$\frac{\partial H(0,0)}{\partial \mu} \neq 0. \tag{4.3}$$

Depending on the signs of the expressions (4.2) and (4.3), there are:

(i) No fixed points near $(0, 0)$ if $\mu < 0$ $(\mu > 0)$.

(ii) Two fixed points near $(0, 0)$ if $\mu > 0$ $(\mu < 0)$.

And the fold bifurcation implies that no fixed point exists for parameter values smaller or larger (depending on the signs of the expressions (4.2) and (4.3)) than the bifurcation value.

***Theorem* 2** *(Pitchfork bifurcation)* [18] — *Let the mapping* $x_{n+1} = H(x_n, \mu), x \in R, \mu \in R$. *Let* H *be* C^2 *and assume that there is a fixed point* $(x^*, \mu_0) = (0, 0)$. *If:*

$$\frac{\partial H(0,0)}{\partial x} = \lambda = 1, \tag{4.4}$$

$$\frac{\partial H^3(0,0)}{\partial x^3} \neq 0, \tag{4.5}$$

$$\frac{\partial^2 H(0,0)}{\partial x \partial \mu} \neq 0. \tag{4.6}$$

Depending on the signs of the expressions (4.5) and (4.6), there are:

(iii) The fixed point x^* *is stable (unstable) for* $\mu < 0$ $(\mu > 0)$.

(iv) The fixed point x^* *becomes unstable (stable) for* $\mu < 0$ $(\mu > 0)$ *and a branch of additional stable (unstable) fixed points* $x(\mu)$ *emerges.*

***Theorem* 3** *(Flip bifurcation)* [18] — *Let* $H_\mu : \mathbb{R} \to \mathbb{R}$ *be a one parameter family of mapping such that* H_{μ_0} *has a fixed point* x^* *with eigenvalue* -1. *If at* (x^*, μ_0) :

$$\frac{\partial H(0,0)}{\partial x} = \lambda = -1, \tag{4.7}$$

$$\left(\frac{\partial H}{\partial \mu}\frac{\partial^2 H}{\partial x^2} + \frac{2\partial^2 H}{\partial x \partial \mu}\right) \neq 0, \tag{4.8}$$

$$\left(\frac{-2\partial H^3}{\partial x^3} - \frac{3\partial^2 H}{\partial x^2}\right) = \alpha \neq 0. \tag{4.9}$$

Depending on the signs of the expressions (4.8) and (4.9), there are:

(v) The fixed point x^* is stable (unstable) for $\mu < \mu_0$ ($\mu > \mu_0$).

(vi) The fixed point x^* becomes unstable (stable) for $\mu < \mu_0$ ($\mu > \mu_0$) and a branch of additional stable (unstable) fixed points of order 2 which enclose x^* emerges.

***Theorem* 4** (Neimark-sacker bifurcation) [18, 22] — *Let the mapping $x_{n+1} = H(x_n, \mu), x \in R^2$, $\mu \in R$, have a smooth family of fixed points $x^*(\mu)$ at which the eigenvalues are complex conjugate. If there is a μ_0 such that:*

$$\operatorname{mod}\lambda(\mu_0) = 1 \textit{ but } \lambda^n(\mu_0) \neq 1,\ n = 1, 2, 3, 4 \tag{4.10}$$

and

$$\frac{d(\operatorname{mod}\lambda(\mu_0))}{d\mu} > 1 \tag{4.11}$$

then there is invariant closed curve bifurcating from $\mu = \mu_0$.

Details and examples on the types of bifurcations can be found in see [8, 10].

The analysis for the local stability and bifurcation of these fixed points consist of four cases.

Case 1 : The trivial solution $E_1(x_1, y_1) = (0, 0)$.

Proposition 2 — The equilibrium solution E_1 is asymptotically stable for $(\mu, \lambda) \in \Omega(E_1)$ where,

$$\Omega(E_1) = \left\{ (\mu, \lambda) \middle| 0 < \mu < 1, 0 < \lambda < \lambda_1(\mu) = \frac{2}{1 + \mu^2} \right\}$$

and E_1 looses its stability via a pitchfork bifurcation when $\mu = 1$ and via flip bifurcation when crossing $\lambda_1(\mu) = \frac{2}{1+\mu^2}$.

PROOF : In order to prove this result we find the eigenvalues of the Jacobian matrix J_{E_1}. The linearization of equation (3.1) about the first fixed point $(0, 0)$ has the Jacobian matrix:

$$J_{E_1} = \begin{bmatrix} 0 & \mu \\ \lambda\mu & 1 - \lambda \end{bmatrix}, \tag{4.12}$$

which has two eigenvalues,

$$\theta_{1,2} = \frac{1}{2}(1 - \lambda) \pm \frac{1}{2}\sqrt{\lambda^2 + 2\lambda(2\mu^2 - 1) + 1.} \tag{4.13}$$

The necessary and sufficient conditions for the stability of Nash equilibrium E_1 are drived by the eigenvalues of the Jacobian matrix J_{E_1}. They are inside the unite circle of the complex plane. This is true iff the Jury criterion are hold:

$$\begin{cases} i)\ 1-T+D=1-(1-\lambda)-\lambda\mu^2>0 \\ ii)\ 1+T+D=2-\lambda\left(1+\mu^2\right)>0 \\ iii)\ 1-D=1+\lambda\mu^2>0 \end{cases},$$

the third condition is always satisfied, whereas the other two conditions (i) and (ii) define a bounded region of stability in the parameters model λ and μ. Then the Nash equilibrium E_1 looses its stability through a pitchfork bifurcation when $\mu=1$ or $\lambda=0$ according to condition (i) and crossing the flip bifurcation for $\lambda=\frac{2}{1+\mu^2}$. □

Case 2 : A positive symmetric fixed point $E_2(x_2,y_2)=(\frac{\mu-1}{\mu},\frac{\mu-1}{\mu})$.

Proposition 3 — The equilibrium solution E_2 is asymptotically stable for $(\mu,\lambda)\in\Omega\left(E_2\right)$ where,

$$\Omega\left(E_2\right)=\left\{(\mu,\lambda)\middle|1<\mu<3, 0<\lambda<\lambda_2\left(\mu\right)=\frac{2}{1+\left(2-\mu\right)^2}\right\}$$

The equilibrium solution E_2 looses stability via a pitchfork bifurcation when $\mu=1$ or $\mu=3$ and via flip bifurcation for $\lambda=\frac{2}{1+(2-\mu)^2}$.

