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MULTIPLE SOLITON SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATION

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Using Morse theory, truncation arguments and an abstract critical point theorem, we obtain the existence of at least three or infinitely many nontrivial solutions for the following quasilinear Schrödinger equation in a bounded smooth domain

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u \Delta_p \left(u^2\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Our main results can be viewed as a partial extension of the results of Zhang *et al.* in [28] and Zhou and Wu in [29] concerning the the existence of solutions to (0.1) in the case of $p = 2$ and a recent result of Liu and Zhao in [21] two solutions are obtained for problem 0.1.

Key words : Quasilinear Schrödinger equation; soliton solution; Morse theory; symmetry mountain pass theorem; truncation arguments.

1. Introduction and Main Results

Consider the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u \Delta_p \left(u^2\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N (N \geq 3)$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the p-Laplacian with $1 < p < \infty$. Quasilinear equations of form (1.1), referred as so-called Modified Nonlinear Schrödinger Equation due to the quasilinear and non-convex term $u\Delta_p u^2$, and have been derived as model of several physical phenomena (see [2, 10, 14] for example).

Problems of type (1.1) were studied primarily in the context of $p = 2$. In this connection, we refer the readers to [5, 6, 7, 18, 19, 20, 25, 28, 29]. One may note that one of the main difficulties of the quasilinear problem (1.1) is that there is no suitable space on which the energy functional is well defined and belongs to C^1 class. To overcome this difficulty, several ideas and techniques were developed, including the constrained minimization argument [6, 18, 25], the Nehari manifold [20, 26], the method of a change of variables [5, 19] and the perturbation method [17].

Recently, there appeared some works dealing with (1.1) when $p \neq 2$. For example, Liu [15] and Liu and Zhao [21] consider problem (1.1) in a bounded smooth domain, to our best knowledge, this is the only results established for the p-Laplacian case in a bounded domain.

Motivated by above results, we devote this paper to establishing multiplicity results for problem (1.1) for the p-Laplacian case in a bounded smooth domain $\Omega \subset \mathbb{R}^N (N \geq 3)$. A major difficulty associated with (1.1) is the following: one may seek to obtain solutions by looking for critical points of the associated "natural" functional: $I(u) : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) := \frac{1}{p}\int_\Omega (1 + p|u|^p)\,|\nabla u|^p \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. However, this functional is not well-defined for all $u \in W_0^{1,p}(\Omega)$, hence it is difficult to apply variational methods directly. To overcome this difficulty, we use the method of changing variables developed in [5, 19] for the case $p = 2$, [15, 21] for p-Laplacian case (i.e. $1 < p < \infty$), and make a new different definition of weak solutions. That is

$$v := g^{-1}(u),$$

where g is defined by

$$\begin{aligned} g'(t) &= \frac{1}{(1 + p|g(t)|^p)^{1/p}},\ \forall\, t \in [0, +\infty), \\ g(t) &= -g(-t),\ \forall\, t \in (-\infty, 0]. \end{aligned} \tag{1.2}$$

We now make use of a change of unknown $v = g^{-1}(u)$, and define an associated equation

$$\begin{cases} -\Delta_p v = f(x, g(v))g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

As shown in [21] and [15] that problem (1.3) is equivalent to our problem (1.1), which takes $u = g(v)$ as its weak solutions. More precisely, we say u is a weak solution for (1.1), if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a critical point of the following functional corresponding to problem (1.3)

$$J(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega F(x, g(v))\, dx.$$

It is well known (see [22]) that the p-homogeneous boundary-value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a first eigenvalue $\lambda_1 > 0$, which is simple, and has an associated eigenfunction $\varphi_1 \in C^{1,\alpha}$ which is positive in Ω. It is also known that λ_1 is an isolated point of $\sigma(-\Delta_p)$, the spectrum of $-\Delta_p$, which contains at least an increasing eigenvalue sequence obtained by the Ljusternik-Schnirelman theory. Now, we list the following assumptions before stating our main results.

(F_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function, $f(x,t)\ t \geq 0$ and

$$|f(x,t)| \leq C\left(1 + |t|^{2(q-1)}\right) \ \text{ a.e. } x \in \Omega,\ t \in \mathbb{R},$$

where $C > 0$ is a constant, $1 < p \leq q < p^* := \frac{Np}{N-p}$.

