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NONLINEAR SCHRÖDINGER EQUATION FOR THE TWISTED LAPLACIAN IN THE CRITICAL CASE

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In [6] and [7], we prove well-posedness of solution to the nonlinear Schrödinger equation associated to the twisted Laplacian on $\mathbb{C}^n$ for a general class of nonlinearities including power type with subcritical case $0 \leq \alpha < \frac{2}{n-1}$. In this paper, we consider the critical case $\alpha = \frac{2}{n-1}$ with $n \geq 2$. Our approach is based on truncation of the given nonlinearity G, which is used in [3]. We obtain solution for the truncated problem. We obtain solution to the original problem by passing to the limit.

Key words : Twisted Laplacian (special Hermite operator); nonlinear Schrödinger equation; Strichartz estimates; well posedness.

1. INTRODUCTION

We consider the initial value problem for the nonlinear Schrödinger equation for the twisted Laplacian $\mathcal{L}$:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = G(z,u), \qquad z \in \mathbb{C}^n,\ t \in \mathbb{R} \tag{1.1}$$

$$u(z,t_0) = f(z). \tag{1.2}$$

Here we consider the nonlinearity G of the form

$$G(z,w) = \psi(x,y,|w|)\, w,\ (x,y,w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}, \tag{1.3}$$

where $z = x + iy \in \mathbb{C}^n, w \in \mathbb{C}$ and $\psi \in C(\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)) \cap C^1(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty))$ satisfy the following inequality

$$|F(x,y,\eta)| \leq C|\eta|^\alpha \tag{1.4}$$

with $F=\psi, \partial_{x_j}\psi, \partial_{y_j}\psi$ $(1\leq j\leq n)$ and $\eta\partial_\eta\psi(x,y,\eta)$, $\alpha=\frac{2}{n-1}$ with $n\geq 2$ and for some constant C.

In [7], we consider subcritical case $0\leq\alpha<\frac{2}{n-1}$ for initial value in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. Sobolev space $\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)$ is introduced in [7]. In this paper, we consider the critical case $\alpha=\frac{2}{n-1}$. In subcritical case $0\leq\alpha<\frac{2}{n-1}$ for each α, we have some $q>2$ such that $(q,2+\alpha)$ be an admissible pair (see Definition 3.1 in [7]), which is not the case when $\alpha=\frac{2}{n-1}$. We overcome this difficulty by considering admissible pair (γ,ρ) and by using embedding theorem (Lemma 2.1), where

$$\rho=\frac{2n^2}{n^2-n+1},\ \ \gamma=\frac{2n}{n-1}.$$

To treat the critical case, we adopt truncation argument of Cazenave and Weissler [3]. To prove local existence, we truncate the nonlinearity G and obtain solution for the truncated problem. Now we obtain solution u for nonlinearity G by using Strichartz estimates and by passing to the limit.

The twisted Laplacian operator $\mathcal{L}$ was introduced by Strichartz [8], and called the special Hermite operator. The twisted Laplacian $\mathcal{L}$ on $\mathbb{C}^n$ is given by

$$\mathcal{L} \;=\; \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j+\overline{Z}_jZ_j\right)$$

where $Z_j=\frac{\partial}{\partial z_j}+\frac{1}{2}\bar{z}_j,\ \overline{Z}_j=-\frac{\partial}{\partial\bar{z}_j}+\frac{1}{2}z_j,\ j=1,2,\ldots n$. Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial\bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial x_j}\mp i\frac{\partial}{\partial y_j}$ respectively. Nonlinear Schrödinger equation for the twisted Laplacian has also been studied in [6, 7, 12]. Spectral theory of twisted Laplacian $\mathcal{L}$ is well known and for this we refer to [9, 10] and for Schrödinger equation we refer to monographs by Cazenave [2] and Tao [11]. Now we state the main theorem of this paper.

***Theorem* 1.1** — *Let $f\in\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and G be as in (1.3) and (1.4) with $\alpha=\frac{2}{n-1}$ and $n\geq 2$. Initial value problem (1.1), (1.2) has maximal solution $u\in C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}})\cap L^{q_1}_{\text{loc}}\left((T_*,T^*),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for every admissible pair (q_1,p_1), where $t_0\in(T_*,T^*)$. Moreover the following properties hold:*

(i) (Uniqueness) : Solution is unique in $C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))\cap L^{\gamma}((T_*,T^*),\tilde{W}^{1,\rho}_{\mathcal{L}})$.

(ii) (Blowup alternative) : If $T^*<\infty$ then $\|u\|_{L^q\left((t_0,T^*),\tilde{W}^{1,p}_{\mathcal{L}}\right)}=\infty$ for every admissible pair (q,p) with $2<p$ and $\frac{1}{q}=n\left(\frac{1}{2}-\frac{1}{p}\right)$. Similar conclusion holds if $T_*>-\infty$.

(iii) (Stability) : If $f_j\to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ then $\|u-\tilde{u_j}\|_{L^q\left(I,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)}\to 0$ as $j\to\infty$ for every admissible pair (q,p) and every interval I with $\overline{I}\subset(T_*,T^*)$, where $u,\tilde{u_j}$ are solutions corresponding to f,f_j respectively.

For $m \geq 1$, consider $G_m(z,u) = \psi_m(z,|u|)u : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, where

$$\psi_m(z,\sigma) = \begin{cases} \psi(z,\sigma) & \text{if } 0 \leq \sigma \leq m \\ m^2\left(\frac{\psi(z,\sigma)}{\sigma^2} - \frac{\psi(z,m)}{\sigma^2} + \frac{\psi(z,m)}{m^2}\right) & \text{if } \sigma \geq m. \end{cases} \tag{1.5}$$

For $m = 0$, we define $G_0(z,u) = G(z,u)$ and $\psi_0(z,|u|) = \psi(z,|u|)$. Note that ψ_m is differentiable at $\sigma = m$ with respect to σ and also note that G_m will satisfy (1.3) and (1.4) with $\alpha = \frac{2}{n-1}$ as well as $\alpha = 0$. For $m \geq 1$, $G_m(z,\cdot) : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz from mean value theorem and

$$|G_m(z,u) - G_m(z,v)| \leq C_m|u-v| \text{ for } m \geq 1 \tag{1.6}$$

where constant C_m depends on $m \in \mathbb{N}$ but independent of $z \in \mathbb{C}^n$ and $u, v \in C$. Moreover by mean value theorem we also see that

$$|G_m(z,u) - G_m(z,v)| \leq C(|u|+|v|)^{\frac{2}{n-1}}|u-v| \text{ for } m \geq 0 \tag{1.7}$$

where constant C is independent of $m \in \mathbb{Z}_{\geq 0}, z \in \mathbb{C}^n$ and $u, v \in C$.

