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A NUMERICAL STUDY ON A NONLINEAR CONSERVATION LAW MODEL PERTAINING TO MANUFACTURING SYSTEM

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This paper deals with a numerical scheme applied to a conservation law model of manufacturing system incorporating yield loss. Yield loss involving a singular term has been considered. Even though an explicit form of the material density in a production system can be obtained under certain assumptions, in general, it is difficult to get an explicit form of the material density. On the other hand, the singular term in a conservation law model imposes severe challenges for the numerical approximations on regular grids. Moreover, an approximate solution often converges to a wrong weak solution. A finite volume type numerical scheme has been studied. The convergence of the numerical solution towards entropy solution (in the Kruzkov sense) is proved. Numerical experiments are presented to get the overview of density distribution and outflux of the system.

Key words : Hyperbolic conservation laws; entropy criteria; yield loss; BV stability; manufacturing system.

1. Introduction

It is observed that several problems of scientific interest are nonhomogeneous in nature [14]. Dynamics of these problems are often represented by the hyperbolic conservation laws with non-vanishing stiff source terms. In general, the appearance of the nonhomogeneous term is either due to the physical effects like exterior forces, release of mass or energy, chemical reacting gas etc. or due to the geometrical effects like axisymmetric or cylindrical problems.

In manufacturing system, raw materials are transformed into finished goods on a large scale. Due to large scale and excessive computational cost for the discrete models, continuous models using partial differential equations (PDE) become indispensable in practice.

One important aspect in manufacturing system is yield loss during the process. Increase in inventories, delay in processing, demand variability and communication gap between customers and suppliers are considered as primary reasons behind the yield loss. If the system is observed more closely, yield loss does not occur throughout the system. At some point in the system there will be certain yield loss such as defective goods which do not make sense in further productions, need to be removed. So the occurrence of Dirac delta function in the yield loss is completely realistic. Frequently, it is seen that several manufacturing items are needed to withdraw from the system with due course of time. Hence the yield loss becomes time dependent.

The model of manufacturing system has been introduced by Armbruster *et al.* in [1]. Thereafter, it has been investigated by several authors in recent years [5, 8, 19, 21]. Incorporating physical effects like yield loss, the conservation law model becomes nonhomogeneous. Monthe [12] and Schroll *et al.* [20] studied the conservation law considering the nonhomogeneous part as function of only unknown variable with appropriate assumptions. Chalabi further considered the problem and analyzed convergence of the solution towards the entropy condition [2]. Several authors have considered singularities in the nonhomogeneous terms. LeVeque and Li incorporated the singularities for elliptic equation [11]. Holden and Risebro [7] studied Riemann problems with a kink. But, for the investigation of singular terms in conservation laws, the literatures are very limited. Diehl has studied sedimentation problems in [4]. He further showed the existence and uniqueness of solutions. A small relaxation term has been considered by Pareschi [13]. Santos *et al.* [16] studied the conservation law with nonhomogeneous part involving space derivatives. Similar problem has been considered by Greenberg *et al.* [6]. But it requires several smoothness properties of the flux function. The flux function in conservation law model of manufacturing system does not possess all those properties. In the direction of manufacturing system, Cutolo *et al.* [3] have focused on the numerics of ODE-PDE model of supply chain.

This paper addresses yield loss as a time dependent quantity along with the presence of a delta function. It is well known that the presence of singular terms in differential equations pose stringent challenges on numerical approximations carried out by standard numerical techniques. Hence a finite volume type numerical scheme obtained from integral representations of the PDE's has been studied.

Stability and convergence to a correct weak solution through entropy criteria have been established. In the numerical implementation, different cases are considered in the production system model. Overview of density distribution and outflux of the system are presented from the prescribed initial conditions.

The paper is organized as follows. In section 2, a conservation law based model of manufacturing system is introduced. The explicit forms of material density are presented under appropriate assumptions. The weak formulation, discretization and motivation behind the numerical study of the model are described in this section. In section 3, a numerical scheme is studied. The stability analysis, entropy condition and convergence of the approximated solution obtained by the numerical scheme are discussed. Section 4 is devoted to the numerical implementation and subsequent discussions. Four cases considering different scenarios in the production system model are investigated numerically.