(F_2) There exist $a > 0$ and $\tau \in (1,p)$ such that

$$\lim_{|t|\to 0} \frac{f(x,t) - a|t|^{\tau-2}t}{|t|^{p-2}t} = 0.$$

(F_3) $\limsup\limits_{|t|\to\infty} \frac{pK_0^{2p}F(x,t)}{|t|^{2p}} < \lambda_1$, where $K_0 = \sqrt{2}p^{-\frac{1}{2p}}$.

(F_4) $\lim\limits_{|t|\to\infty} \frac{pK_0^{2p}F(x,t)}{|t|^{2p}} = \lambda_1$, where $K_0 = \sqrt{2}p^{-\frac{1}{2p}}$.

(F_5) $\lim\limits_{|t|\to\infty} (f(x,t)t - 2pF(x,t)) = +\infty$.

Motivated by [15, 16, 21, 29], our main results read as follows.

***Theorem* 1.1** — *Suppose that (F_1), (F_2) and (F_3) are satisfied. Then (1.1) has at least three nontrivial solutions.*

***Theorem* 1.2** — *Suppose that (F_1), (F_2), (F_4) and (F_5) are satisfied. Then (1.1) has at least three nontrivial solutions.*

Recently, Zhou and Wu [29] obtain infinitely many small solutions of problem (1.1) for the case $p = 2$ by a cutoff technique used in [12, 29] and a variant version of symmetry mountain pass theorem due to Kajikiya [12]. Inspired by their results, we will show that their results also hold for the $p-$Laplacian case. To obtain infinitely many solutions, we need the following assumptions.

Assume that $f : \Omega\times \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for almost all $x \in \Omega$ and

(S_1) f is bounded on bounded sets.

(S_2) There exists a constant $\delta > 0$ such that $f(x,-t) = -f(x,t)$ for $|t| \leq \delta$ and all $x \in \Omega$.

(S_3) $\lim\limits_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = +\infty$ uniformly for $x \in \Omega$.

(S_4) There exist constants $r > 0$ and $\alpha \in (0,p)$ such that $f(x,t)t \leq \alpha F(x,t)$ for $|t| \leq r$ and all $x \in \Omega$.

***Theorem* 1.3** — *Suppose that (S_1), (S_2), (S_3) and (S_4) are satisfied. Then (1.1) admits a weak solution sequence $\{u_n\}$ such that $u_n \neq 0,\ u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ J(u_n) < 0,\ J(u_n) \to 0^-$, and $u_n \to 0$ in $W_0^{1,p}(\Omega)$.*

***Theorem* 1.4** — *Suppose that (S_1), (S_2) and (S_3) are satisfied. Then either (i) or (ii) below holds.*

(i) The conclusion of Theorem (1.3) holds.

(ii) Problem (1.1) has a weak solution sequence $\{u_n\}$ such that $u_n \neq 0$,

$$u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ J(u_n) = 0,\ and\ u_n \to 0\ in\ W_0^{1,p}(\Omega).$$

Remark 1.5 : In a recent paper, Liu and Zhao [21] obtain two solutions for (1.1) via Morse theory if $(F_1),\ (F_2)'$ and (F_3) (respectively $(F_1),\ (F_2)'$, (F_4) and (F_5)) are satisfied, where $(F2)'$ is defined as follows:

$(F2)'$ There exist $r > 0$, $\hat{\lambda}_1,\ \hat{\lambda}_2 \in (\lambda_1, \lambda_2)$ such that $\hat{\lambda}_1 < \hat{\lambda}_2$ and $|t| \leq r$ implies

$$\hat{\lambda}_1 |t|^p \leq pF(x,t) \leq \hat{\lambda}_2 |t|^p.$$

It enriches and develops the results of Liu and Zhao [21].

Remark 1.6 : Theorem 1.3 is a generalization of the results in [28, 29]. Firstly, we pass from the Laplacian case (i.e. $p = 2$) to the p-Laplacian case (i.e. $1 < p < +\infty$). Secondly, compare Theorem 1.3 with the results of Zhou and Wu [29], we know that the assumptions $(S1) - (S3)$ are enough to obtain infinitely solutions, the subcritical growth condition imposed on f in [28] can be removed. In a recent paper [15], Liu uses Fountain theorem to obtain infinitely many solutions for (1.1) with f satisfying symmetry, subcritical growth and (AR) conditions, which are different from our assumptions made in this paper, see [15] for details.