PROOF : We consider the Jacobian matrix J_{E_2}, evaluate in E_2 that is,

$$J_{E_2}=\begin{bmatrix} 0 & 2-\mu \\ \lambda\left(2-\mu\right) & 1-\lambda \end{bmatrix}, \tag{4.14}$$

the corresponding eigenvalues of equation (4.14) are,

$$\begin{aligned} \theta_1 &= \frac{1}{2}\left(1-\lambda\right)+\frac{1}{2}\sqrt{\left(1-\lambda\right)^2+4\lambda\left(2-\mu\right)^2} \\ \theta_2 &= \frac{1}{2}\left(1-\lambda\right)-\frac{1}{2}\sqrt{\left(1-\lambda\right)^2+4\lambda\left(2-\mu\right)^2}. \end{aligned} \tag{4.15}$$

The stability condition for the fixed point E_2 is given by:

$$\begin{cases} i)\ 1-T+D=1-(1-\lambda)-\lambda\left(2-\mu\right)^2>0 \\ ii)\ 1+T+D=1+(1-\lambda)-\lambda\left(2-\mu\right)^2>0 \\ iii)\ 1-D=1+\lambda\left(2-\mu\right)^2>0 \end{cases}, \tag{4.16}$$

From (4.16) it can easily be seen that condition (iii) fulfilled while conditions (i) and (ii) can be violated. Therefore the Nash-equilibria E_2 can loose stability through pitchfork bifurcation when $\mu=1$ or $\mu=3$ according to condition (i) when $(2-\mu)^2<1$. And looses the stability through a flip (period-doubling) bifurcation when $\lambda=\frac{2}{1+(2-\mu)^2}$ from condition (ii). □

Case 3 : Nash equilibria, given by:

$E_3(x_3, y_3) = \left(\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right)$.

Proposition 4 — The equilibrium solution E_3 is asymptotically stable for $(\mu, \lambda) \in \Omega(E_3)$, $\Omega(E_3) = \Omega(E_{31}) \cup \Omega(E_{32})$ where,

$$\begin{aligned}\Omega(E_{31}) &= \left\{(\mu,\lambda) \middle| 3<\mu<1+\sqrt{5}, 0<\lambda<\lambda_{31}(\mu) = \frac{2}{5+2\mu-\mu^2}\right\} \\ \Omega(E_{32}) &= \left\{(\mu,\lambda) \middle| \mu>1+\sqrt{5}, 0<\lambda<\lambda_{32}(\mu) = \frac{1}{\mu^2-2\mu-4}\right\}\end{aligned}$$

The equilibrium solution E_3 looses stability via a flip (period-doubling) bifurcation when $\lambda = \lambda_{31}(\mu) = \frac{2}{5+2\mu-\mu^2}$, $3 < \mu < 1+\sqrt{5}$ and via a Neimark-Sacker bifurcation for $\lambda = \lambda_{32}(\mu) = \frac{1}{\mu^2-2\mu-4}$, $\mu > 1+\sqrt{5}$.

PROOF : The linearization of equation (3.1) about this fixed point given by the Jacobian matrix J_{E_3}.

$$J_{E_3} = \begin{bmatrix} 0 & \mu\left(1-\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{\mu}\right) \\ \lambda\mu\left(1-\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{\mu}\right) & 1-\lambda \end{bmatrix}, \tag{4.17}$$

the eigenvalues of matrix (4.17) are,

$$\begin{aligned}\theta_1 &= \frac{1}{2}(1-\lambda) + \frac{1}{2}\sqrt{(1-\lambda)^2 + 4\lambda(4-\mu^2+2\mu)} \\ \theta_2 &= \frac{1}{2}(1-\lambda) - \frac{1}{2}\sqrt{(1-\lambda)^2 + 4\lambda(4-\mu^2+2\mu)}.\end{aligned} \tag{4.18}$$

The stability condition for the fixed point E_3 is given by:

$$\begin{cases} i)\ 1-T+D = 1-(1-\lambda)-\lambda\left(4-\mu^2+2\mu\right) > 0 \\ ii)\ 1+T+D = 1+(1-\lambda)-\lambda\left(4-\mu^2+2\mu\right) > 0 \\ iii)\ 1-D = 1-\lambda\left(4-\mu^2+2\mu\right) > 0 \end{cases}, \tag{4.19}$$

according to Jury criterion (4.19), the Nash equilibrium point E_3 is satisfied the condition (i) while conditions (ii) and (iii) can be violated. Then E_3 looses it stability through period-doubling bifurcation when $\lambda = \frac{2}{5+2\mu-\mu^2}$ from (ii) and via a Neimark-Sacker bifurcation for $\lambda = \frac{1}{\mu^2-2\mu-4}$ according to (iii).

Next, the Neimark-Sacker bifurcation at E_3 is to be discussed. Consider the case that λ is an bifurcation parameter. The eigenvalues of J_{E_3} are $\theta_{1,2} = \frac{1}{2}(1-\lambda) \pm \frac{1}{2}i\sqrt{-(1-\lambda)^2 - 4\lambda(4-\mu^2+2\mu)}$, and

$$
\begin{aligned}
|\theta| &= \frac{1}{2}\sqrt{(1-\lambda)^2-(1-\lambda)^2-4\lambda\left(4-\mu^2+2\mu\right)} \\
&= \frac{1}{2}\sqrt{-4\lambda\left(4-\mu^2+2\mu\right)} \\
&= \sqrt{\lambda\left(\mu^2-2\mu-4\right)}.
\end{aligned}
$$

Let $\lambda=\frac{1}{\mu^2-2\mu-4}$, then

$$
\left.\frac{d\,|\theta|}{d\lambda}\right|_{\lambda=\frac{1}{\mu^2-2\mu-4}}=\frac{\mu^2-2\mu-4}{2\sqrt{\lambda\left(\mu^2-2\mu-4\right)}}=\frac{2}{\lambda}>0,
$$

$|\theta(\lambda)|=1$ and

$$
\begin{aligned}
\theta_{1,2}(\lambda) &= \left.\frac{1}{2}\left[(1-\lambda)\pm i\sqrt{-(1-\lambda)^2-4\lambda\left(4-\mu^2+2\mu\right)}\right]\right|_{\lambda=\frac{1}{\mu^2-2\mu-4}} \\
&= \frac{1}{2}\left[(1-\lambda)\pm i\sqrt{4-(1-\lambda)^2}\right].
\end{aligned}
$$

If $\lambda\neq 1$, $\theta_{1,2}^n\neq 1$, $n=1,2,3,4$. □

Case 4 : Nash equilibria, given by:

$E_4\left(x_4,y_4\right)=\left(\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu},\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right)$.

