Indian J. Pure Appl. Math., **48**(1): 147-161, March 2017

DOI: 10.1007/s13226-017-0216-9

QUANTITATIVE UNCERTAINTY PRINCIPLES FOR THE SHORT TIME FOURIER TRANSFORM AND THE RADAR AMBIGUITY FUNCTION

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(*Received* 11 *April* 2016; *after final revision* 17 *July* 2016; *accepted* 30 *September* 2016)

Logarithmic uncertainty principle and Beckner's uncertainty principle in terms of entropy are proved for the short time Fourier transform and the radar ambiguity function, also a Heisenberg inequality for generalized dispersion and Price's local uncertainty principle are obtained.

Key words : Short time Fourier transform; Radar ambiguity function; uncertainty principle; time-frequency analysis; localization; logarithmic Beckner's theorem; entropy; generalized dispersion; Heisenberg inequality; Price's theorem; local uncertainty principle.

1. INTRODUCTION

The uncertainty principles in harmonic analysis state that a nonzero function f and its Fourier transform $\widehat{f}$ cannot be at the same time simultaneously and sharply localized, that is, it's impossible for a nonzero function and its Fourier transform to be simultaneously small. There are many formulations of this general fact where the smallness and the localization have been interpreted differently and by several ways. For more details about uncertainty principles, we refer the reader to [4, 8]. For an arbitrary function $f \in L^2(\mathbb{R}^d)$ and a nonzero function $g \in L^2(\mathbb{R}^d)$ called a window function, the short time Fourier transform (STFT) of f with respect to g is defined on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ by [6]

$$\mathcal{V}_g(f)(x,\omega) = \int_{\mathbb{R}^d} f(z)\overline{g(z-x)e^{i\langle z|\omega\rangle}}d\mu_d(z), \tag{1.1}$$

where $\langle . \,|\, . \rangle$ is the classical inner product on $\mathbb{R}^d$ defined by $\langle z|\omega\rangle = \sum_{i=1}^{d} z_i\omega_i$ and $d\mu_d(z) = \frac{dz}{(2\pi)^{\frac{d}{2}}}$ is the normalized Lebesgue measure.

The STFT plays an important role in time-frequency analysis namely by providing an interesting way to study the local frequency spectrum of signals. Relation (1.1) shows that unlike the classical Fourier transform, the STFT gives a simultaneous representation of the space and the frequency variables. In signal analysis, the short time Fourier transform is closely related to other common and known time frequency distributions as the radar ambiguity function defined on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, by [6]

$$\mathcal{A}(f)(x,\omega) = \int_{\mathbb{R}^d} f\left(z + \frac{x}{2}\right) \overline{f\left(z - \frac{x}{2}\right)} e^{-i\langle z|\omega\rangle} d\mu_d(z). \tag{1.2}$$

The radar ambiguity function occurs naturally in many radar applications, for more details about its physical aspect, we refer the reader to [6]. In fact, a standard change of variables shows that

$$\mathcal{A}(f)(x,\omega) = e^{\frac{i}{2}\langle x|\omega\rangle} \mathcal{V}_f(f)(x,\omega). \tag{1.3}$$

Roughly speaking, the uncertainty principles for the STFT say that for a given function $f \in L^2(\mathbb{R}^d)$, the STFT $\mathcal{V}_g(f)$ cannot be concentrated in the time-frequency plane. In this context, Lieb [14] proved an analogue of Donoho-Strak uncertainty principle for the STFT and the radar ambiguity function. We cite Fernandez, Galbis and Wilczok [5, 13] who studied the annihilating sets for the STFT. Other uncertainty principles have been also showed for the STFT namely by Grochening and Zimmerman [7] who established an analogue of Hardy and Benedick's theorem for the STFT. Heisenberg inequality, Cowling-price theorem as well as Gelfand-Shilov theorem have been also showed for the STFT and the radar ambiguity function by Bonami, Demange, and Jaming [1]. Recently Lamouchi and Omri [10] proved a quantitative version of Shapiro's and the umbrella theorems for the STFT.

Our purpose in this work is to prove two logarithmic uncertainty principles due to Beckner for both of the STFT and the radar ambiguity function. We also generalize the Heisenberg inequality proved by Bonami, Demange and Jaming [1] and we prove Price's local uncertainty principle for these two transforms.