Remark 1.7 : Compare Theorem 1.3 with Theorem 1.4, we find that the energy of weak solutions obtained in Theorem 1.3 is negative, but we do not know whether the same fact is true in Theorem 1.4. Thus, the condition $(S3)$ ensures the existence of negative energy small solution of (1.1).

This article is organized as follows. In Section 2, we give the proofs of Theorem 1.1 and Theorem 1.1; In section 3, we sketch out the proofs of Theorem 1.3 and Theorem 1.4 since its proof are similar to that of [29, Theorem 1.1 and Theorem 1.2].

Throughout this paper, we make use of the following notation. $L^p(\Omega), 1 \leq p \leq \infty$, denotes Lebesgue space; the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$; the norm in $W_0^{1,p}(\Omega)$ is denoted by $\|\cdot\|$; $C, C_0, C_1, C_2, ...$ denote (possibly different) positive constants.

2. Proofs of Theorem 1.1 and Theorem 1.2

Before we proceed, let us recall some basic definitions and facts from Morse theory (critical groups) which we will need in the sequel. Let J be a C^1-functional defined on a Banach space X, then the critical groups of J at an isolated critical point u with $J(u) = c$ is defined by

$$C_q(J, u) := H_q(J^c, J^c \backslash \{u\}; \mathcal{G}), \quad q \in \mathbb{N}_0 := \{0, 1, 2...\},$$

where H_q is the singular relative homology with coefficients in an Abelian group $\mathcal{G}$ and $J^c := J^{-1}(-\infty, c]$, see [3, 8, 23] for details.

Let us summarize some properties of the function g defined by (1.2). For its proof, we refer to [15, 21].

Lemma 2.1 — The function g defined by (1.2) satisfies the following conditions:

(1) $g(0) = 0$;

(2) g is uniquely defined, C^∞ and invertible;

(3) $0 < g'(t) \leq 1$ for all $t \in \mathbb{R}$;

(4) $\frac{1}{2}g(t) \leq tg'(t) \leq g(t)$ for all $t > 0$;

(5) $g(t)/t \nearrow 1$, as $t \to 0+$;

(6) $|g(t)| \leq |t|$ for all $t \in \mathbb{R}$;

(7) $g(t)/\sqrt{t} \nearrow K_0 := \sqrt{2}p^{-1/(2p)}$, as $t \to +\infty$;

(8) $|g(t)| \leq K_0|t|^{1/2}$ for all $t \in \mathbb{R}$;

(9) $g^2(t) - g(t)g'(t)t \geq 0$ for all $t \in \mathbb{R}$;

(10) There exists a positive constant C such that $|g(t)| \geq C|t|$ for $|t| \leq 1$ and $|g(t)| \geq C|t|^{1/2}$ for $|t| \geq 1$;

(11) $|g(t)g'(t)| < K_0^2$ for all $t \in \mathbb{R}$;

(12) $g''(t) < 0$ when $t > 0$ and $g''(t) > 0$ when $t < 0$.

Lemma 2.2 [27] — Suppose $h \in L^q(\Omega)$ for some $q > N$. Then the Dirichlet problem

$$\begin{cases} -\Delta_p u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C_0^1(\overline{\Omega})$. Moreover, if $h \geq 0$ is nontrivial, then

$$u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} > 0 \text{ on } \partial\Omega,$$

where ν is the interior unit normal on $\partial\Omega$.

Now let

$$f_+(x,t) := \begin{cases} f(x,t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \qquad F_+(x,t) = \int_0^t f_+(x,s)\mathrm{d}s.$$

Then the critical points of the C^1-functional $J_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_+(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega F_+(x, g(v))\mathrm{d}x, \ \ v \in W_0^{1,p}(\Omega),$$

are solution of the truncation problem

$$\begin{cases} -\Delta_p v = f_+(x, g(v))g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_+)$$

Inspired by [11] and [16], we can prove the following lemma, which shows the origin **0** is not a local minimizer of J.