Proposition 5 — The equilibrium solution E_4 looses stability via a period-doubling bifurcation when $\lambda=\frac{2}{5+2\mu-\mu^2}$ and via a Neimark-Sacker bifurcation for $\lambda=\frac{1}{\mu^2-2\mu-4}$.

PROOF : The Jacobian matrix J_{E_4} about this fixed point given by,

$$
J_{E_4}=\begin{bmatrix} 0 & \mu\left(1-\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{\mu}\right) \\ \lambda\mu\left(1-\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{\mu}\right) & 1-\lambda \end{bmatrix} \tag{4.20}
$$

The eigenvalues of matrix (4.20) are

$$
\begin{aligned}
\theta_1 &= \frac{1}{2}(1-\lambda)+\frac{1}{2}\sqrt{(1-\lambda)^2+4\lambda\left(4-\mu^2+2\mu\right)} \\
\theta_2 &= \frac{1}{2}(1-\lambda)-\frac{1}{2}\sqrt{(1-\lambda)^2+4\lambda(2-\mu)^2}.
\end{aligned}
$$

We see that the fixed point E_4 has the same eigenvalues as the fixed point E_3 with $(x,y)\longmapsto(y,x)$ transformation. Then the stability conditions for the case 4 are the same as case 3. □

5. Numerical Simulations

The main purpose of this section is to show the qualitative behavior of the solutions of the duopoly game with heterogeneous players described by the dynamic equation (3.1). To provide some numerical evidence for the chaotic behavior of system (3.1), we present several numerical results, including bifurcation diagrams, strange attractors and sensitive dependence on initial conditions.

We fix the parameter $\mu = 3.8$ and assume that λ varies. In Figure (2) the bifurcation diagram for system (3.1) is plotted as a function of parameter λ and the state variable x while the bifurcation diagram with respect to λ and variable state y is plotted in Figure (3). From these Figures show that the system is stable for $\lambda < 0.3$, and loses its stability at a Neimark-Sacker bifurcation in region $\Omega\left(E_{3,4}\right)$, the eigenvalues are complex for parameter $\mu = 3.8$ and $\lambda = 0.3$, the fixed point E_3 becomes unstable and Neimark-Sacker bifurcation occurs. To study the behavior of the model (3.1) when the parameter λ varied in the interval $[0, 1]$ we consider the initial condition (x_0, y_0) situated in the basin of attraction of fixed point E_3. For fix parameter $\mu = 3.8$ and λ varies, the stability of a periodic solution may be lost through various types of bifurcations. The phase space portraits are considered in the following cases:

Figure (4) shows that the fixed point E_3 is stable attractor at $\lambda = 0.28$. For this parameter value, E_3 occurs at $x^* = 0.8894199730$, $y^* = 0.3737379218$ and the associated complex conjugate values are $\theta_{\pm} = 0.3600000000 \pm 0.8158431220i$, then $|\theta_{\pm}| = 0.8917398722$. This means that E_3 is asymptotically stable. The behavior of the model (3.1) before a Neimark-Sacker bifurcation at $\lambda = 0.35$ is shown in Figure (5). When $\lambda = 0.356$ demonstrates the behavior of the model after a Neimark-Sacker bifurcation shown in Figure (6). From these Figures we deduce that the fixed point E_3 loses its stability through Hopf bifurcation (a Neimark-Sacker bifurcation) when λ varies 0.35 to 0.356. Increasing the parameter $\lambda = 0.42$, leads E_3 to unstable and invariant closed curve was created a round the fixed point which $x^* = 0.8894199730$, $y^* = 0.3737379218$ and the associated complex conjugate values are $\theta_{\pm} = 0.2900000000 \pm 1.052948242i$, then $|\theta_{\pm}| = 1.092153835$ shown in Figure (7). As λ is further increasing, we can see the portrait starts to create new phenomena due to breakdown of the invariant closed curve, exists for $\lambda = 0.5$ in Figure (8). When $\lambda = 0.51$, a chaotic attractor appears in the phase portrait, moreover increasing $\lambda > 0.570$ the behavior of this model becomes very complicated, including many chaotic bands which exhibit fractal structure see in Figures (9 and 10).

5.1 *Sensitive dependence on initial conditions*

To demonstrate the sensitivity to initial conditions of system (3.1), we compute two orbits with initial points $(x_0, y_0) = (0.1, 0.2)$ and $(x_0 + 0.0001, y_0) = (0.1001, 0.2)$ at the parameter values $\mu = 3.8$, $\lambda = 0.650$ respectively. The difference between the two $x-$ coordinate is 0.0001, while the other coordinate is kept to has the same value. We can get two firms' game results which are shown in (11). At the beginning the results are indistinguishable, but after a number of games the difference between them builds up rapidly. The same to variable y. Figure (12) shows the sensitivity dependence on initial conditions, $y-$coordinate of the two orbits.

6. Controlling Chaos in Cournot-Kopel Model

One way to generate such a periodic output from a chaotic system is to use the OGY method, a feedback-based approach published by Ott *et al.*, [24] in 1990. It can be explained as follows:

Feedback controls are used in many aspects of our lives, the method has been used by engineers for many years. However, the systematic study of stabilization by state feedback control is of more recent origin see [16, 25, 28].

6.1 *State feedback control*

The idea of state feedback is simple: It is assumed that the state vector $x(n)$ can be directly measured and the control $u(n)$ is adjusted based on this information. Consider time-invariant control system whose equation is,

$$x(n+1) = A\,x(n) + B\,u(n), \tag{6.1}$$

where, A is a $k \times k$ matrix and B a $k \times m$ matrix. Suppose we apply linear feedback $u(n) = -Hx(n)$, where H is a real $m \times k$ matrix called the state feedback or gain state matrix. The resulting system obtained by substituting $u = -Hx$ into equation (6.1) is:

$$x(n+1) = A\,x(n) - B\,H\,u(n),$$

or

$$x(n+1) = (A - BH)x(n). \tag{6.2}$$

The objective of feedback control is to choose H in such a way such that, H must be chosen so that all the eigenvalues of $A - BH$ lie inside the unit disk.