2. Preliminaries

The nonlinear conservation law model of manufacturing system to be investigated, is described as follows:

$$\partial_t \rho(x,t) + \partial_x f(x,t,\rho) + y_l(x,t) = 0, \quad x \in (0,1],\ t > 0, \tag{2.1}$$

where the flux function $f(x,t,\rho)$ is given by

$$f(x,t,\rho) = \min\{\mu(x,t), v(x,t)\rho(x,t)\}.$$

Here completion of the product within the supplier is represented by the continuous variable x. In (2.1), $\rho(x,t)$ represents the density of goods at stage x and time t. Manufacturing system or serial supply chain is parameterized by the interval $[0,1]$. The yield loss phenomena is expressed by $y_l(x,t)$. For convenience, we shall use $y_l(x,t) = -g(x,t)$. Further, we assume that yield loss $L(t)$ occurs at some point $c \in (0,1)$. Hence $g(x,t) = L(t)\delta(x-c)$. Influx and initial situation in the system are prescribed below. Influx provides the information about the arrival rate of the raw materials in the system and initial condition gives an adequate information of the suppliers at the beginning of the process.

$$\text{Initial condition:} \qquad \rho(x,0) = \rho_0(x), \quad x \in [0,1]. \tag{2.2}$$

$$\text{Influx condition:} \qquad f(0,t,\rho(0,t)) = f^{in}(t), \quad t > 0. \tag{2.3}$$

Maximal capacity $\mu(x,t)$ in the model (2.1) can be taken as a constant or piecewise constant function. The form of the velocity function is given in [18] as $v(x,t) = v(W(t))$, where $W(t)$

represents work in progress of the suppliers at time t, i.e., $W(t) = \int_0^1 \rho(\xi, t)d\xi$. Velocity of a product, being a function of $W(t)$, carries the information of the products ahead and behind this product.

It is well known that solutions of a nonlinear conservation law may form discontinuities even for smooth flux function and smooth input data [10]. By treating the problem in weak sense, it is possible to deal with the discontinuous solutions. A function $\rho(x, t)$ is called a weak solution of (2.1) if the following integral formulation holds for every C^1 function $\phi(x, t)$ having compact support in $[0, 1] \times [0, T]$:

$$\begin{aligned}\int_0^1 \int_0^T [\rho(x,t)\phi_t(x,t)+f(x,t,\rho)\phi_x(x,t)]dxdt + \int_0^1 \rho(x,0)\phi(x,0)dx \\ + \int_0^T \phi(0,t)f^{in}(t)dt + \int_0^T L(t)\phi(c,t)dt = 0.\end{aligned} \tag{2.4}$$

The product density $\rho(., t)$ lies in $BV(\mathbb{R})$ for $\rho_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the subspace of L^1_{loc} consisting of functions with bounded total variation, i.e.,

$$BV(\mathbb{R}) = \{v \in L^1_{loc} : \text{T.V.}(v) < \infty\},$$

where

$$\text{T.V.}(v) = \sup_{h \neq 0} \int_{\mathbb{R}} \left| \frac{v(x+h) - v(x)}{h} \right| dx.$$

Further, it is observed that in the case of a nonhomogeneous conservation law, along the characteristics the solution $\rho(x, t)$ is not necessarily a constant. Hence by making appropriate assumptions, we shall try to construct a solution of (2.1) via characteristics.

2.1 *Explicit Form-I*

We seek an explicit form of the solution assuming certain features in the production system model (2.1)-(2.3). We have considered the following assumptions:

(i) Velocity of the material is constant inside the network;

(ii) Flux in the supplier does not exceed its maximal capacity;

(iii) Yield loss term is considered as independent of time;

(iv) Influx in the system and initial conditions of the suppliers are differentiable functions.

We endeavor into characteristics analysis to construct the solution. The main difficulty comes from the presence of a singular term. In the distributional sense we know that $\frac{d}{dx}H_c(x) = \delta(x - c)$, where $H_c(x)$ is the Heaviside function defined by

$$H_c(x) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x > c. \end{cases}$$

We define the characteristic curve $\xi = \xi(s; x, t)$, which passes through the point (x, t) by the solution of the differential equation $\dfrac{d\xi}{ds} = v$, $\xi(t) = x$. It is not difficult to establish that the solution for $x < vt$ will be affected by the boundary condition only. Whenever $x < vt$, the explicit form of density $\rho(x, t)$ is given by

$$\rho(x,t) = \frac{1}{v}\big(H_c(t) - H_c(t - \frac{x}{v})\big) + \frac{1}{v} f^{in}\big(t - \frac{x}{v}\big). \tag{2.5}$$

The solution will be influenced by the initial conditions whenever $x > vt$, in which case the explicit form of $\rho(x, t)$ is presented by

$$\rho(x,t) = \frac{1}{v}(H_c(x) - H_c(x - vt)) + \rho_0(x - vt). \tag{2.6}$$

It can be shown that the solution constructed above satisfies the equation (2.1) in weak sense (2.4).