Since F_0 satisfies estimate (1.4) with $\alpha = \frac{2}{n-1}$, we conclude that

$$|F_m(z,\sigma)| \leq C\sigma^{\frac{2}{n-1}}, \tag{1.8}$$

where $F_m = \psi_m, \partial_{x_j}\psi_m, \partial_{y_j}\psi_m, \sigma\partial_\sigma\psi_m(x,y,\sigma)$ with $1 \leq j \leq n$ and constant C is independent of m.

From Lemma 6.1 in [7] and in order to find solution for given IVP (1.1), (1.2) with G replaced by G_m and initial value $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, it is sufficient to find the solution of the following equation

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}}f(z) - i\int_{t_0}^t e^{-i(t-s)\mathcal{L}}G_m(z,u(z,s))ds.$$

This reduces the existence theorem for the solution to the nonlinear Schrödinger equation to a fixed point theorem for the operator with $m \geq 0$

$$\mathcal{H}_m(u)(z,t) = e^{-i(t-t_0)\mathcal{L}}f(z) - i\int_{t_0}^t e^{-i(t-s)\mathcal{L}}G_m(z,u(z,s))ds. \tag{1.9}$$

2. Some Auxilliary Estimates

Lemma 2.1 [Sobolev Embedding Theorem] — We have the continuous inclusion

$$\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n) \hookrightarrow L^{p_2}(\mathbb{C}^n) \quad \begin{array}{ll} \text{for } p_1 \leq p_2 \leq \frac{2np_1}{2n-p_1} & \text{if } p_1 < 2n \\ \text{for } p_1 \leq p_2 < \infty & \text{if } p_1 = 2n \\ \text{for } p_1 \leq p_2 \leq \infty & \text{if } p_1 > 2n. \end{array}$$

PROOF : Let $f \in \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)$ and $\epsilon > 0$. Consider $u_\epsilon = e^{-\epsilon\mathcal{L}} f$. Then $u_\epsilon \in \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$ and we have

$$2|u_\epsilon| \frac{\partial}{\partial x_j} |u_\epsilon| = \frac{\partial}{\partial x_j} (\overline{u_\epsilon} u_\epsilon) = 2\Re e \left(\overline{u_\epsilon} \frac{\partial}{\partial x_j} u_\epsilon \right) = 2\Re e \left(\overline{u_\epsilon} (\frac{\partial}{\partial x_j} - \frac{i y_j}{2}) \, u_\epsilon \right).$$

Hence on the set $A = \{z \in \mathbb{C}^n | u_\epsilon(z) \neq 0\}$, we have

$$\left| \frac{\partial}{\partial x_j} |u_\epsilon| \right| = \left| \Re e \left(\frac{\overline{u_\epsilon}}{|u_\epsilon|} (\frac{\partial}{\partial x_j} - \frac{i y_j}{2}) \, u_\epsilon \right) \right| \leq \frac{1}{2} (|Z_j u_\epsilon| + |\bar{Z}_j u_\epsilon|).$$

Similarly $\left| \frac{\partial}{\partial y_j} |u_\epsilon| \right| \leq \frac{1}{2}(|Z_j u_\epsilon| + |\bar{Z}_j u_\epsilon|)$ on A. Note that $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} = \|u_\epsilon \chi_A\|_{L^{p_2}(\mathbb{C}^n)}$. By usual Sobolev embedding on $\mathbb{C}^n$ and above observations, we have inequality $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} \leq C \| |u_\epsilon \chi_A| \|_{W^{1,p_1}} \leq C \, \|u_\epsilon\|_{\tilde{W}^{1,p_1}_{\mathcal{L}}}$. Since $u_\epsilon = e^{-\epsilon \mathcal{L}} f \to f$ in $\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)$ and also in $L^{p_2}(\mathbb{C}^n)$ as $\epsilon \to 0$ (see [7]), we have $\|f\|_{L^{p_2}(\mathbb{C}^n)} \leq C \|f\|_{\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)}$, where constant C is a generic constant independent of f. Hence Lemma is proved. □

Lemma 2.2 — Let $u, v \in L^\gamma \left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n) \right)$ for some interval I, then the following estimate holds for each $m \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} \|G_m(z,u) - G_m(z,v)\|_{L^{\gamma'}\left(I, L^{\rho'}(\mathbb{C}^n)\right)} &\leq C \|u - v\|_{L^\gamma(I, L^\rho(\mathbb{C}^n))} \times \\ &\left(\|u\|_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|v\|_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \right)^{\frac{2}{n-1}} \end{aligned} \tag{2.1}$$

where constant C is independent of u, v, m, t_0 and I.

PROOF : Since $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{n-1}{n^2} = \frac{1}{\rho} + \frac{2}{n-1} \cdot \frac{n-1}{n\gamma}$, by using Hölder's inequality in the z-variable in (1.7) and by embedding theorem (Lemma 2.1), we get

$$\begin{aligned} & \|G_m(\cdot, u(\cdot,t)) - G_m(\cdot, v(\cdot,t))\|_{L^{\rho'}(\mathbb{C}^n)} \\ \leq \quad & C \|(u-v)(\cdot,t)\|_{L^\rho(\mathbb{C}^n)} \left(\|u(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} + \|v(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} \right)^{\frac{\gamma}{n}} \\ \leq \quad & C \|(u-v)(\cdot,t)\|_{L^\rho(\mathbb{C}^n)} \left(\|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} \right)^{\frac{\gamma}{n}} \end{aligned} \tag{2.2}$$

for each $t \in I$. Since $\frac{1}{\gamma'} = \frac{1}{\gamma} + \frac{1}{n}$, by taking $L^{\gamma'}$ norm in t-variable in this inequality and then by using Hölder's inequality we get the desired estimate (2.1). □

Lemma 2.3 — Let I be a bounded interval and $u \in L^\infty(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^\gamma \, (I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))$, then the following estimate holds

$$\begin{aligned} & \|G_m(z, u(z,t)) - G(z, u(z,t))\|_{L^{\gamma'}\left(I, L^{\rho'}(\mathbb{C}^n)\right)} \\ & \leq C |I|^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \|u\|^{\frac{n^2-n+1}{n(n-1)}}_{L^\infty\left(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)} \|u\|^{\frac{2}{n-1}}_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \end{aligned}$$

for all $m \geq 1$, where constant C is independent of m, u and I.