We fix the parameters $\mu = 3.8$, $\lambda = 0.570$ in such context the system exhibits a chaotic attractor. We take λ as the control parameter which allows for external adjustment but is restricted to lie in a small interval $|\lambda - \lambda_0| < \delta$, $\delta > 0$, around the nominal value $\lambda_0 = 0.570$. The system becomes,

$$\begin{aligned} f \quad &: \quad x_{t+1} = 3.8\,y_t\,(1 - y_t) \qquad (6.3)\\ g \quad &: \quad y_{t+1} = (1-\lambda)y_t + 3.8\,\lambda\,x_t\,(1 - x_t)\,. \end{aligned}$$

We consider the stabilization of the unstable period one orbit $E_3 \cong (0.88941, 0.37373)$. The map (6.3) can be approximated in the neighborhood of the fixed point by the following linear map:

$$\begin{pmatrix} x_{t+1} - x^* \\ y_{t+1} - y^* \end{pmatrix} \cong A \begin{pmatrix} x_t - x^* \\ y_t - y^* \end{pmatrix} + B\left(\lambda - \lambda_0\right), \tag{6.4}$$

where,

$$A = \begin{pmatrix} \frac{\partial f(x^*,y^*)}{\partial x_t} & \frac{\partial f(x^*,y^*)}{\partial y_t} \\ \frac{\partial g(x^*,y^*)}{\partial x_t} & \frac{\partial g(x^*,y^*)}{\partial y_t} \end{pmatrix}, \tag{6.5}$$

and

$$B = \begin{pmatrix} \frac{\partial f(x^*,y^*)}{\partial \lambda} \\ \frac{\partial g(x^*,y^*)}{\partial \lambda} \end{pmatrix}. \tag{6.6}$$

Are the Jacobian matrices with respect to the control state variable (x_t, y_t) and to the control parameter λ. The partial derivatives are evaluated at the nominal value λ_0 and at (x^*, y^*). In our case we get:

$$\begin{pmatrix} x_{t+1} - 0.88941 \\ y_{t+1} - 0.37373 \end{pmatrix} \cong \begin{pmatrix} 0 & 0.959652 \\ -1.686924120 & 0.430 \end{pmatrix} \begin{pmatrix} x_t - 0.88941 \\ y_t - 0.37373 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.0000374372 \end{pmatrix} (\lambda - 0.570). \tag{6.7}$$

If we assume a linear feedback rule control for the parameter of the form:

$$(\lambda - \lambda_0) = -H \begin{pmatrix} x_t - x^* \\ y_t - y^* \end{pmatrix}, \tag{6.8}$$

where $H := \begin{bmatrix} h_1 & h_2 \end{bmatrix}$, then the linearized map becomes,

$$\begin{pmatrix} x_{t+1} - x^* \\ y_{t+1} - y^* \end{pmatrix} \cong (A - BH) \begin{pmatrix} x_t - x^* \\ y_t - y^* \end{pmatrix} \tag{6.9}$$

that is

$$\begin{pmatrix} x_{t+1} - 0.88941 \\ y_{t+1} - 0.37373 \end{pmatrix} \cong \begin{pmatrix} 0 & 0.959652 \\ -1.686924120 - 0.0000374372h_1 & 0.430 - 0.00000374372h_2 \end{pmatrix} \tag{6.10}$$

$$\bullet \begin{pmatrix} x_t - 0.88941 \\ y_t - 0.37373 \end{pmatrix}$$

which shows that the fixed point will be stable provided that $A - BH$ is that all its eigenvalues have modulus smaller than one. The characteristic equation of the matrix $[A - BH]$ to find the eigenvalues is defined as

$$((A - BH) - \gamma I) = \gamma^2 + \gamma\,(-0.430 + 0.00000374372h_2) + 1.618860106 + 0.0000359266h_1. \tag{6.11}$$

To stabilize the system about the fixed points through decaying the error, we must choose values of h_1 and h_2 such that both eigenvalues, γ_1 and γ_2 have magnitudes less than 1. Another way to interpret this is that the product of both eigenvalues must be less than 1, and that $\gamma_1 < 1$ and $\gamma_1 > -1$.

To meet the condition that $\gamma_1\gamma_2 < 1$, we can factor out the characteristic equation (6.11) into $(\gamma - \gamma_1)(\gamma - \gamma_2) = 0$. This means $\gamma_1\gamma_2 = 1.618860106 + 0.0000359266h_1$, thus the boundary when $\gamma_1\gamma_2$ transitions from below to above 1 equals $1.618860106 + 0.0000359266h_1 = 1$. We can find the parameter ranges such that $\gamma_1 = 1$, which form the boundary for when γ_1 transitions from below 1 to above 1. We can locate the boundary by trying to factor out $(\gamma - 1)$ from characteristic equation (6.11). Through long division, we can get the the boundary for $\gamma_1 = 1$ on the h_1, h_2 plane is a line defined as: $2.188860106 + 0.00003592668385h_1 + 0.00000374372h_2$. We use a similar approach in factoring out $(\gamma + 1)$ to find the boundary for $\gamma_1 = -1$. The boundary where $\gamma_1 = -1$ transitions from below -1 to above is a line defined by: $3.048860106 + 0.00003592668385h_1 - 0.00000374372h_2$. So the boundaries state as,

$$\begin{aligned}
\text{Blue Line} \quad &: \quad \gamma_1\gamma_1 < 1 : h_1 = \frac{1 - 1.618860106}{0.00003592668385} \\
\text{Red line} \quad &: \quad \gamma_1 < 1 : h_1 = \frac{-3.048860106 + 0.00000374372h_2}{0.00003592668385} \\
\text{Black line} \quad &: \quad \gamma_1 > -1 : h_1 = \frac{-2.188860106 - 0.00000374372h_2}{0.00003592668385}
\end{aligned} \tag{6.12}$$

In Figure (13) has the three boundaries, a triangle is defined by equation (6.12). To stabilize the system about the fixed point, we must choose a value of h_1 and h_2 within that triangle. In Figures (14 and 15), we show the time series of the chaotic trajectory starting from the initial point $(x_0, y_0) = (0.1, 0.2)$ for parameters $\mu = 3.8$, $\lambda = 0.570$ which we have chosen to control. In contrast, Figures (16 and 17) present the controlled orbit converging to the stabilized fixed point when the feedback matrix H is chosen such that a value of h_1 and h_2 within that triangle. This implies that for chosen parameter $H = (-40000, 40000)$.