Lemma 2.3 — If f satisfies (F_1) and (F_2), then we have

$$C_q(J,\mathbf{0}) \cong 0, \ \ \forall q \in \mathbb{N}_0.$$

PROOF : From (F_2), it is easy to see that

$$\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{\tau-2}t} = a. \tag{2.1}$$

It follows from condition (F_1) and (2.1) that

$$F\left(x,g(t)\right) \geq C_0|g(t)|^{\tau} - C_1|g(t)|^{2q}, \ \ \forall (x,t) \in \Omega\times\mathbb{R}.$$

Hence for $v \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $s>0$, we have

$$\begin{aligned} J(sv) =& \frac{s^p}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega F\left(x,g(v)\right)\mathrm{d}x \\ \leq& \frac{s^p}{p}\|v\|^p - C_0\int_\Omega |g(sv)|^{\tau}\mathrm{d}x + C_1\int_\Omega |g(sv)|^{2q}\mathrm{d}x \\ \leq& \frac{s^p}{p}\|v\|^p - C_0\int_\Omega |g(sv)|^{\tau}\mathrm{d}x + C_1K_0^{2q}s^q|v|_q^q. \end{aligned}$$

The last inequality follows from the fact that $|g(t)| \leq K_0|t|^{1/2}$ for all $t\in\mathbb{R}$ (see Lemma 2.1). Since in one dimensional space span $\{v\}$, all norms are equivalent, note that $\tau < p \leq q$ and $|g(t)| \geq C|t|$ for $|t|\leq 1$, for the given $v\in W_0^{1,p}(\Omega)\setminus\{0\}$, there exists a small $s_0 = s_0(v) \in (0,1)$ such that

$$\begin{aligned} J(sv) \leq& \frac{s^p}{p}\|v\|^p - C_2 s^{\tau}|v|_\tau^\tau + C_1K_0^{2q}s^q|v|_q^q \\ <&0 \ \text{ for } s\in(0,s_0). \end{aligned} \tag{2.2}$$

On the other hand, from (F_2), we have

$$\lim_{|t|\to 0}\frac{\tau F(x,t) - f(x,t)t}{|t|^p} = 0.$$

Note that

$$\begin{aligned} |F\left(x,g(v)\right) - \frac{1}{\tau}f\left(x,g(v)\right)g(v)| \leq& C\left(1+|g(v)|^{2q}\right) \\ \leq& C\left(1+K_0^{2q}|v|^q\right). \end{aligned}$$

Since $g(t)$ behaves like t near 0 (see Lemma 2.1), so we have

$$\int_\Omega \left[F(x,g(v)) - \frac{1}{\tau} f(x,g(v))g(v)\right] \mathrm{d}x = o\left(\|v\|^p\right) \text{ as } \|v\| \to 0.$$

Thus by a direct computation we obtain

$$\begin{aligned}\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=1} J(sv) =& J(v) + \left(\frac{1}{\tau} - \frac{1}{p}\right)\int_\Omega |\nabla v|^p \mathrm{d}x \\ &+ \int_\Omega \left(F(x,g(v)) - \frac{1}{\tau} f(x,g(v))g'(v)v\right)\mathrm{d}x \\ =& J(v) + \left(\frac{1}{\tau} - \frac{1}{p}\right)\int_\Omega |\nabla v|^p \mathrm{d}x + o(\|v\|^p),\end{aligned}$$

where we have used the fact that $\frac{g(t)}{t} \to 1$ as $t \to 0^+$. Hence, there exists $\rho > 0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=1} J(sv) > 0, \;\; \forall v \in J^{-1}([0,+\infty)) \cap B_\rho \setminus \{0\}, \tag{2.3}$$

where $B_\rho := \{v \in W_0^{1,p}(\Omega) : \|v\| \le \rho\}$.

Next, we follow the arguments in the proof of [11, Proposition 2.1]. It follows by the monotonicity arguments from (2.3) that

$$J(sv) < 0 \text{ for } s \in (0,1), \;\; v \in J^{-1}(-\infty,0) \cap B_\rho. \tag{2.4}$$

From (2.2)-(2.4), for $v \in B_\rho$, if $J(v) > 0$, then there exists a unique $T(v) \in (0,1)$ such that

$$J\left(T(v)v\right) = 0,$$

$$J(sv) < 0, \; \forall s \in (0, T(v)) \text{ and } J(sv) > 0, \; \forall s \in (T(v), 1). \tag{2.5}$$