PROOF : Note that

$$G_m(z, u(z,t)) - G(z, u(z,t)) = (u\chi_{|u(z,t)|>m}(z,t))(\psi_m(z,|u|) - \psi(z,|u|)).$$

Therefore $|G_m(z, u(z,t)) - G(z, u(z,t))| \leq C|u\chi_{|u(z,t)|>m}(z,t)|\,|u|^{\frac{2}{n-1}}$. By Taking $L^{\rho'}$-norm in the z-variable, we have

$$\begin{aligned} \|G_m(z,u) - G(z,u)\|_{L^{\rho'}(\mathbb{C}^n)} &\leq C\|u\chi_{|u|>m}(\cdot,t)\|_{L^{\rho}(\mathbb{C}^n)}\|u(\cdot,t)\|^{\frac{\gamma}{n}}_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} \\ &\leq C\|u\chi_{|u|>m}(\cdot,t)\|_{L^{\rho}(\mathbb{C}^n)}\|u(\cdot,t)\|^{\frac{\gamma}{n}}_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)}. \end{aligned} \tag{2.3}$$

Now we observe the following

$$\begin{aligned} \|u\chi_{|u|>m}(\cdot,t)\|^{\rho}_{L^{\rho}(\mathbb{C}^n)} &= \int_{\mathbb{C}^n} |u|^{\rho}\chi_{|u|>m}(z,t)dz \\ &\leq \int_{\mathbb{C}^n} m^{-\frac{\rho}{n(n-1)}}|u|^{\frac{2n}{n-1}}dz \\ &\leq m^{-\frac{\rho}{n(n-1)}}\|u\|^{\frac{(n^2-n+1)\rho}{n(n-1)}}_{L^{\frac{2n}{n-1}}(\mathbb{C}^n)} \\ &\leq m^{-\frac{\rho}{n(n-1)}}\|u\|^{\frac{(n^2-n+1)\rho}{n(n-1)}}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \\ \|u\chi_{|u|>m}(\cdot,t)\|_{L^{\rho}} &\leq m^{-\frac{1}{n(n-1)}}\|u\|^{\frac{(n^2-n+1)}{n(n-1)}}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}. \end{aligned}$$

By taking L^{γ}-norm in the t-variable we have

$$\|u\chi_{|u|>m}\|_{L^{\gamma}(I,L^{\rho}(\mathbb{C}^n))} \leq |I|^{\frac{n-1}{2n}}m^{-\frac{1}{n(n-1)}}\|u\|^{\frac{(n^2-n+1)}{n(n-1)}}_{L^{\infty}\left(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)}. \tag{2.4}$$

By taking $L^{\gamma'}$-norm in the t-variable in estimate (2.3) and using Hölder's inequality, we get

$$\|G_m(z,u) - G(z,u)\|_{L^{\gamma'}\left(I,L^{\rho'}\right)} \leq C\|u\chi_{|u|>m}\|_{L^{\gamma}(I,L^{\rho})}\|u\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}.$$

By using inequality (2.4) in the above inequality, we get the desired estimate. □

Lemma 2.4 — Let $u \in L^{\gamma}\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for some interval I. Then for each $m \in \mathbb{Z}_{\geq 0}$, $G_m(z, u(z,t)) \in L^{\gamma'}\left(I, \tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)\right)$ and the following estimates hold:

$$\|SG_m(z,u(z,t))\|_{L^{\gamma'}\left(I,L^{\rho'}(\mathbb{C}^n)\right)} \leq C\|u\|^{\frac{n+1}{n-1}}_{L^{\gamma}\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \tag{2.5}$$

$$\|G_m(z,u(z,t))\|_{L^{\gamma'}\ \ I,\tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)} \leq C\|u\|^{\frac{n+1}{n-1}}_{L^{\gamma}\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \tag{2.6}$$

where $S = Z_j, \overline{Z}_j$ or Id, $1 \le j \le n$ and the constant C is independent of u and I.

PROOF : Since $\psi_m, \partial_{x_j}\psi_m, \partial_{y_j}\psi_m, |u|\partial_{|u|}\psi_m$ satisfy estimate (1.8), we have

$$|SG_m(z,u)| \le C|u|^{\frac{2}{n-1}}(|u| + |Z_j u| + |\bar{Z}_j u|)$$

where $S = Z_j, \overline{Z}_j$ $(1 \le j \le n)$ or Id, see Lemma 3.4 in [7]. Now estimate (2.5) follows from Hölder's inequality and embedding theorem (Lemma 2.1) as we used in the proof of Lemma 2.2. Estimate (2.6) is a consequence of estimate (2.5). □

Proposition 2.5 — Let $t_0 \in \mathbb{R}$ and I be a bounded interval such that $t_0 \in \overline{I}$.

(i) If $u, v \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, then $\mathcal{H}_m u - \mathcal{H}_m v \in L^q(I, L^p(\mathbb{C}^n))$ for every admissible pair (q,p), for every $m \in \mathbb{Z}_{\ge 0}$ and the following estimate holds:

$$\begin{aligned} \|\mathcal{H}_m u \quad - \quad & \mathcal{H}_m v\|_{L^q(I,L^p(\mathbb{C}^n))} \\ & \le C\|u-v\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))}\left(\|u\|_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|v\|_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}}. \end{aligned} \tag{2.7}$$

(ii) If $u \in L^\infty\left(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right) \cap L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, then $\mathcal{H}_m u - \mathcal{H} u \in L^q(I, L^p(\mathbb{C}^n))$ for every admissible pair (q,p), for every $m \in \mathbb{Z}_{\ge 1}$ and the following estimate holds

$$\begin{aligned} \|\mathcal{H}_m u \quad & - \mathcal{H} u\|_{L^q(I,L^p(\mathbb{C}^n))} \\ & \le C|I|^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \|u\|^{\frac{n^2-n+1}{n(n-1)}}_{L^\infty\left(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)} \|u\|^{\frac{2}{n-1}}_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \end{aligned} \tag{2.8}$$

where constant C is independent of u, v, m and t_0.