6.2 *State delayed feedback control*

Bifurcation and chaos can be controlled via a delayed feedback control method. The characterization of the stabilization by linear and non linear feedback of the Cournot equilibrium points of a duopoly model is analyzed in see [6, 17]. In this subsection, we apply the DFC to the state (the production of the firm) and the parameters of the Cournot–Kopel model for the Nash-equilibria $E_4\,(x_4, y_4)$. Assume the duopoly model has the form

$$\begin{aligned} x_{t+1} &= f_1\,(y_t) \\ y_{t+1} &= (1-\lambda)\,y_t + \lambda f_2\,(x_t) \end{aligned} \tag{6.13}$$

where production of both firms in time t is x_t and y_t , and f_1, f_2 are the reaction functions of the market. This yields the specified reaction functions in equation (6.13) as follows:

$$\begin{aligned} f_1\,(y_t) &= \mu y_t\,(1-y_t) \\ f_2\,(x_t) &= \mu x_t\,(1-x_t) \end{aligned} \tag{6.14}$$

and let $f_1^{\acute{}}\,(y_4) = a$ and $f_2^{\acute{}}\,(x_4) = b$.

Now, we introduce a controlled form of the duopoly model (6.13) as follows:

$$\begin{cases} x_{t+1} = f_1\,(y_t) + u_t \\ y_{t+1} = (1-\lambda)\,y_t + \lambda f_2\,(x_t) \\ u_t = k(x_{t+1-T} - x_{t+1}), \quad t > T \end{cases} \tag{6.15}$$

where T is the time delay and k is the feedback gain (controlling parameter). We set $T = 1,$ then the controlled system can be expressed as follows:

$$\begin{cases} x_{t+1} = \frac{f_1(y_t)}{k+1} + \frac{k}{k+1} x_t \\ y_{t+1} = (1-\lambda)\,y_t + \lambda f_2\,(x_t)\,. \end{cases} \tag{6.16}$$

Proposition 6 — The Nash-equilibria $E_4\,(x_4, y_4)$ of the controlled system described by (6.16) is stabilized by delay feedback if and only if

$$x + \frac{1}{\lambda} > k > \frac{\lambda(x-1)-2}{2(2-\lambda)} \tag{6.17}$$

where $x = a.b.$

PROOF : The Jacobian matrix of the controlled system at the equilibrium point $E_4\,(x_4, y_4)$ is

$$J = \begin{bmatrix} \frac{k}{k+1} & \frac{a}{k+1} \\ \lambda b & 1-\lambda \end{bmatrix} \tag{6.18}$$

which has characteristic polynomial given by

$$p(\theta) = \theta^2 + \frac{\lambda(k+1) - 2k - 1}{k+1}\,\theta - \frac{\lambda(x+k) - k}{k+1} \tag{6.19}$$

where $x = a.b$, we can see that $p(\theta)$ has roots with modulus smaller than one if and only if the following three conditions are fulfilled [6]:

$(s1)$ $p(0) < 1$,

$(s2)$ $p(1) > 0$,

$(s3)$ $p(-1) > 0$.

The condition $(s1)$ reduces to $\lambda(k - x) < 1$. The condition $(s2)$ is $\lambda(1 - x) > 0$, which is fulfilled when $x < 1$. Finally $p(-1) = 2(2 - \lambda)k - (x - 1)\lambda + 2$. Then the controlled parameter k given by equation (6.17). □

Now we have performed some numerical simulations to see how the state delayed feedback method controls the unstable Cournot-Kopel equilibrium E_4. Parameters values are fixed as $\lambda = 0.7$ and $\mu = 3.52$, the initial condition $x(0) = 0.01$, $y(0) = 0.02$ and the feedback gain $-1.4020 < k < 0.07$. The chaotic trajectory is stabilized on the Cournot-Kopel equilibrium and the control parameter $k = -0.0021$ eventually tends to zero is shown in Figures (18 and 19).

7. Conclusion

This paper presented Kopel duopoly game, which contains two types of heterogeneous players: naive and adaptive. We have investigated the local stability of equilibria, bifurcation and chaotic behaviors of the duopoly game. The equilibrium points and corresponding local stable regions of the model are obtained. The basic properties of the system have been analyzed by meaning of bifurcation diagrams, phase portraits and sensitive dependence. By investigating the local stability and bifurcations of the equilibrium points we found that the dynamic may become chaotic so, the stabilization of the chaotic behavior can be obtained by applying a feedback control method. The results show that the adjustment speed of adaptive player has an obvious impact on the stability of the competition model.

Acknowledgement

The authors thank the reviewers and the editor for their careful reading of the original manuscript and many valuable comments and suggestions that greatly improved the presentation of this paper. Also the authors wish to thank professors A. M. A. El-Sayed and Ali Azzam for the help.