If $J(v) \le 0$, we set $T(v) = 1$. It follows from (2.3),(2.4) and the implicit function theorem that the mapping $T : B_\rho \to [0,1]$ is continuous in v. Define a mapping $\eta : [0,1] \times B_\rho \to B_\rho$ by

$$\eta(s,v) = (1-s)v + sT(v)v, \; s \in [0,1], \; v \in B_\rho.$$

It is easy to see that η is a continuous function from $(B_\rho, B_\rho \setminus \{0\})$ to $(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{0\})$. By the homotopy invariance of homology group, we have

$$C_q(J,\mathbf{0}) = H_q\left(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{\mathbf{0}\}\right) \cong H_q\left(B_\rho, B_\rho \setminus \{\mathbf{0}\}\right) \cong 0, \; \forall q \in \mathbb{N}_0,$$

since $B_\rho \setminus \{\mathbf{0}\}$ is contractible. The proof is completed. □

Lemma 2.4 — Assume (F_1) and (F_2), then **0** is not a local minimizer of J_+.

PROOF : Similar as the previous proof, from (F_2), it is easy to see that

$$\lim_{|t|\to 0} \frac{f_+(x,t)}{|t|^{\tau-2}t} = a.$$

Thus, from (F_1), we obtain

$$F_+(x, g(t)) \geq C_0|g(t)|^\tau - C_1|g(t)|^{2q}, \ \ \forall (x,t) \in \Omega \times \mathbb{R}.$$

Hence for $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $s > 0$, we have

$$\begin{aligned} J_+(sv) =& \frac{s^p}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega F_+(x, g(v))\,\mathrm{d}x \\ \leq& \frac{s^p}{p}\|v\|^p - C_0\int_\Omega |g(sv)|^\tau \mathrm{d}x + C_1\int_\Omega |g(sv)|^{2q}\mathrm{d}x \\ \leq& \frac{s^p}{p}\|v\|^p - C_0\int_\Omega |g(sv)|^\tau \mathrm{d}x + C_1K_0^{2q}s^q|v|_q^q, \end{aligned}$$

where we have used the fact $|g(t)| \leq K_0|t|^{1/2}$ for all $t \in R$ (see Lemma 2.1). Since in one dimensional space span $\{v\}$, all norms are equivalent, note that $\tau < p < q$ and $|g(t)| \geq C|t|$ for $|t| \leq 1$, for the given $v \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a small $s_0 = s_0(v) \in (0,1)$ such that

$$\begin{aligned} J_+(sv) \leq& \frac{s^p}{p}\|v\|^p - C_2s^\tau|v|_\tau^\tau + C_1K_0^{2q}s^q|v|_q^q \\ <& 0 \ \text{ for } s \in (0, s_0). \end{aligned}$$

Hence **0** is not a local minimizer of J_+. □

The next lemma can be found in [21].

Lemma 2.5 — Assume (F_1), if f satisfies (F_3) (or substitute (F_4) and (F_5) for (F_3)), then

(i) J_+ is coercive on $W_0^{1,p}(\Omega)$, that is $J_+(v) \to +\infty$ as $\|v\| \to \infty$;

(ii) J_+ satisfies the (PS) condition.

From $(F1)$ and Lemma 2.1, it is easy to see that

$$|f_+(x, g(v))| \leq C\left(1 + |g(v)|^{2(q-1)}\right) \leq C(1 + |v|^{q-1}),$$

where $1 < p \leq q < p^*$. Hence, a direct consequence of [7, Theorem 1.2] (see also [9, Proposition 2.6]), we have

Lemma 2.6 — If $v \in W_0^{1,p}(\Omega)$ is a local minimizer of J_+ in $C_0^1(\overline{\Omega})$, then v is a local minimizer of J_+ in $W_0^{1,p}(\Omega)$.

We are now ready to prove Theorem 1.1 and Theorem 1.2.

PROOF THEOREM 1.1 By Lemma 2.5 and Weierstrass theorem, there exists $v_+ \in W_0^{1,p}(\Omega)$ such that

$$J_+(v_+) = \inf_{v \in W_0^{1,p}(\Omega)} J_+(v). \tag{2.6}$$

Using Lemma 2.4, we have $v_+ \neq 0$. Hence v_+ is a nonzero solution of $(P+)$. By multiplying the negative part of v_+ to the equation $(P+)$ and then integrating by parts, it is easy to see that $v_+ \geq 0$. Then by the regularity results for p-Laplacian equation [13, 24, 27], we get

$$v_+ \in C_0^1(\overline{\Omega}).$$

From $v_+ \in C_0^1(\overline{\Omega})$, condition $(F1)$ and Lemma 2.1, we have

$$f_+(x, g(v_+)) \geq 0, \ \ -\Delta_p v_+ = f_+(x, g(v_+))g'(v_+) \in L^\infty(\Omega).$$