PROOF : Estimate (2.7) follows from Strichartz estimates (Theorem 3.3 in [7]) and Lemma 2.2, whereas estimate (2.8) follows from Strichartz estimates and Lemma 2.3. □

Now we state the following Proposition, which is useful in proving stability.

Proposition 2.6 — Let Φ be a continuous complex valued function on $\mathbb{C}$ such that $|\Phi(w)| \le C|w|^{\frac{2}{n-1}}$ with $n \ge 2$. Let $\{u_m\}$ be a bounded sequence in $L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for some interval I. If $u_m \to u$ in $L^\gamma(I, L^\rho(\mathbb{C}^n))$ then $u \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ and $[\Phi(u_m) - \Phi(u)]Su \to 0$ in $L^{\gamma'}\left(I, L^{\rho'}(\mathbb{C}^n)\right)$, for $S = \text{Id}, Z_j, \overline{Z}_j; 1 \le j \le n$.

PROOF : First we will prove $u \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$. By duality argument (also see Lemma A.2.1 in [4]), we have

$$\begin{aligned}
\|Su\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} &= \sup_{\phi\in C_c^\infty(\mathbb{C}^n\times I),\|\phi\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{C}^n))}\le 1} \left|\langle Su,\phi\rangle_{z,t}\right| \\
&= \sup_\phi \left|\langle u, S^*\phi\rangle_{z,t}\right| \\
&= \sup_\phi \lim_{m\to\infty} \left|\langle u_m, S^*\phi\rangle_{z,t}\right| \\
&= \sup_\phi \lim_{m\to\infty} \left|\langle Su_m, \phi\rangle_{z,t}\right| \\
&\le \sup_\phi \liminf_{m\to\infty} \|Su_m\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))}\|\phi\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{C}^n))} \\
&\le \liminf_{m\to\infty} \|Su_m\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \qquad (2.9)
\end{aligned}$$

for $S = Z_j, \bar{Z}_j$; $1 \le j \le n$. Therefore,

$$\|u\|_{L^\gamma(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} \le \liminf_{m\to\infty} \|u_m\|_{L^\gamma(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} < \infty.$$

Since $u_m \to u$ in $L^\gamma(I, L^\rho(\mathbb{C}^n))$, we can extract a subsequence still denoted by u_k such that

$$\|u_{k+1} - u_k\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \le \frac{1}{2^k}$$

for all $k \ge 1$ and $u_k(z,t) \to u(z,t)$ a.e. (z,t). Hence by continuity of Φ,

$$[\Phi(u_k) - \Phi(u)]Su \to 0 \quad \text{for a.e } (z,t) \in \mathbb{C}^n \times I. \qquad (2.10)$$

We establish the norm convergence by appealing to a dominated convergence argument in z and t variables successively.

Consider the function $H(z,t) = \sum_{k=1}^\infty |u_{k+1}(z,t) - u_k(z,t)|$. Clearly $H \in L^\gamma(I, L^\rho(\mathbb{C}^n))$. Also for $l > k$, $|(u_l - u_k)(z,t)| \le |u_l - u_{l-1}| + \cdots + |u_{k+1} - u_k| \le H(z,t)$, hence $|u_k - u| \le H$. This leads to the pointwise almost everywhere inequality

$$|u_k(z,t)| \le |u(z,t)| + H(z,t) = v(z,t).$$

Hence

$$|\,[\Phi(u_k) - \Phi(u)]\, Su(z,t)|^{\rho'} \le C[v^{\frac{2}{n-1}} + |u|^{\frac{2}{n-1}}]^{\rho'} \;\; |Su(z,t)|^{\rho'}. \qquad (2.11)$$

Since $u, v \in L^{\gamma}(I, L^{\rho}(\mathbb{C}^n))$, using Hölder's inequality with $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{n-1}{n^2} = \frac{1}{\rho} + \frac{2}{n-1} \cdot \frac{n-1}{n\gamma}$ and Lemma 2.1, we get

$$\begin{aligned} &\int_{\mathbb{C}^n} [v^{\frac{2}{n-1}} + |u|^{\frac{2}{n-1}}]^{\rho'} \ |Su(z,t)|^{\rho'} dz \qquad (2.12)\\ &\leq (\|v(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)})^{\frac{\rho'\gamma}{n}} \|Su(\cdot,t)\|^{\rho'}_{L^{\rho}(\mathbb{C}^n)}. \\ &\leq (\|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)})^{\frac{\rho'\gamma}{n}} \|Su(\cdot,t)\|^{\rho'}_{L^{\rho}(\mathbb{C}^n)} < \infty \end{aligned}$$

for a.e. $t \in I$. Thus in view of (2.11), (2.12) and using dominated convergence theorem in the z-variable, we see that

$$\| \, [\Phi(u_k) - \Phi(u)] \, Su(\cdot,t)\|_{L^{p'}(\mathbb{C}^n)} \to 0 \qquad (2.13)$$

as $k \to \infty$, for a.e. t.

Again, in view of (2.11) and (2.12), we get

$$\begin{aligned} &\|[\Phi(u_k) - \Phi(u)] \, Su(\cdot,t)\|_{L^{\rho'}(\mathbb{C}^n)} \\ &\leq C(\|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)})^{\frac{\gamma}{n}} \|Su(\cdot,t)\|_{L^{\rho}(\mathbb{C}^n)}. \end{aligned}$$

Since $\frac{1}{\gamma'} = \frac{1}{\gamma} + \frac{1}{n}$, an application of the Hölder's inequality in the t-variable shows that

$$\begin{aligned} &\|[\Phi(u_k) - \Phi(u)] \, Su\|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))} \\ &\leq C(\|v\|_{L^{\gamma}(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} + \|u\|_{L^{\gamma}(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))})^{\frac{\gamma}{n}} \|Su\|_{L^{\gamma}(I, L^{\rho}(\mathbb{C}^n))}. \end{aligned}$$

Hence a further application of dominated convergence theorem with (2.13) shows that $\| \, (\Phi(u_k) - \Phi(u)) \, Su\|_{L^{\gamma'}(I, L^{\rho'})} \to 0$, as $k \to \infty$.