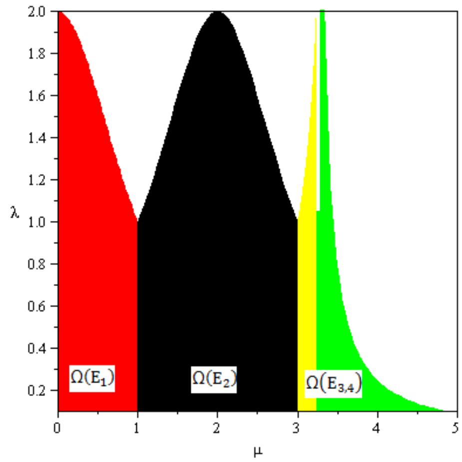

Fig. 1: Stable regions of equilibrium points

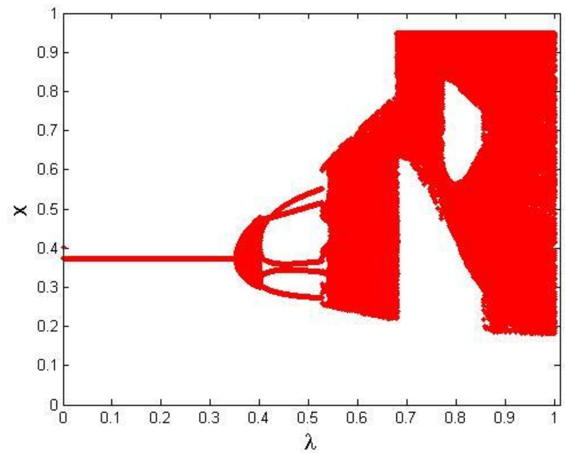

Fig. 2: Bifurcation diagram of x output vs λ

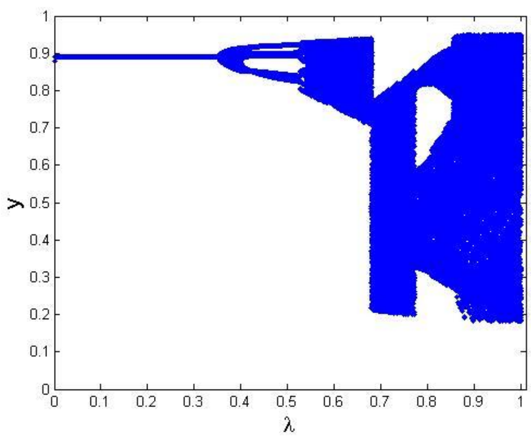

Fig. 3: Bifurcation diagram of y output vs λ

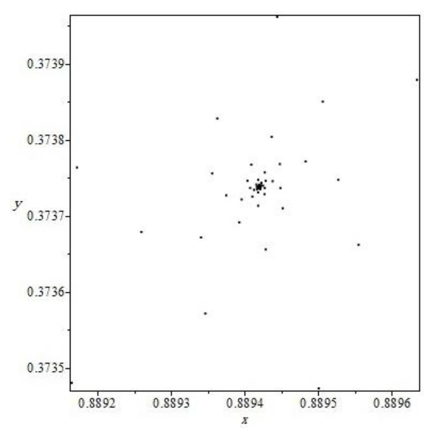

Fig. 4: A stable fixed point for $\lambda = 0.28$

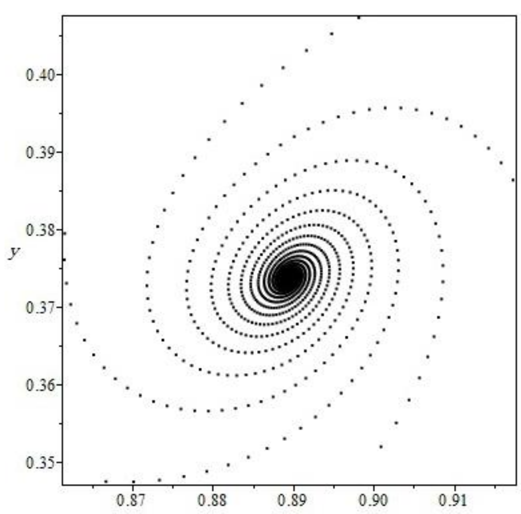

Fig. 5: Phase portrait before Neimark-Sacker bifurcation when $\lambda = 0.35$

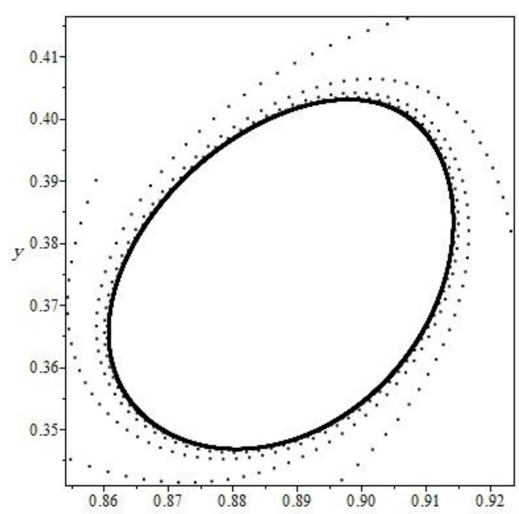

Fig. 6: Phase portrait after Neimark-Sacker bifurcation when $\lambda = 0.356$

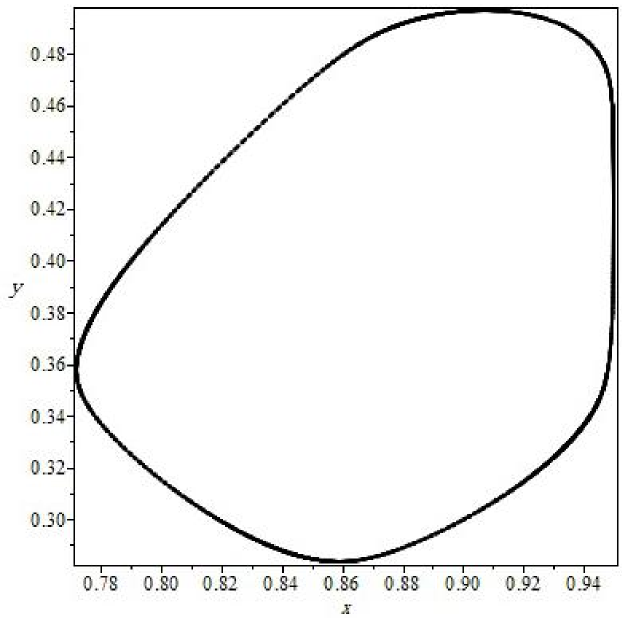

Fig. 7: The invariant closed curve which exists for $\lambda = 0.42$

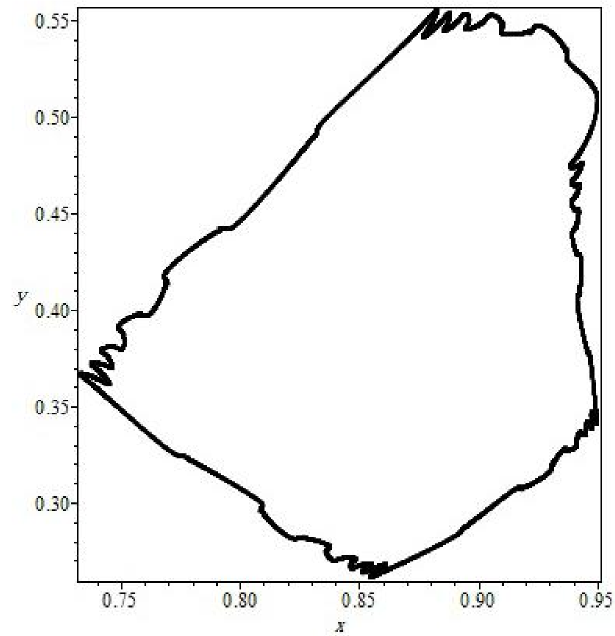

Fig. 8: The breakdown of the invariant closedcurve which exists for $\lambda = 0.5$