It follows from Theorem 2.2 that

$$v_+ > 0, \ \text{in}\ \Omega, \quad \frac{\partial v_+}{\partial \nu} > 0, \ \text{on}\ \partial\Omega, \tag{2.7}$$

where ν is the interior normal on $\partial\Omega$. Hence v_+ is a positive solution of (P). From (2.6), v_+ is also a local minimizer of J_+ in $C_0^1(\overline{\Omega})$, we can find some $r > 0$ such that

$$J_+(v_+) \leq J_+(v), \ \ \text{for } v \in C_0^1(\overline{\Omega}) \text{ with } \|v - v_+\|_{C^1} \leq r. \tag{2.8}$$

By (2.7) we may choose $r_1 < r$ such that for $v \in C_0^1(\overline{\Omega})$ with $\|v - v_+\|_{C^1} \leq r_1$, then $v(x) > 0$ for all $x \in \Omega$. However, if $v > 0$, then $F(x, g(v)) = F_+(x, g(v))$. Hence $J(v) = J_+(v)$. Thus for $v \in C_0^1(\overline{\Omega})$ with $\|v - v_+\|_{C^1} \leq r_1$, we have

$$J(v_+) = J_+(v_+) \leq J_+(v) = J(v).$$

That is to say, v_+ is a local minimizer of J restrict to $C_0^1(\Omega)$, in the C^1-topology. Applying Lemma 2.6, we know that v_+ is a local minimizer of J in $W_0^{1,p}(\Omega)$ topology. We may assume that v_+ is an isolated critical point of J, thus

$$C_q(J, v_+) = \mathcal{G} \text{ for } q = 0, \quad C_q(J, v_+) = 0 \text{ for } q \neq 0. \tag{2.9}$$

Similarly, by considering the functional

$$J_-(u) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega F_-(x, g(v))\mathrm{d}x, \;\; v \in W_0^{1,p}(\Omega)$$

with

$$f_-(x,t) := \begin{cases} 0, & \text{if } t \geq 0, \\ f(x,t), & \text{if } t < 0, \end{cases} \qquad F_-(x,t) = \int_0^t f_-(x,s)\mathrm{d}s,$$

we can obtain a negative solution u_- of the problem (P), which is a local minimizer of J_- in $W_0^{1,p}(\Omega)$ with

$$C_q(J, v_-) = \mathcal{G} \text{ for } q = 0, \quad C_q(J, v_-) = 0 \text{ for } q \neq 0. \tag{2.10}$$

Now we recall the Morse inequality (see [3, 8, 23]).

Proposition 2.7 — Let X be a Banach space. Suppose that $J \in C^1(X, \mathbb{R})$ satisfies the (PS) condition, has only isolated critical points, and the critical values of J are bounded from below by some $b \in \mathbb{R}$. Then we have

$$\sum_{q=0}^{\infty}(-1)^q M_q = \sum_{q=0}^{\infty}(-1)^q \beta_q,$$

where $M_q = \sum_{I'(u)=0} \text{rank } C_q(J,u), \beta_q = \text{rank } H_q(X, J_b)$.

Now we can obtain the third nonzero critical point of J: If $\mathbf{0}$, v_+ and v_- are the only critical points of J, we fix some

$$b < \inf_{u \in W_0^{1,p}(\Omega)} J(v),$$

then $J_b = \emptyset$ and

$$H_q(W_0^{1,p}(\Omega), J_b) = \mathcal{G} \text{ for } q = 0, \quad H_q(W_0^{1,p}(\Omega), J_b) = 0 \text{ for } q \neq 0. \tag{2.11}$$

By (2.9) and (2.10), the Morse inequality yields

$$0 + 1 \times (-1)^0 + 1 \times (-1)^0 = (-1)^0.$$

This is impossible. Thus J has at least three nonzero critical points. So the problem (P) has at least three nontrivial solutions. □

3. Proofs of Theorem 1.3 and Theorem 1.4

In this section, we will use a cut off technique used in [12] and [29] together with the following abstract critical point theorem to prove Theorem 1.3 and Theorem 1.4. Our ideas are inspired by Zhou and Wu [29], we want to show that their results are also true for the $p-$Laplacian case (i.e. $1 < p < \infty$) under weaker assumptions on $f(x, u)$. Note that in [29], the authors suppose that g is in $C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ which is stronger than our assumption $(S1)$ (Indeed, the condition $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ in there paper should be replaced by $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, otherwise the estimates (3.2) in [29] can not be obtained). To state the symmetric mountain pass theorem, we need the notion of genus [1, 8].