Thus we have shown that $[\Phi(u_{m_k}) - \Phi(u)] \, Su \to 0$ in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$ for some subsequence u_{m_k} whenever $u_m \to u$ in $L^{\gamma}(I, L^{\rho}(\mathbb{C}^n))$. But the above arguments are also valid if we had started with any subsequence of u_m. It follows that any subsequence of $[\Phi(u_m) - \Phi(u)] \, Su$ has a subsequence that converges to 0 in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$. From this we conclude that the original sequence $[\Phi(u_m) - \Phi(u)] \, Su$ converges to zero in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$, hence the proposition. □

3. PROOF OF THEOREM 1.1

PROOF (of Theorem 1.1): (Local existence): Since $G_m(z, \cdot) : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz for each $m \geq 1$, see (1.6), from [7] it follows that there exists a unique global solution $u_m \in C(\mathbb{R}, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$

of the initial value problem

$$i\partial_t v(z,t) - \mathcal{L}v(z,t) = G_m(z,v), \qquad z\in\mathbb{C}^n,\ t\in\mathbb{R} \tag{3.1}$$

$$v(\cdot,t_0) = f. \tag{3.2}$$

Furthermore $\mathcal{H}_m u_m = u_m$ and $u_m \in L^q_{\text{loc}}(\mathbb{R}, \tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair (q,p). We deduce from Lemma 2.4 and Strichartz estimates (Theorem 3.3 in [7]) that

$$\begin{aligned}&\|u_m\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)}\\ &\le \|e^{-i(t-t_0)\mathcal{L}}f\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} + C\|u_m\|^{\frac{n+1}{n-1}}_{L^\gamma\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}.\end{aligned} \tag{3.3}$$

Let $l \ge m$, we see that

$$u_m - u_l = (\mathcal{H}_m(u_m) - \mathcal{H}_m(u_l)) + (\mathcal{H}_m(u_l) - \mathcal{H}(u_l)) + (\mathcal{H}(u_l) - \mathcal{H}_l(u_l)).$$

From Proposition 2.5, we deduce that

$$\begin{aligned}\|u_m - u_l\|_{L^q((t_0,t_0+T),L^p(\mathbb{C}^n))} &\le C\left(\|u_m\|_{L^\gamma((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}})} + \|u_l\|_{L^\gamma((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}})}\right)^{\frac{2}{n-1}}\\ &\times\left(\|u_m - u_l\|_{L^\gamma((t_0,t_0+T),L^\rho)} + T^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \|u_l\|^{\frac{n^2-n+1}{n(n-1)}}_{L^\infty((t_0,t_0+T),\tilde{W}^{1,2}_{\mathcal{L}})}\right).\end{aligned} \tag{3.4}$$

We choose $T \le \pi$, so that we can take constant C to be independent of T. Let $\tilde{C}$ be larger than the constant C that appear in (3.3), (3.4), (2.7), (2.8) and in Strichartz estimates for the particular choice of the admissible pair $(q,p) = (\gamma,\rho)$. Fixed δ small enough so that

$$\tilde{C}(4\delta)^{\frac{2}{n-1}} < \frac{1}{2}. \tag{3.5}$$

We claim that if $0 < T \le \pi$ is such that

$$\|e^{-i(t-t_0)\mathcal{L}}f\|_{L^\gamma\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \le \delta \tag{3.6}$$

then

$$\sup_{m\ge 1}\|u_m\|_{L^\gamma\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \le 2\delta \tag{3.7}$$

$$\sup_{m\ge 1}\|u_m\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty \tag{3.8}$$

for every admissible pair (q,p). Let $\theta_m(t) = \|u_m\|_{L^\gamma\left((t_0,t_0+t),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}$. From (3.3), we see that

$$\theta_m(t) \le \delta + \tilde{C}\theta_m(t)^{\frac{n+1}{n-1}}.$$

If $\theta_m(t) = 2\delta$ for some $t \in (t_0, t_0 + T]$, then

$$2\delta \leq \delta + \tilde{C}(2\delta)^{\frac{n+1}{n-1}} < 2\delta$$

which is a contradiction. Since θ_m is a continuous function with $\theta_m(t_0) = 0$, we conclude that $\theta_m(t) < 2\delta$ for all $t \in (t_0, t_0 + T]$, which proves (3.7). From (3.3), we see that

$$\begin{aligned} \sup_m \|u_m\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} &\leq \|e^{-i(t-t_0)\mathcal{L}} f\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} + C(2\delta)^{\frac{n+1}{n-1}} \\ &\leq C(q,p,n,\delta,f) < \infty. \end{aligned}$$

This proves (3.8). Put $(q, p) = (\gamma, \rho)$ in (3.4), we see that

$$\begin{aligned} \|u_m - u_l\|_{L^\gamma((t_0,t_0+T),L^\rho(\mathbb{C}^n))} &\leq \frac{1}{2}\left(\|u_m - u_l\|_{L^\gamma((t_0,t_0+T),L^\rho(\mathbb{C}^n))} + CT^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}}\right) \\ &\leq 2CT^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \to 0 \text{ as } m \to \infty. \end{aligned}$$

This shows that u_m is a cauchy sequence in $L^\gamma\left((t_0, t_0 + T), L^\rho(\mathbb{C}^n)\right)$ and from (3.4) it is also cauchy in $L^q\left((t_0, t_0 + T), L^p(\mathbb{C}^n)\right)$ for every admissible pair (q, p). Let u be its limit, then $u_m \to u$ in $L^q\left((t_0, t_0 + T), L^p(\mathbb{C}^n)\right)$ for every admissible pair (q, p). By duality argument (see (2.9)) and from estimates (3.7), (3.8), we have

$$\|u\|_{L^\gamma\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq 2\delta \tag{3.9}$$

$$\|u\|_{L^q\left((t_0,t_0+T),\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty. \tag{3.10}$$

From Lemma 2.4, $G_m(z, u(z, t)) \in L^{\gamma'}\left((t_0, t_0 + T), \tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for each $m \geq 0$. From Strichartz estimates (Theorem 3.3 in [7]) and (1.9), $\mathcal{H}u \in L^q((t_0, t_0 + T), \tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair (q, p).