2.2 *Explicit Form-II*

Since the inventories and demands of the customers vary with the time, we shall consider that the yield loss term is function of t also, i.e., we need to replace the assumption (iii) in section 2.1. Moreover, demand becomes discontinuous due to market failures, quality of materials, excessive competition and so on. In this context, kinked demand curve is well studied in economical problems. Evidently, the nature of the demand has an impact on the yield loss. Hence the yield loss function considered here is justifiable. With this yield loss and the above described assumptions, explicit form of the material density is presented by the following two cases.

Case 1 : For $x < vt$,

$$\rho(x,t) = \begin{cases} \frac{1}{v} f^{in}(t - \frac{x}{v}) & \text{if } t < t_0 \\ \frac{1}{v} L(\frac{t_0 - (t - \frac{x}{v})}{v}) + \frac{1}{v} f^{in}(t - \frac{x}{v}) & \text{if } t_0 \le t < t_0 + \frac{x}{v} \\ \frac{1}{v} f^{in}(t - \frac{x}{v}) & \text{if } t \ge t_0 + \frac{x}{v}, \end{cases}$$

where t_0 is given by the relation $c = vt_0$.

Case 2 : For $x > vt$,

$$\rho(x,t) = \begin{cases} \rho_0(x - vt) & \text{if } x < c \\ \frac{1}{v} L(\frac{c - (x - vt)}{v}) + u_0(x - vt) & \text{if } c \le x < c + vt \\ \rho_0(x - vt) & \text{if } x \ge c + vt. \end{cases}$$

It can be shown that the explicit form of solution $\rho(x,t)$ constructed above is indeed a weak solution in the sense of (2.4).

It is evident from the above discussion that for a non-constant velocity function to construct a solution analytically can become a formidable task. Hence our objective is to study the model (2.1) numerically. We shall start with the discretization of the domain.

2.3 *Discretization*

The spatial domain is divided into cells $I_j = [x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2})$ with centers at the point $x_j = j\Delta x,\ j \in \mathbb{Z}$. Similarly, the time domain is discretized by $t^n = n\Delta t,\ n \in \mathbb{N}$. Time strip is denoted by $J^n = [t^n, t^{n+1})$. Let χ_j^n be the characteristic function for the rectangle $R_j^n = I_j \times J^n$.

A weak solution of the problem, described in section 2.1, satisfies the integral formulation

$$\begin{aligned}\int_{I_j} \rho(x,t^{n+1})dx = \int_{I_j} \rho(x,t^n)dx + \int_{J^n} [v\rho(x_{j-\frac{1}{2}},t) - H_c(x_{j-\frac{1}{2}})]dt \\ - \int_{J^n} [v\rho(x_{j+\frac{1}{2}},t) - H_c(x_{j+\frac{1}{2}})]dt,\end{aligned} \tag{2.7}$$

where $x_{j\pm\frac{1}{2}} = x_j \pm \frac{\Delta x}{2}$.

For the more general case, a weak solution of the PDE model (2.1) satisfies the following integral formulation

$$\begin{aligned}\int_{I_j} \rho(x,t^{n+1})dx = \int_{I_j} \rho(x,t^n)dx - \Big[\int_{J^n} f(\rho(x_{j+\frac{1}{2}},t))dt - \int_{J^n} f(\rho(x_{j-\frac{1}{2}},t))dt\Big] \\ + \Big[\int_{J^n} H_c(x_{j+\frac{1}{2}})L(t)dt - \int_{J^n} H_c(x_{j-\frac{1}{2}})L(t)dt\Big].\end{aligned} \tag{2.8}$$

A weak solution $\rho(x,t)$ is approximated by a function $\rho^\Delta(x,t)$ defined as

$$\rho^\Delta(x,t) = \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \chi_j^n(x,t)\rho_j^n, \tag{2.9}$$

where ρ_j^n are generated by specified time marching numerical scheme. The initial data is projected onto the space of piecewise constant functions by

$$\rho^\Delta(x,0) = \sum_j \chi_j(x)\rho_j^0, \quad \rho_j^0 = \frac{1}{\Delta x}\int_{I_j} \rho_0(x)dx,$$

where $\chi_j(x)$ is the characteristic function of the space grid I_j for $j \in \mathbb{Z}$. Several numerical approaches have been studied and analyzed for conservation law [15, 10]. Generalization of these numerical approaches for the nonhomogeneous case encounters specific problems. We shall demonstrate this in the following subsection.