More precisely, our first main result will be the logarithmic Beckner's uncertainty principle for the radar ambiguity function. Indeed for every nonzero function $f \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{A}(f) \in \mathcal{S}(\mathbb{R}^{2d})$

and

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|(x,\omega)||\mathcal{A}(f)(x,\omega)|^4 d\mu_{2d}(x,\omega) \geqslant \frac{1}{2}\left(\psi\left(\frac{d}{2}\right)+\ln 2\right)\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{A}(f)(x,\omega)|^4 d\mu_{2d}(x,\omega), \tag{1.4}$$

where ψ denotes the logarithmic derivative of Euler's function Γ. Next, we will prove an analogue of logarithmic Beckner's uncertainty principle in terms of entropy for the STFT and the radar ambiguity function, that is for every $f,g\in L^2(\mathbb{R}^d)$ such that g is nonzero, we have

$$E(|\mathcal{V}_g(f)|^2)\geqslant \|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}\left(d-\ln\left(\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}\right)\right).$$

As consequence of the uncertainty principle in terms of entropy, we will prove a Heisenberg type inequality for generalized dispersion by showing that for every positive real numbers p,q, there is a nonnegative constant $D_{p,q}$ such that for every $f,g\in L^2(\mathbb{R}^d)$, we have

$$\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\right)^{\frac{q}{p+q}}\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\right)^{\frac{p}{p+q}} \geqslant D_{p,q}\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}.$$

Finally, we will prove Price's uncertainty principle for the STFT and the radar ambiguity function, that is for every finite measurable subset Σ of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and for every $\xi,p\in\mathbb{R},\ 0<\xi<d,\ p>1$, there is a nonnegative constant $M_{\xi,p}$ such that for every $f,g\in L^2(\mathbb{R}^d)$ such that g is nonzero, we have

$$\int\int_{\Sigma}|\mathcal{V}_g(f)(x,\omega)|^p d\mu_{2d}(x,\omega) \leqslant M_{\xi,p}\left(\mu_{2d}(\Sigma)\right)^{\frac{1}{(p+1)}}\left\||(x,\omega)|^\xi\mathcal{V}_g(f)\right\|^{\frac{2pd}{(d+\xi)(p+1)}}_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\|f\|^{p-\frac{2pd}{(d+\xi)(p+1)}}_{2,\mathbb{R}^d}\|g\|^{p-\frac{2pd}{(d+\xi)(p+1)}}_{2,\mathbb{R}^d}.$$

2. The Short Time Fourier Transform

According to relation (1.1), fix a window function $g\in L^2(\mathbb{R}^d)$, for every $f\in L^2(\mathbb{R}^d)$ the short time Fourier transform of f with respect to g, is defined on the time-frequency plane $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ by [6]

$$\mathcal{V}_g(f)(x,\omega)=\int_{\mathbb{R}^d}f(z)\overline{g(z-x)e^{i\langle z|\omega\rangle}}d\mu_d(z).$$

For every $x,\omega\in\mathbb{R}^d$, we denote by $\mathcal{M}_\omega$ and $\mathcal{T}_x$ the modulation and the translation operators defined respectively on $L^2(\mathbb{R}^d)$ by

$$\forall z\in\mathbb{R}^d,\ \mathcal{M}_\omega h(z)=e^{i\langle z|\omega\rangle}h(z), \tag{2.1}$$

and

$$\forall z \in \mathbb{R}^d,\ \mathcal{T}_x h(z) = h(z-x). \tag{2.2}$$

Then by relations (2.1) and (2.2), we deduce that

$$\forall z \in \mathbb{R}^d,\ \mathcal{M}_\omega(\mathcal{T}_x h)(z) = e^{i\langle z|\omega\rangle} h(z-x), \tag{2.3}$$

and

$$\forall z \in \mathbb{R}^d,\ \mathcal{T}_x(\mathcal{M}_\omega h)(z) = e^{-i\langle x|\omega\rangle} e^{i\langle z|\omega\rangle} h(z-x). \tag{2.4}$$

Again by relation (2.2), the STFT may be expressed as

$$\mathcal{V}_g(f)(x,\omega) = \widehat{f\overline{\mathcal{T}_x g}}(\omega). \tag{2.5}$$

and by relations (1.1) and (2.1), we have

$$\begin{aligned}\mathcal{V}_g(f)(x,\omega) &= \overline{\int_{\mathbb{R}^d} \overline{f(z)} e^{i\langle z|\omega\rangle} g(z-x) d\mu_d(z)} \\ &= \overline{\mathcal{M}_\omega \overline{f} * g(x)},\end{aligned}$$

where $*$ denotes the usual convolution product on $\mathbb{R}^d$.