From Lemma 2.2 $\|G_m(z, u_m) - G_m(z, u)\|_{L^{\gamma'}\left((t_0,t_0+T),L^{\rho'}(\mathbb{C}^n)\right)} \to 0$ and from Lemma 2.3, $\|G_m(z, u) - G(z, u)\|_{L^{\gamma'}\left((t_0,t_0+T),L^{\rho'}(\mathbb{C}^n)\right)} \to 0$ as $m \to \infty$. Therefore

$$\|G_m(z, u_m) - G(z, u)\|_{L^{\gamma'}((t_0,t_0+T),L^{\rho'})} \to 0 \quad \text{as} \quad m \to \infty.$$

Since $u_m = \mathcal{H}_m u_m$ for each $m \geq 1$, from Strichartz estimates we deduce that

$$\begin{aligned} \|u_m - \mathcal{H}u\|_{L^q((t_0,t_0+T),L^p(\mathbb{C}^n))} &= \|\mathcal{H}_m u_m - \mathcal{H}u\|_{L^q((t_0,t_0+T),L^p(\mathbb{C}^n))} \\ &\leq C\|G_m(z, u_m) - G(z, u)\|_{L^{\gamma'}((t_0,t_0+T),L^{\rho'})} \to 0 \end{aligned}$$

as $m \to \infty$. Therefore for $t \in (t_0, t_0 + T)$

$$u = \mathcal{H}u = e^{-i(t-t_0)\mathcal{L}} f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z, u(z, s))ds. \tag{3.11}$$

From Strichartz estimates and estimate (3.10), $u \in C([t_0, t_0+T], \tilde{W}_{\mathcal{L}}^{1,2}) \cap L^q((t_0, t_0+T), \tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n))$ for every admissible pair (q,p). In view of Lemma A.1 in [6] and Lemma 6.1 in [7], u is also a solution to the initial value problem (1.1), (1.2). Similarly solution exists on the interval $[t_0 - T', t_0]$ for some $T' > 0$. Now we continue this process with initial time t_0+T and t_0-T'. By continuing this process, we get maximal interval (T_*, T^*) and solution $u \in C((T_*, T^*), \tilde{W}_{\mathcal{L}}^{1,2}) \cap L^q_{\text{loc}}\left((T_*, T^*), \tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)\right)$ for every admissible pair (q,p).

Uniqueness : Uniqueness in $C((T_*, T^*), \tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)) \cap L^\gamma_{\text{loc}}\left((T_*, T^*), \tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)$ will follow from estimate (2.7) with $m = 0$ in Proposition 2.5, see uniqueness in [6].

Blowup alternative : We prove blowup alternative by method of contradiction. Let us assume that $T^* < \infty$ and $u \in L^q((t_0, T^*), \tilde{W}_{\mathcal{L}}^{1,p})$ for some admissible pair (q,p) with $2 < p$ and $\frac{1}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$. Since $2 < p < \frac{2n}{n-1}$, $n \geq 2$, so $p < 2n$. We choose admissible pair (q_1, p_1) as follows

$$\frac{1}{p_1'} = \frac{1}{p_1} + \frac{2}{n-1}\left(\frac{1}{p} - \frac{1}{2n}\right), \quad \frac{1}{q_1'} = \frac{1}{q_1} + \frac{2}{n-1}\frac{1}{q}.$$

Let us choose s and t such that $t_0 \leq s < t < T^*$. Since $|S_j G(z, u(z,t))| \leq C|u|^{\frac{2}{n-1}}(|u| + |Z_j u| + |\overline{Z}_j u|)$ for $S_j = Id, Z_j, \overline{Z}_j$ ($1 \leq j \leq n$) (see Lemma 3.4 in [7]), by Lemma 2.1 and Hölder's inequality we see that

$$\|G(z, u(z,\tau))\|_{L^{q_1'}((s,t),\tilde{W}_{\mathcal{L}}^{1,p_1'})} \leq C\|u\|_{L^{q_1}((s,t),\tilde{W}_{\mathcal{L}}^{1,p_1})} \|u\|^{\frac{2}{n-1}}_{L^q((s,t),\tilde{W}_{\mathcal{L}}^{1,p})}. \tag{3.12}$$

Since (t_0, T^*) is a bounded interval, so we can choose constant C independent of s and t, where $t_0 \leq s < t < T^*$. Now we see that

$$u(z,\tau) = e^{-i(\tau - s)\mathcal{L}} u(\cdot, s)(z) - i\int_s^\tau e^{-i(\tau - s_1)\mathcal{L}} G(z, s_1, u(z, s_1)) ds_1.$$

Therefore we deduce from Strichartz estimates that

$$\|u\|_{L^{q_1}\ \ (s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}} \leq C\|u(\cdot,s)\|_{\tilde{W}_{\mathcal{L}}^{1,2}} + C\|u\|_{L^{q_1}\ \ (s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}} \|u\|^{\frac{2}{n-1}}_{L^q\left((s,t),\tilde{W}_{\mathcal{L}}^{1,p}\right)}.$$

where constant C is independent of s and t. Since $p \neq 2$, so $q < \infty$ and $u \in L^q\left((t_0, T^*), \tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)\right)$, we choose s sufficiently close to T^* such that

$$C\|u\|^{\frac{2}{n-1}}_{L^q\left((s,T^*),\tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)\right)} \leq \frac{1}{2}.$$

Therefore we get

$$\|u\|_{L^{q_1}\ \ (s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)} \leq 2C\|u(\cdot,s)\|_{\tilde{W}_{\mathcal{L}}^{1,2}}.$$

Since RHS is independent of $t \in (s, T^*)$, so we have $u \in L^{q_1}\left((s,T^*), \tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)\right)$. Therefore $u \in L^{q_1}\left((t_0,T^*), \tilde{W}_{\mathcal{L}}^{1,p_1}\right)$ and $G(z,u(z,\tau)) \in L^{q_1'}\left((t_0,T^*), \tilde{W}_{\mathcal{L}}^{1,p_1'}\right)$ follows from (3.12). Now from Strichartz estimates and (3.11), $u \in L^{\tilde{q}}((t_0,T^*), \tilde{W}_{\mathcal{L}}^{1,\tilde{p}}(\mathbb{C}^n)) \cap C([t_0,T^*], \tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n))$ for every admissible pair $(\tilde{q},\tilde{p})$. Now by considering T^* as a initial time and by local existence argument, we get contradiction to maximality of T^*.