2.4 *Motivation*

It is well known that standard numerical methods [16] derived by a direct discretization of the differential form fail to converge even for the simpler problem described in section 2.2. Let us consider the following explicit time marching scheme

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + v\frac{\rho_j^n - \rho_{j-1}^n}{\Delta x} = L(t^n)\delta(x_j - c). \tag{2.10}$$

We also consider the initial situations of the suppliers as there are no materials present inside the system. Letting $\Delta x = v\Delta t$, we obtain the following expression from (2.10):

$$\rho_j^{n+1} = \rho_{j-1}^n + \Delta t L(t^n)\delta(x_j - c).$$

Through the successive iterations, we can show that

$$\rho_j^n = \Delta t \sum_{m=0}^{n-1} L(t^m)\delta(x_{j+m-(n-1)} - c). \tag{2.11}$$

We observe from (2.11) that $\rho_j^n \to 0$ as $\Delta t, \Delta x \to 0$. In a similar way, if we choose the influx as a constant, it is observed that the solution will also approach a constant as we refine the grids. Therefore the numerical solution does not converge to the correct weak solution. To obtain the physically relevant weak solution, it is necessary to study a different numerical scheme which ensures the convergence of the approximate solution towards the entropy solution.

3. Study on Numerical Scheme

In this section a conservative scheme is considered for the model of manufacturing system incorporating yield loss involving a delta function. We shall study the stability of the numerical scheme. In order to examine the physically relevant solution we study entropy conditions in Kruzkov sense. Further, we show that the approximate solution obtained by numerical scheme will converge to a entropy admissible solution. To do this we need to study the entropy condition in discrete sense.

In order to derive an appropriate numerical scheme which provides admissible weak solution, we would like to get back to the integral formulation (2.8). We consider the following conservative scheme:

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right] + \frac{\Delta t}{\Delta x}\left[G_{j+\frac{1}{2}}^n - G_{j-\frac{1}{2}}^n\right], \tag{3.1}$$

where approximation ρ_j^n is given by $\rho_j^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(x,t^n)dx$. Numerical homogeneous fluxes are considered as $F_{j+\frac{1}{2}}^n = F(\rho_j^n, \rho_{j+1}^n) \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(\rho(x_{j+\frac{1}{2}},t))dt$ and $G_{j+\frac{1}{2}}^n = G(p_j^n, p_{j+1}^n) \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} p(x_{j+\frac{1}{2}},t)dt$ respectively. Normalization has been done with respect to the grid sizes. In order to complete the numerical scheme we need to define

$$F_{\frac{1}{2}}^n := F(\bar{\rho}_0^n, u_1^n), \quad \bar{\rho}_0^n := \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \rho_b(t)dt.$$

The term $\rho_b(t)$ can be obtained from the prescribed influx condition. Comparing with (2.8), $p(x,t)$ can be expressed as $L(t)H_c(x)$. It is easily observed that the scheme, presented in (3.1), satisfies the following properties:

(i) Consistent: $F(s,s) = f(s)\ \forall s \in \mathbb{R}$ and $G(p,p) = p\ \forall p \in \mathbb{R}$.

(ii) F and G are both locally Lipschitz continuous function from $\mathbb{R}^2$ to $\mathbb{R}$.

In order to study the convergence of the scheme (3.1), we need to analyze some form of stability.

3.1 *Stability analysis*

We like to point out the following issues about stability.

(i) For linear problems the Lax Equivalence theorem ensure convergence. Since the approach heavily depends on linearity, the theorem no longer holds for nonlinear problems. So we would like to seek an alternative approach.

(ii) It is observed [10] that total variation stability or TV-stability of a consistent and conservative method is enough to guarantee convergence.

The numerical method (3.1) is said to be TV-stable if the approximation ρ^Δ for $\Delta < \Delta_0$ lies in some fixed set of the form

$$K = \big\{\rho \in L_{1,T} : TV_T(\rho) \le C \text{ and } supp(u(.,t)) \subset [-M,M],\ \forall\, t \in [0,T]\big\},$$

where C and M are constants. The space $L_{1,T}$ is given by

$$L_{1,T} = \{u(x,t) : \|u\|_{1,T} < \infty\} \text{ where } \|u\|_{1,T} = \int_0^T \int_0^1 |u(x,t)|dxdt.$$

We state the following theorems which ensure the bounded variation stability estimate in space for approximated solution along with TV-stability of the numerical method.

***Theorem* 3.1 — (Discrete space BV estimate).** *Assume that $\rho_0 \in BV(\mathbb{R})$ and $T.V.(p^n)$ is bounded by some constant. Let $\{\rho_j^n : j \in \mathbb{Z}, n \in \mathbb{N}\}$ be obtained by numerical scheme (3.1). Then under the CFL condition*

$$\frac{\Delta t}{\Delta x}\max_{q,r}\left[|F(\rho,q)-F(\rho_1,q)|+|F(r,\rho)-F(r,\rho_1)|\right] \leq |\rho-\rho_1|, \tag{3.2}$$

$T.V.(\rho^n) \leq C$ for some constant C.