Let X be a Banach space and A a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k\backslash\{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

Lemma 3.1 [12, Theorem 1] — Let X be an infinite dimensional Banach space and $J \in C^1(X, \mathbb{R})$ an even functional with $J(0) = 0$. Suppose that J satisfies

(1) J is bounded from below and satisfies the (PS) condition.

(2) For each $k \in N$, there exists a $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} J(u) < 0$.

Then we conclude that

(a) (Clark [4]) $c_k = \inf\limits_{A \in \Gamma_k} \sup\limits_{u \in A} J(u) < 0$ is a crtical value of J for every $k \in \mathbb{N}$, and $c_k \to 0^-$ as $k \to \infty$, and

(b) (Kajikiya [12]) either (i) or (ii) below holds.

(i) There exists a sequence $\{w_k\}$ such that $J'(w_k) = 0,\ J(w_k) < 0$ and $\lim_{k\to\infty} w_k = 0$.

(ii) There exist two sequences $\{w_k\}$ and $\{v_k\}$ such that $J'(w_k) = 0,\ J(w_k) = 0,\ w_k \neq 0,\ \lim_{k\to\infty} w_k = 0,\ J'(v_k) = 0,\ J(v_k) < 0,\ \lim_{k\to\infty} J(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.

Let l be a small constant satisfying $0 < l \leq \frac{1}{2}\min\{\delta, r\}$, where δ and r are the constants appearing in $(S2)$ and $(S4)$, respectively. Let $h \in C^1(\mathbb{R}, [0, 1])$ be an even function satisfying $h(t) = 1$, for $|t| \leq l,\ h(t) = 0$ for $|t| \geq 2l$ and h is decreasing in $[l, 2l]$. Consider the cutoff equation of (1.1)

$$\begin{cases} -\Delta_p v = f(x, g(v))g'(v)h(g(v)) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

If v is a weak solution of (3.1) satisfying $|v|_\infty \leq l$, we conclude that $|g(v)|_\infty \leq l$ (note that $|g(t)| \leq |t|$) and $h(g(v)) = 1$. Hence v solves actually problem (1.3) and $u = g(v)$ is a weak solution of (1.1). We set $f_h(x, v) = f(x, v)h(v)$ and $F_h(x, v) = \int_0^v f_h(x, t)\mathrm{d}x$. Consider the C^1-functional for (3.1) defined by

$$J_h(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega F_h(x, g(v))\mathrm{d}x, \;\; v \in W_0^{1,p}(\Omega).$$

Lemma 3.2 — J_h is bounded from below and satisfies the (PS) condition.

PROOF : By virtue of the hypothesis $(S1)$ and the definition of f_h, we can find a constant $C_1 > 0$ such that

$$|f_h(x, v)| \leq C_1$$

for all $(x, v) \in \Omega \times \mathbb{R}$. It follows from Lemma 2.1 and Sobolev embedding theorem that

$$J_h(v) \geq \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - C_1\int_\Omega |g(v)|\mathrm{d}x \geq \frac{1}{p}\|v\|^p - C_1\int_\Omega |v|\mathrm{d}x \geq \frac{1}{p}\|v\|^p - C_1\|v\| \tag{3.2}$$

for any $v \in W_0^{1,p}(\Omega)$, which implies that J_h is bounded from below. We shall prove that J_h satisfies (PS) condition. Let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a (PS) sequence of J_h, i.e.,

$$|J_h(v_n)| \leq C_2, \;\; \text{and} \;\; J_h'(v_n) \to 0 \text{ in } W_0^{1,p}(\Omega)^* \text{ as } k \to \infty \tag{3.3}$$

for some $C_2 > 0$ and $W_0^{1,p}(\Omega)^*$ is the dual space of $W_0^{1,p}(\Omega)$. Then $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$ by (3.2) and (3.3). Thus, up to a subsequence, we have $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, $v_n \to v$ in $L^1(\Omega)$. Using Lemma 2.1 and the standard inequality in $\mathbb{R}^N$ given by