Stability : We prove stability in the following two steps.

Step 1 : Let $f_k \to f$ in $\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$. Then for each $T>0$,

$$\|e^{-i(t-t_0)\mathcal{L}}(f-f_k)\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} \leq C\|f-f_k\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)} \to 0 \text{ as } k\to\infty$$

where $I_T = (t_0-T, t_0+T)$. Therefore for given $\delta>0$ in (3.5), choose $T(\delta)$ sufficiently small such that

$$\|e^{-i(t-t_0)\mathcal{L}}f\|_{L^\gamma(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho})} \leq \frac{\delta}{2} \tag{3.13}$$

and choose k sufficiently large so that

$$\|e^{-i(t-t_0)\mathcal{L}}(f-f_k)\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} \leq C\|f-f_k\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)} \leq \frac{\delta}{2}.$$

Therefore choose $k_0(T)$ so large such that

$$\|e^{-i(t-t_0)\mathcal{L}}f_k\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} \leq \delta \tag{3.14}$$

for $k \geq k_0(T)$.

Let u and $\tilde{u_k}$ are solutions corresponding to initial values f and f_k at time t_0 respectively for $k\geq 1$. In view of estimates (3.9) and (3.10), $u, \tilde{u_k}$ will satisfy following estimates

$$\|u\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} \leq 2\delta \tag{3.15}$$

$$\|u\|_{L^q\left((t_0,t_0+T),\tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)\right)} < \infty \tag{3.16}$$

$$\sup_{k\geq k_0(T)} \|\tilde{u}_k\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} \leq 2\delta \tag{3.17}$$

$$\sup_{k\geq k_0(T)} \|\tilde{u}_k\|_{L^q\left(I_T,\tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)\right)} < \infty \tag{3.18}$$

where (q,p) be any admissible pair. Now from Strichartz estimates and Lemma 2.2, we see that

$$\begin{aligned}
\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)} &= \|\mathcal{H}u - \mathcal{H}\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)} \\
&\leq C\|f-f_k\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)} + C\|G(z,u)-G(z,\tilde{u_k})\|_{L^{\gamma'}(I_T,L^{\rho'})} \\
&\leq C\|f-f_k\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)} + C\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho(\mathbb{C}^n))} \times \\
&\quad \left(\|u\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)} + \|\tilde{u_k}\|_{L^\gamma\left(I_T,\tilde{W}_{\mathcal{L}}^{1,\rho}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}}.
\end{aligned}$$

From (3.5) and (3.9), we observe that

$$C\left(\|u\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}+\|\tilde{u_k}\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}}\leq\frac{1}{2}.$$

Therefore $\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)}\leq 2C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}\to 0$ as $k\to\infty$. Since $\{\tilde{u_k}\}$ is a bounded sequence in $L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, from Lemma 2.2 with $m=0$, $\|G(z,u(z,t))-G(z,\tilde{u_k}(z,t))\|_{L^{\gamma'}\left(I_T,L^{\rho'}(\mathbb{C}^n)\right)}\to 0$ as $j\to\infty$. Since $\mathcal{H}u=u,\mathcal{H}\tilde{u}_k=\tilde{u}_k$, from Strichartz estimates, $\|u-\tilde{u_k}\|_{L^q(I_T,L^p(\mathbb{C}^n))}\to 0$ as $k\to\infty$ for every admissible pair (q,p).

Note that $(\partial_{x_j}-\frac{iy_j}{2})=\frac{1}{2}(Z_j-\overline{Z}_j)$ and $(\partial_{y_j}+\frac{ix_j}{2})=\frac{i}{2}(Z_j+\overline{Z}_j)$. For $S=(\partial_{x_j}-\frac{iy_j}{2}),(\partial_{y_j}+\frac{ix_j}{2})$ and using the notation $\psi_{(k)}=\psi\left(z,|\tilde{u}_k(z,t)|\right)$ (see (4.17) in [6]), we have

$$\begin{aligned}S(G_{(k)}-G)&=\psi_{(k)}S(\tilde{u}_k-u)+(\psi_{(k)}-\psi)Su+(\partial_j\psi_{(k)})(\tilde{u}_k-u)\\&\quad+(\partial_j\psi_{(k)}-\partial_j\psi)u+(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\mathfrak{Re}\left(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}S(\tilde{u}_k-u)\right)\\&\quad+\left(\partial_{2n+1}\psi_{(k)}\right)\tilde{u}_k\mathfrak{Re}\left(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}Su\right)-(\partial_{2n+1}\psi)u\mathfrak{Re}\left(\frac{\overline{u}}{|u|}Su\right)\end{aligned}\tag{3.19}$$

where $\partial_j=\partial_{x_j}$ for $S=(\partial_{x_j}-\frac{iy_j}{2})$ and $\partial_j=\partial_{y_j}$ for $S=(\partial_{y_j}+\frac{ix_j}{2}),\ 1\leq j\leq n$.

Using the assumption (1.4) on ψ, Lemma 2.1, and by similar computations as used in Lemma 2.2 and Proposition 2.6, we have

$$\|\psi_{(k)}S(\tilde{u}_k-u)\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}\leq C\|S(\tilde{u}_k-u)\|_{L^\gamma(I_T,L^\rho)}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}$$

$$\|(\partial_j\psi_{(k)})(\tilde{u}_k-u)\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}\leq C\|\tilde{u}_k-u\|_{L^\gamma(I_T,L^\rho)}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}$$

$$\begin{aligned}&\|(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\mathfrak{Re}(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}S(\tilde{u}_k-u))\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}\\&\qquad\leq C\|S(\tilde{u}_k-u)\|_{L^\gamma(I_T,L^\rho)}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}.\end{aligned}$$