Theorem 3.1 ensures bounded variation of the numerical solution obtained by (3.1). Proof of Theorem 3.1 is not difficult. Taking into account the CFL condition (3.2) and the numerical scheme (3.1), bounded variation of ρ^n can be established. We omit the details of the proof (a proof can be carried out in a similar way as described in [17]). With the help of Theorem 3.1 we have the following theorem on TV-stability.

***Theorem* 3.2** — *Consider a conservative numerical method (3.1) with a homogeneous numerical flux F and numerical loss flux G which are Lipschitz continuous. Suppose for each initial data ρ_0 and nonhomogeneous part $p(x,t)$, there exist some positive constants k_0, C_1, C_2 such that $T.V.(\rho^n) \leq C_1$ and $T.V.(p^n) \leq C_2 \; \forall \; n, k$ with $k < k_0, \; nk < T$. Then the method is TV-stable.*

Thanks to bounded variation of ρ^n and p^n, it can be shown that $\|\rho^{n+1} - \rho^n\| \leq C\Delta t$, for some constant C. Then the Theorem 3.2 follows in straight forward way. Theorem 3.2 ensures that the numerical scheme (3.1) is TV-stable. Having established TV-stability, we look for the convergence. We state the following convergence theorem which requires total variation stability along with the consistency.

***Theorem* 3.3** — *Let ρ^Δ be generated by conservative numerical scheme (3.1) with Lipschitz continuous homogeneous numerical flux F and numerical loss flux G. If the method (3.1) is TV-stable then the method is convergent, i.e., $\inf_{\widetilde{\rho}\in\widetilde{W}} \|\rho^\Delta - \widetilde{\rho}\|_{1,T} \to 0$ as $\Delta t \to 0$, where $\widetilde{W} = \{\widetilde{\rho} : \widetilde{\rho}(x,t)$ is a weak solution$\}$.*

Once again we omit the details of the proof of Theorem 3.3. Basically, the proof of Theorem 3.3 can be carried out in a similar way as in [9]. The proof becomes simpler if we approach through the contradiction. Having analyzed the convergence of the approximated solution obtained by (3.1), we want to enquire about the physical relevance of the weak solution through entropy condition.

3.2 *Entropy condition*

In general, the weak solution defined in section 2 is not unique. Hence the physically relevant solution is characterized by the entropy condition. If a strictly convex function $\eta(\rho)$ and corresponding entropy flux $\psi(\rho)$ are chosen such that $\psi^{'}(\rho) = \eta^{'}(\rho)f^{'}(\rho)+\eta^{''}(\rho)g$, then the entropy solution $\rho(x,t)$ satisfies the following inequality in the distributional sense:

$$\frac{\partial}{\partial t}\eta(\rho(x,t)) + \frac{\partial}{\partial t}\psi(\rho(x,t)) \leq \frac{\partial}{\partial x}(\eta^{'}(\rho(x,t))L(t)H_c(x)). \tag{3.3}$$

Consider the case $\eta(\rho) = ((\rho-k)^2+\delta^2)^{\frac{1}{2}},\ \delta > 0$, for some constant k. By letting $\delta \to 0$ we can extend the analysis to the case where $\eta(\rho) = |\rho - k|$. Subsequently, we can find the entropy flux $\psi(\rho)$.

Taking into account entropy pair (η, ψ), the entropy condition in Kruzkov sense can be represented by

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \Big(|\rho(x,t)-k|\phi_t + \text{sign}(\rho(x,t)-k)(f(\rho)-f(k))\phi_x\Big)dxdt \\ + \int_{\mathbb{R}} \big(|\rho_0(x)-k|\phi(x,0) - |\rho(x,T)-k|\phi(x,T)\big)dx \\ \geq \int_0^T \int_{\mathbb{R}} \text{sign}(\rho(x,t)-k)L(t)H_c(x)\phi_x(x,t)dxdt, \forall\, k \in \mathbb{R}.\end{aligned} \tag{3.4}$$

It is important to investigate that the approximate solution, obtained from numerical scheme (3.1), satisfies the entropy condition in discrete form. The following proposition provides entropy inequality which must be satisfied by an approximate solution.