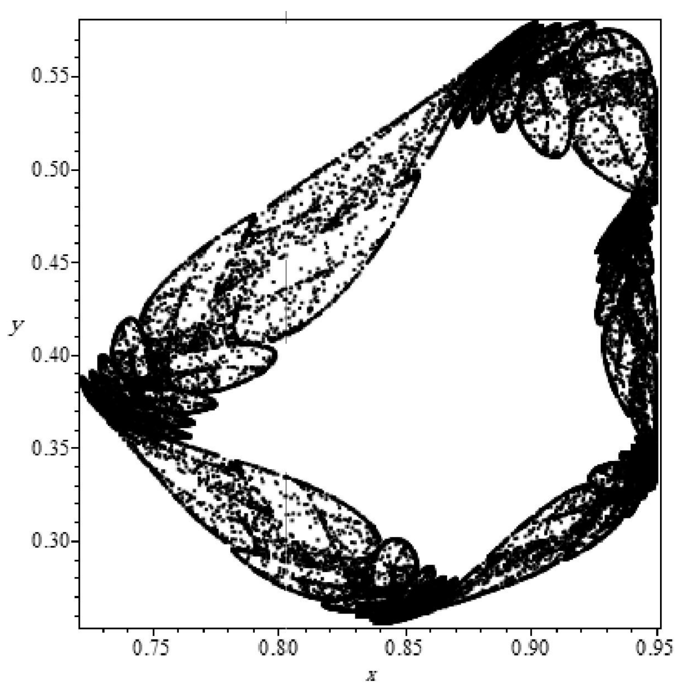

Fig. 9: Chaotic attractor for the model which exists for $\lambda = 0.51$

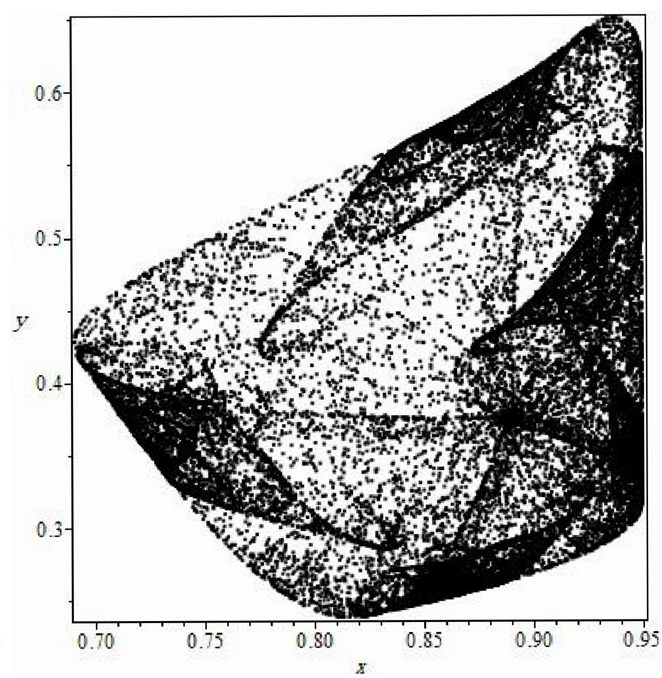

Fig. 10: Strong chaotic attractor for the model which exists for $\lambda = 0.570$

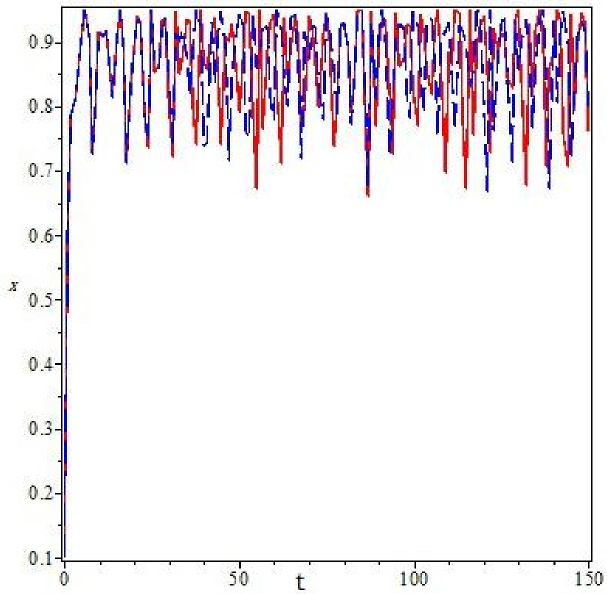

Fig. 11: Sensitive dependence on initial conditions, the two orbits of x-coordinates

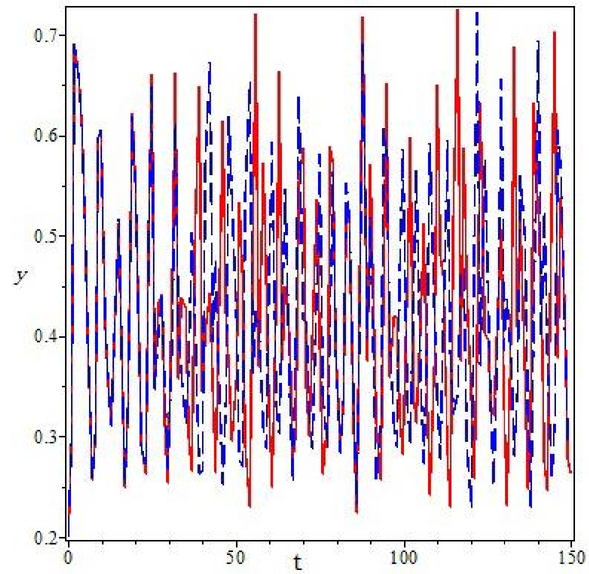

Fig. 12: Sensitive dependence on initial conditions, the two orbits of y-coordinates