Since $\|\tilde{u}_k-u\|_{L^\gamma(I_T,L^\rho(\mathbb{C}^n))}\to 0$ and $\{\tilde{u}_k\}$ is a bounded sequence in $L^\gamma(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}})$, by second inequality in the above estimates, $(\partial_j\psi_{(k)})(\tilde{u}_k-u)\to 0$ as $k\to\infty$ in $L^{\gamma'}\left(I_T,L^{\rho'}(\mathbb{C}^n)\right)$. Since G is C^1, so in view of the condition (1.4) on ψ and Proposition 2.6, the sequences $(\psi_{(k)}-\psi)Su,,(\partial_j\psi_{(k)}-\partial_j\psi)u$ and $(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\mathfrak{Re}(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}Su)-(\partial_{2n+1}\psi)u\mathfrak{Re}(\frac{\overline{u}}{|u|}Su)$ converges to zero in $L^{\gamma'}(I_\tau,L^{\rho'})$ as $k\to\infty$. Using these observations in (3.19), we get

$$\|S(G_{(k)}-G)\|_{L^{\gamma'}(I_T,L^{\rho'})}\leq C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}\|S(\tilde{u}_k-u)\|_{L^\gamma(I_T,L^\rho(\mathbb{C}^n))}+a_k$$

where $S = (\partial_{x_j} - \frac{iy_j}{2}), (\partial_{y_j} + \frac{ix_j}{2})$ $(1 \leq j \leq n)$ and $a_k \to 0$ as $k \to \infty$. Since $(\partial_{x_j} - \frac{iy_j}{2}) = \frac{1}{2}(Z_j - \overline{Z}_j)$ and $(\partial_{y_j} + \frac{ix_j}{2}) = \frac{i}{2}(Z_j + \overline{Z}_j)$, we have

$$\|G_{(k)} - G\|_{L^{\gamma'}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho'})} \leq C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho}\right)} \|\tilde{u}_k - u\|_{L^{\gamma}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho})} + a_k. \tag{3.20}$$

Now from Strichartz estimates and above estimate, we have

$$\|\tilde{u}_k - u\|_{L^{\gamma}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho})} \leq C\|f_k - f\|_{\tilde{W}_{\mathcal{L}}^{1,2}} + C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho}\right)} \|\tilde{u}_k - u\|_{L^{\gamma}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho})} + a_k. \tag{3.21}$$

Now we choose $\delta > 0$ sufficiently small such that it satisfies condition (3.5) and

$$C(2\delta)^{\frac{2}{n-1}} \leq \frac{1}{2}$$

where constant C is appearing in the inequality (3.21). Note that T depends on δ through (3.13). Therefore from estimates (3.17) and (3.21), we have

$$\|\tilde{u}_k - u\|_{L^{\gamma}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho})} \leq 2C\|f_k - f\|_{\tilde{W}_{\mathcal{L}}^{1,2}} + 2a_k \to 0$$

as $k \to \infty$. Now from estimates (3.20), (3.15) and (3.17)

$$\|G_{(k)} - G\|_{L^{\gamma'}(I_T, \tilde{W}_{\mathcal{L}}^{1,\rho'})} \to 0$$

as $k \to \infty$. From Strichartz estimates, $\|\tilde{u}_k - u\|_{L^q(I_T, \tilde{W}_{\mathcal{L}}^{1,p})} = \|\mathcal{H}\tilde{u}_k - \mathcal{H}u\|_{L^q(I_T, \tilde{W}_{\mathcal{L}}^{1,p})} \to 0$ as $k \to \infty$ for every admissible pair (q, p).

Step 2 : Let $(T_{*,k}, T_k^*)$ be the maximal interval for the solutions $\tilde{u}_k$ and $I \subset (T_*, T^*)$ be a compact interval. The key idea is to extend the local stability result proved in step 1 to the interval I by covering it with finitely many intervals obtained by successive application of step 1. This is possible provided $\tilde{u}_k$ is defined on I, for all but finitely many k. In fact, we prove $I \subset (T_{*,k}, T_k^*)$ for all but finitely many k.

Without loss of generality, we assume that $t_0 \in I = [a, b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T_{k_m}^* \leq b$ and let $c = \liminf T_{k_m}^*$. Then for $\epsilon > 0$, $[t_0, c - \epsilon] \subset [t_0, T_{k_m}^*)$ for all k_m sufficiently large and $\tilde{u}_{k_m}$ are defined on $[t_0, c - \epsilon]$.

By compactness, the stability result proved in step 1 can be extended to the interval $[t_0, c - \epsilon]$.

For given $\delta > 0$, choose $\epsilon > 0$ sufficiently small such that

$$\begin{aligned}\|e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c-\epsilon) - e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon, c+\epsilon), \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\ \leq C\|u(\cdot, c-\epsilon) - u(\cdot, c)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \frac{\delta}{6} \\ \|e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c) - e^{-i(t-c)\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon, c+\epsilon), \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\ \leq C\|e^{-i\epsilon t\mathcal{L}}u(\cdot, c) - u(\cdot, c)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \frac{\delta}{6} \\ \|e^{-i(t-c)\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon, c+\epsilon), \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \leq \frac{\delta}{6}.\end{aligned}$$

Now we choose $k_0(\epsilon)$ such that following estimate holds for all $k \geq k_0$

$$\begin{aligned}\|e^{-i(t-(c-\epsilon))}\tilde{u}_{k_m}(\cdot, c-\epsilon) - e^{-i(t-(c-\epsilon))}u(\cdot, c-\epsilon)\|_{L^\gamma\left((c-\epsilon, c+\epsilon), \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\ \leq C\|\tilde{u}_{k_m}(\cdot, c-\epsilon) - u(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \tfrac{\delta}{2}.\end{aligned}$$

Therefore $\|e^{-i(t-(c-\epsilon))}\tilde{u}_{k_m}(\cdot, c-\epsilon)\|_{L^\gamma\left((c-\epsilon, c+\epsilon), \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \leq \delta$ for all $k_m \geq k_0$. Now by local existence argument (see (3.6)), $\tilde{u}_{k_m}$ is defined on $(t_0, c+\epsilon)$ and therefore $T^*_{k_m} \geq c+\epsilon$ for all $k_m \geq k_0$, hence contradicts the fact that $\liminf T^*_{k_m} = c$.

Similarly we can show that $[a, t_0] \subset (T_{*,k}, t_0]$ for all but finitely many k which completes the proof of stability. □

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