Proposition 3.4 (**Discrete Entropy Inequality**) — Let ρ^{Δ} be the approximate solution obtained from the numerical scheme (3.1). Under the CFL condition (3.2), the following inequality holds:

$$\begin{aligned}(|\rho_j^{n+1}-k| - |\rho_j^n - k|) + \frac{\Delta t}{\Delta x}[F(\rho_j^n \vee k, \rho_{j+1}^n \vee k) - F(\rho_j^n \wedge k, \rho_{j+1}^n \wedge k) \\ -F(\rho_{j-1}^n \vee k, \rho_j^n \vee k) + F(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k)] \leq \frac{\Delta t}{\Delta x}[\xi(\rho_j^n, \rho_{j+1}^n)G(p_j^n, p_{j+1}^n) \\ -\xi(\rho_{j-1}^n, \rho_j^n)G(p_{j-1}^n, p_j^n)], \quad \forall\, j \in \mathbb{Z},\ \forall\, n \in \mathbb{N} \text{ and } \forall\, k \in \mathbb{R},\end{aligned} \tag{3.5}$$

where ξ is Lipschitz continuous function with respect to each variable along with $\xi(\rho,\rho) = \eta^{'}(\rho)$ and G is Lipschitz continuous with respect to each variable along with $G(p,p) = p(x,t)$.

Taking into account CFL condition (3.2) and numerical scheme (3.1), inequality (3.5) can be easily obtained. With the help of discrete entropy inequality (3.5), we show that the approximate solution converges to the entropy solution.

3.3 *Convergence towards entropy solution*

***Theorem* 3.5** — *Assume that $\rho_0 \in BV(\mathbb{R})$ and $T.V.(p^n)$ is finite. Let $\{\rho_j^n : j \in \mathbb{Z}, n \in \mathbb{N}\}$ be obtained by numerical scheme (3.1). If the numerical scheme is consistent with the entropy condition (3.5), then the numerical solution will converge to a weak solution which satisfies the entropy condition (3.4).*

PROOF : Let ϕ be a nonnegative differentiable function on $\mathbb{R}\times[0,T]$. Multiply (3.5) by $\Delta x \phi(x_j, t^n)$ and taking summation over all $j \in \mathbb{Z}$ and $n \geq 0$, we obtain $B_1 + B_2 \leq B_3$.

Using discrete integration by parts and setting $\eta(\rho) = |\rho - k|$ we obtain the following:

$$B_1 = \Delta x \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \left(\eta(\rho_j^{n+1}) - \eta(\rho_j^n)\right)\phi_j^n = \Delta x \Delta t \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \frac{\eta(\rho_j^{n+1}) - \eta(\rho_j^n)}{\Delta t}\phi_j^n$$
$$= \Delta x \left(- \sum_{j\in\mathbb{Z}} \eta(\rho_j^0)\phi_j^0 - \Delta t \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} \eta(\rho_j^n) \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} + \sum_{j\in\mathbb{Z}} \eta(\rho_j^{N+1})\phi_j^N \right).$$

Setting $\bar{F}(v,w) = F(\rho \vee k, v \vee w) - F(\rho \wedge k, w \wedge k)$ we express the terms B_2 and B_3 as following:

$$B_2 = \Delta t \Delta x \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \frac{1}{\Delta x}\left(\bar{F}(\rho_j^n, \rho_{j+1}^n) - \bar{F}(\rho_{j-1}^n, \rho_j^n)\right)\phi_j^n$$
$$= -\Delta t \Delta x \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \bar{F}(\rho_j^n, \rho_{j+1}^n) \frac{\phi_{j+1}^n - \phi_j^n}{\Delta x},$$
$$B_3 = \Delta t \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \Big[(\xi(\rho_j^n, \rho_{j+1}^n) G(p_j^n, p_{j+1}^n) - \xi(\rho_{j-1}^n, \rho_j^n) G(p_{j-1}^n, p_j^n)) \Big] \phi_j^n$$
$$= -\Delta t \Delta x \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \xi(\rho_j^n, \rho_{j+1}^n) \frac{\phi_{j+1}^n - \phi_j^n}{\Delta x} G(p_j^n, p_{j+1}^n).$$

Now let $\Delta t \to 0$. It is straightforward, using the 1-norm convergence of $\rho^{\Delta t}$ to ρ and smoothness of ϕ, to show that as $\Delta t \to 0$

$$B_1 \to -\int_{\mathbb{R}} \eta(\rho_0)\phi(x,0)dx - \int_0^T \int_{\mathbb{R}} \eta(\rho)\phi_t(x,t)dxdt + \int_{\mathbb{R}} \eta(\rho(x,T))\phi(x,T)dx.$$

Employing consistency of F along with the Lipschitz properties of F we can show that

$$\bar{F}(\rho_j^n, \rho_{j+1}^n) \to sign(\rho - k)(f(\rho) - f(k)).$$

Taking into account the properties of ξ and G, letting $\Delta t \to 0$ we will end up with

$$B_2 \to -\int_0^T \int_{\mathbb{R}} \text{sign}(\rho - k)(f(\rho) - f(k))\phi_x(x,t)dxdt,$$
$$B_3 \to -\int_0^T \int_{\mathbb{R}} \text{sign}(\rho - k)L(t)H_c(x)\phi_x(x,t)dxdt.$$

Therefore we can conclude that $\rho(x,t)$ satisfies entropy condition in the sense (3.4) as $\Delta t \to 0.\Box$

4. Numerical implementation and discussion

In this section, we present various numerical investigations to demonstrate the performance of the numerical scheme (3.1). We concentrate on four different situations arising in the model of production system. We consider the same form of yield loss for all cases.