$$|\zeta - \eta|^p \leq \begin{cases} C_p(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta)(\zeta - \eta), & p \geq 2, \\ C_p(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta)(\zeta - \eta)(|\zeta| + |\eta|)^{2-p}, & 1 < p < 2, \end{cases}$$

where C_p is a positive constant. We have

$$\begin{aligned} o(1) =& \langle J_h'(v_n) - J_h'(v), v_n - v\rangle \\ =& \int_\Omega \left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\right)(\nabla v_n - \nabla v)\,\mathrm{d}x \\ &- \int_\Omega \left[f_h(x, g(v_n))g'(v_n) - f_h(x, g(v))g'(v)\right](v_n - v)\mathrm{d}x \\ \geq& \int_\Omega \left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\right)(\nabla v_n - \nabla v)\,\mathrm{d}x - 2C_1\int_\Omega |v_n - v|\mathrm{d}x \\ \geq& C_p\|v_n - v\|^p + o(1), \end{aligned}$$

which implies that $v_n \to v$ in $W_0^{1,p}(\Omega)$. The proof is completed. □

Lemma 3.3 — For any $k \in \mathbb{N}$, there exists a closed symmetric subset $A_k \subset W_0^{1,p}(\Omega)$ such that the genus $\gamma(A_k) \geq k$ and $\sup_{v \in A_k} J_h(v) < 0$.

PROOF : Since the proof is essentially the same as [29, Lemma 3.2], we omit it. □

Using similar techniques as [28, Lemma 3.3], we can prove the following Lemma.

Lemma 3.4 — Assume that $\{v_n\}$ is a sequence of critical points of J_h satisfying $v_n \to 0$ in $W_0^{1,p}(\Omega)$. Then $v_n \to 0$ in $L^\infty(\Omega)$ as $n \to \infty$.

PROOF OF THEOREM 1.3 : Obviously, the functional J_h is even and $J_h(0) = 0$. Moreover, Lemma 3.1, Lemma 3.2 and Lemma 3.3 imply that J_h has a critical sequence $\{v_n\}$ such that $J_h(v_n) < 0$ and $J_h(v_n) \to 0^-$. By the assumption $(S4)$ and Lemma 2.1, we have

$$\begin{aligned} o(1) =& \frac{1}{\alpha}\langle J_h'(v_n), v_n\rangle - J_h(v_n) \\ =& \left(\frac{1}{\alpha} - \frac{1}{p}\right)\int_\Omega |\nabla v_n|^p \mathrm{d}x + \int_\Omega \left[F_h(x, g(v_n)) - \frac{1}{\alpha} f_h(x, g_h(v_n))g'(v_n)v_n\right]\mathrm{d}x \\ \geq& \left(\frac{1}{\alpha} - \frac{1}{p}\right)\int_\Omega |\nabla v_n|^p \mathrm{d}x. \end{aligned}$$

Hence $v_n \to 0$ in $W_0^{1,p}(\Omega)$. By Lemma 3.4, we can assume that, without loss of generality, $\|v_n\|_\infty \leq l$ for all $n \in \mathbb{N}$. Thus, $J(v_n) = J_h(v_n)$ and $J'(v_n) = 0$ for all $n \in \mathbb{N}$. Let $u_n = f(v_n)$. Then $\{u_n\}$ is a weak solution sequence of (1.1) and $I(u_n) = J(v_n)$. Hence $I(u_n) < 0, I(u_n) \to 0^-$ and $u_n \neq 0$. Moreover,

$$\int_\Omega (1 + p u_n^p)|\nabla u_n|^p \mathrm{d}x = \int_\Omega |\nabla v_n|^p \mathrm{d}x$$

implies $u_n \to 0$ in $W_0^{1,p}(\Omega)$.The proof is completed. □

PROOF OF THEOREM 1.4 By Lemma 3.1, Lemma 3.2 and Lemma 3.3, J_h has a critical sequence $\{v_n\}$ such that $J_h(v_n) \leq 0$ and $\lim\limits_{n\to\infty} v_n = 0$, then

$$\frac{1}{p}\int_\Omega |\nabla v_n|^p \mathrm{d}x \leq \int_\Omega F_h(x, g(v_n))\mathrm{d}x \to 0,$$

that is to say $v_n \to 0$ in $W_0^{1,p}(\Omega)$. Then the conclusion of Theorem 1.4 follows by an argument as in the proof of Theorem 1.3. □

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