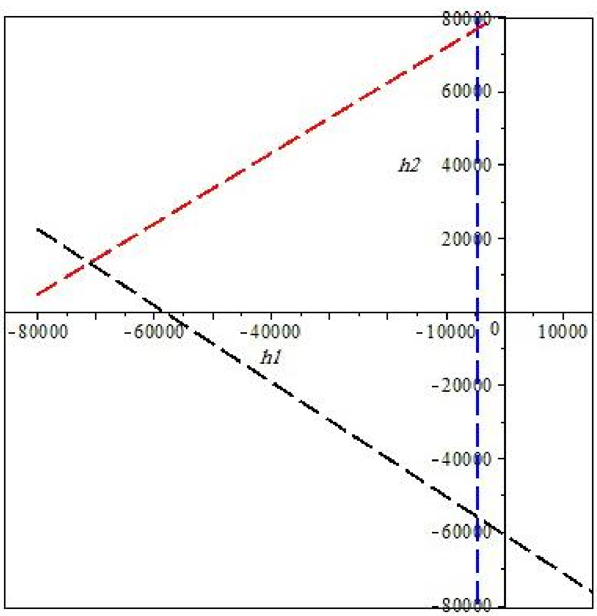

Fig. 13: Boundary for control parameters

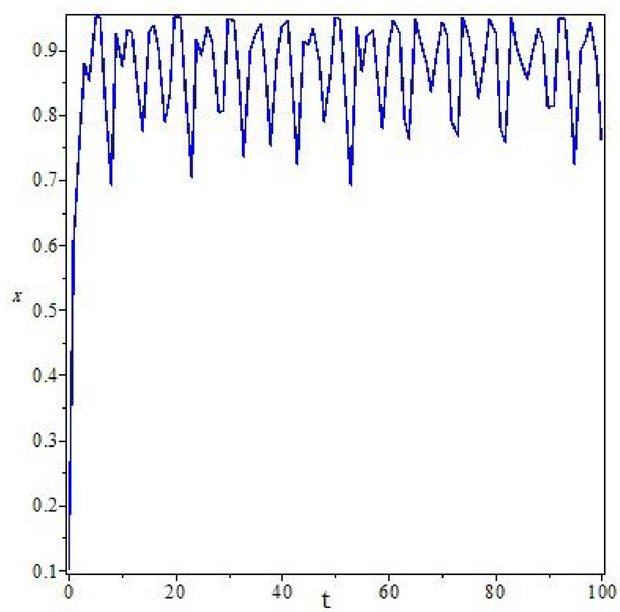

Fig. 14: Original chaotic orbit of the state variable $x(t)$

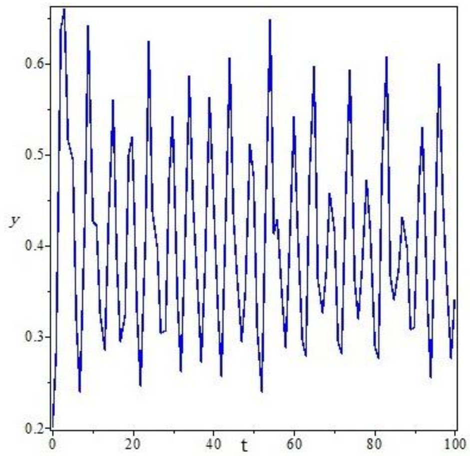

Fig. 15: Original chaotic orbit of the state variable $y(t)$

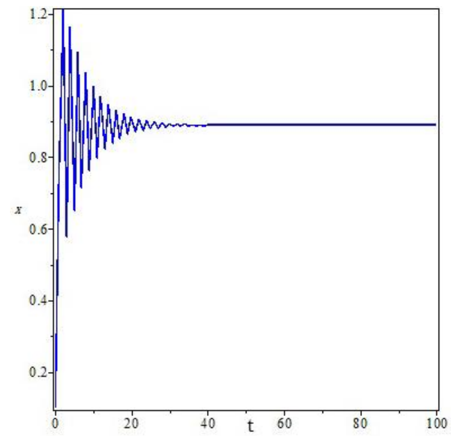

Fig. 16: Controlled chaotic orbit of the state variable $x(t)$ for $h_1 = -40000$, $h_2 = 40000$

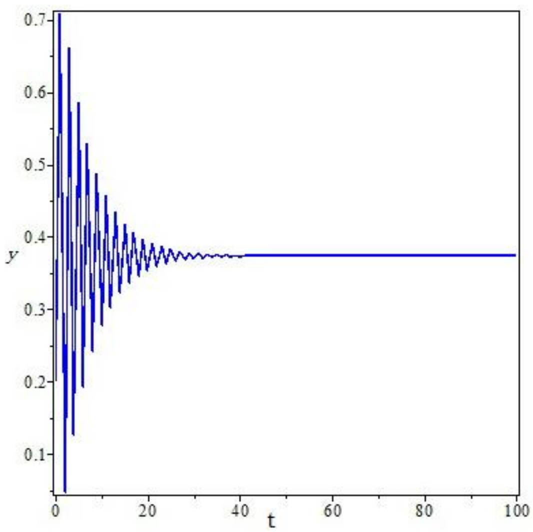

Fig. 17: Controlled chaotic orbit of the state variable $y(t)$ for $h_1 = -40000$, $h_2 = 40000$

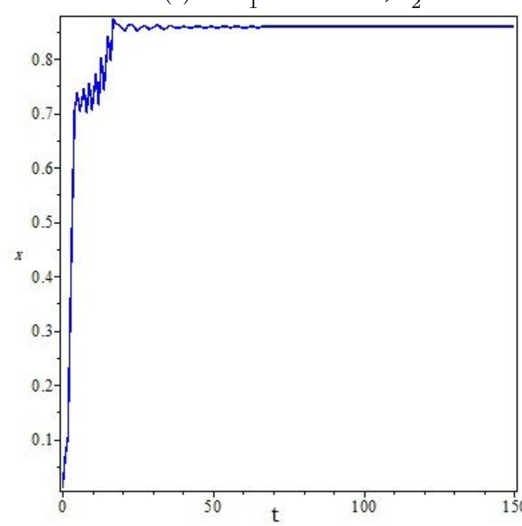

Fig. 18: Controlled chaotic orbit of the state variable $x(t)$ in delayed feedback control method when $k = -0.0021$.

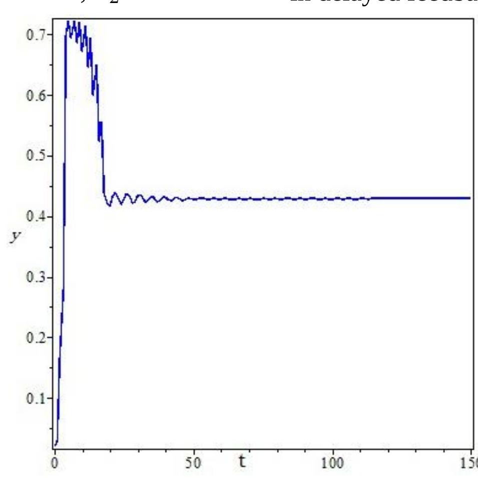

Fig. 19: Controlled chaotic orbit of the state variable $y(t)$ in delayed feedback control method when $k = -0.0021$

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