4.1 *Test Case-I*

In the first case, we would like to validate the numerical results with explicit solution obtained in section 2. We have considered the velocity of materials in production system as constant ($v = 1$). Also we assume that flux in production system does not exceed maximal capacity. Influx in the system is taken as a constant throughout the process. Suppose there are no materials inside the suppliers at start of the process. For discretization of the PDE (2.1), we use a generalization of Lax-Friedrichs method [10]. The method can be written in the conservative form by taking

$$F^n_{j+\frac{1}{2}} = \frac{h}{2k}(\rho^n_j - \rho^n_{j+1}) + \frac{1}{2}(f(\rho^n_j) + f(\rho^n_{j+1})), \quad G^n_{j+\frac{1}{2}} = \frac{1}{2}(p^n_j + p^n_{j+1}).$$

The yield loss term is considered as $sin(\pi t)\delta(x - 0.2)$. With the described set up, we know the explicit form of the solution (2.5)-(2.6). In Figure 1, explicit solution is represented by continuous line while the numerical solution is represented with circles and dashed line. The discretization lengths have been taken as $\Delta x = 0.05$ and $\Delta t = 0.05$. We have compared analytical and numerical results at time $T = 10$ in Figure 1.

4.2 *Test Case-II*

A production system receives raw materials in a discrete way. Using piecewise linear interpolation, influx in the system can be considered as in Figure 2. It is quite reasonable to choose the influx like this since one can hardly expect influx function to be smooth. Yield loss term and numerical scheme are same as in Test Case-I. Velocity is taken as constant ($v = 0.5$). A snapshot of the density distribution throughout the suppliers is presented in Figure 2 at time $T = 200$.

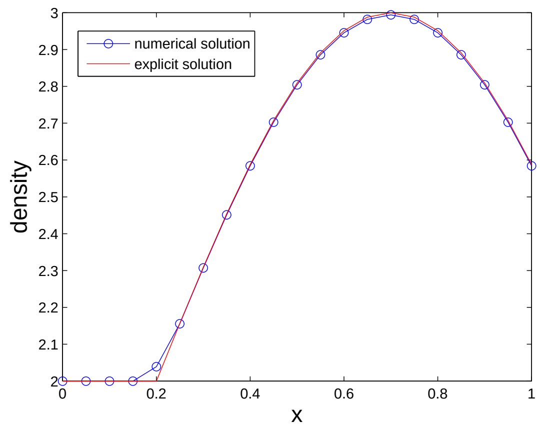

Figure 1: Density distribution of the production system at $T = 10$ considering constant influx

It is observed from Figure 2 that initially material density struggles to incorporate with the influx but later it becomes regular despite the non-smooth behavior of the influx.

4.3 *Test Case-III*

Having analyzed the effect of constant and non-constant influx on the density and outflux of the system, we want to concentrate on maximal capacity of the suppliers. In this case, we consider a chain of three suppliers having different maximal processing rates. We assume that the completion interval is equal to unity. The suppliers will represent the intervals [0, 0.2], [0.2, 0.8] and [0.8, 1] respectively. We consider the maximal processing rate as the following:

$$\mu(x,t) = \begin{cases} 15 & \text{for } 0 \leq x \leq 0.2 \\ 10 & \text{for } 0.2 < x \leq 0.8 \\ 15 & \text{for } 0.8 < x \leq 1. \end{cases}$$

The influx is prescribed in Figure 2. Flux function is apprehended as $f(x,t) = \min\{\mu(x,t), v\rho(x,t)\}$, where velocity is taken as 0.2.

Figure 3 presents the outflux of production system at time $T = 200$. In the beginning outflux is increasing with the time. After that there is a sudden decline in the outflux. This happens due to the sudden drop in the maximal capacity of the second supplier. So far we have considered the velocity as a constant. Hence we shall carry out the analysis for nonconstant velocity in the next case.

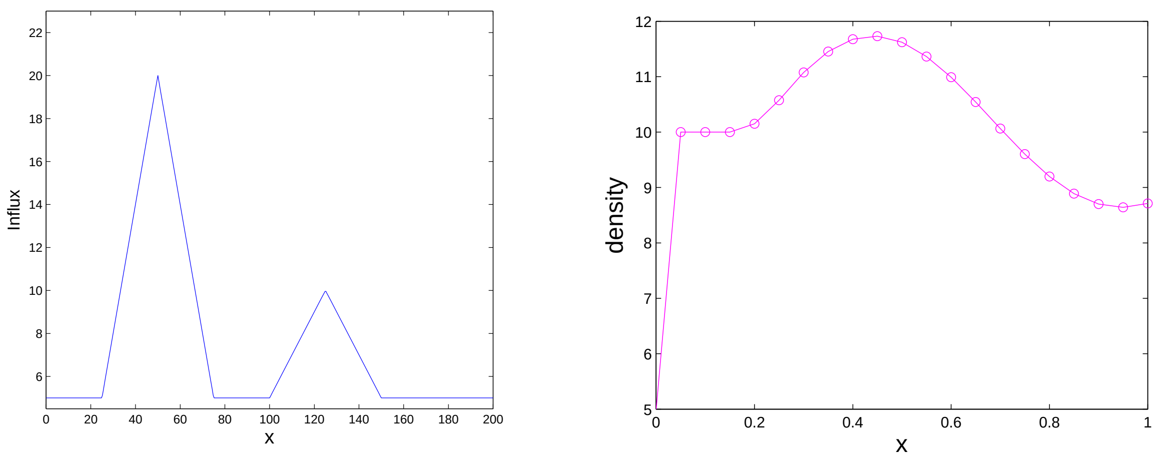

Figure 2: Influx and density distribution of production system

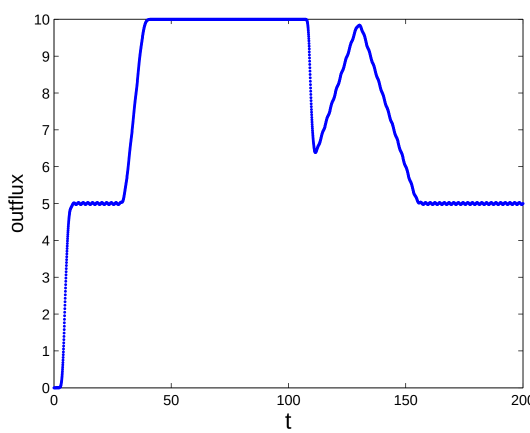

Figure 3: Outflux in the production system at $T = 200$

4.4 *Test Case-IV*

In this case, we have considered a chain of three suppliers as in Test Case-III. The maximal capacity also remain same for the suppliers. The velocity form is considered here as

$$v(x,t) = \frac{v_{max}}{P + P.W(t)}, \quad W(t) = \int_0^1 \rho(\xi, t) d\xi,$$

where parameters v_{max} and P are taken as 4 and 1.5 respectively.

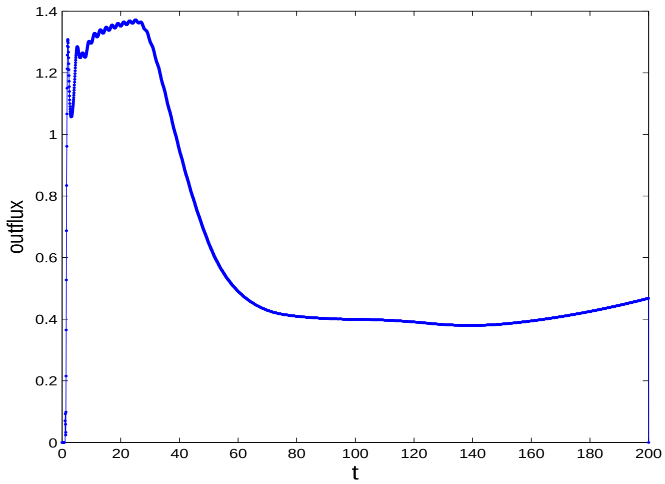

Figure 4: Outflux in production system at $T = 200$ considering velocity as function of work in progress

The yield loss is considered same as in Test Case-I. Again the discretization lengths have been taken as $\Delta x = 0.05$ and $\Delta t = 0.05$.

Figure 4 shows a snapshot of outflux in the production system. Initially outflux declines when the influx increases. At the beginning of the process, production line competes for machine availability with all the other requests. The outflux becomes reasonably stable as the time progresses.

Thus, we have analyzed the density and outflux of the manufacturing system for a constant influx and then for a non-differentiable influx function. Further, the numerical simulation has been carried out for different maximal capacity of the suppliers and for the velocity function depending on the work in progress of the suppliers.

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