It's known [6] that for every $f, g \in L^2(\mathbb{R}^d)$ the STFT $\mathcal{V}_g(f)$ is uniformly continuous and bounded on the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and satisfies

$$\|\mathcal{V}_g(f)\|_{\infty, \mathbb{R}^d \times \widehat{\mathbb{R}}^d} \leqslant \|f\|_{2,\mathbb{R}^d} \|g\|_{2,\mathbb{R}^d}. \tag{2.6}$$

Moreover according to [6], for all $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ the functions $\mathcal{V}_{g_1}(f_1)$ and $\mathcal{V}_{g_2}(f_2)$ belong to $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and we have the following orthogonality relation

$$\langle \mathcal{V}_{g_1}(f_1) | \mathcal{V}_{g_2}(f_2) \rangle_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} = \langle f_1 | f_2 \rangle_{\mathbb{R}^d} \overline{\langle g_1 | g_2 \rangle_{\mathbb{R}^d}}. \tag{2.7}$$

In particular, for every $f, g \in L^2(\mathbb{R}^d)$ we get

$$\|\mathcal{V}_g(f)\|_{2, \mathbb{R}^d \times \widehat{\mathbb{R}}^d} = \|f\|_{2,\mathbb{R}^d} \|g\|_{2,\mathbb{R}^d}. \tag{2.8}$$

Given $f, g \in L^2(\mathbb{R}^d)$ and $\xi, \lambda, y, z \in \mathbb{R}^d$, then by relations (2.3) and (2.4) we have

$$\begin{aligned}\mathcal{V}_{M_\xi \mathcal{T}_z g}(M_\lambda \mathcal{T}_y f)(x,\omega) &= \int_{\mathbb{R}^d} (M_\lambda \mathcal{T}_y f)(u) \overline{\mathcal{T}_x\left(M_\xi \mathcal{T}_z g\right)(u)} e^{-i\langle \omega|u\rangle} d\mu_d(u) \\ &= e^{i\langle x|\xi\rangle} \int_{\mathbb{R}^d} f(u-y) \overline{g(u-(x+z))} e^{-i\langle u|\omega-\lambda+\xi\rangle} d\mu_d(u) \\ &= e^{i\langle x|\xi\rangle} e^{-i\langle y|\omega-\lambda+\xi\rangle} \int_{\mathbb{R}^d} f(t) \overline{g(t-(x-y+z))} e^{-i\langle t|\omega-\lambda+\xi\rangle} d\mu_d(t) \\ &= e^{i\langle x|\xi\rangle} e^{-i\langle y|\omega-\lambda+\xi\rangle} \mathcal{V}_g(f)(x-y+z, \omega-\lambda+\xi).\end{aligned} \tag{2.9}$$

Relation (2.9) shows in particular that for every $(y,\lambda),(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d$

$$\mathcal{V}_{M_\lambda \mathcal{T}_y g}(M_\lambda \mathcal{T}_y f)(x,\omega)=e^{i\langle x|\lambda\rangle}e^{-i\langle y|\omega\rangle}\mathcal{V}_g(f)(x,\omega). \tag{2.10}$$

We have also the following switching property

$$\begin{aligned}\mathcal{V}_f(g)(x,\omega) &= \int_{\mathbb{R}^d} g(z)\overline{f(z-x)}e^{-i\langle z|\omega\rangle}d\mu_d(z)\\ &= e^{-i\langle x|\omega\rangle}\overline{\int_{\mathbb{R}^d} f(y)\overline{g(y+x)}e^{-i\langle y|-\omega\rangle}}d\mu_d(z)\\ &= e^{-i\langle x|\omega\rangle}\overline{\mathcal{V}_g(f)(-x,-\omega)}. \end{aligned}\tag{2.11}$$

Let λ be a positive real number, for every measurable function f, we denote by f_λ the dilate of f defined by $f_\lambda(x)=f(\lambda x)$, then for every $f\in L^2(\mathbb{R}^d)$, f_λ belongs to $L^2(\mathbb{R}^d)$ and by a standard computation we deduce that $\mathcal{V}_g(f_\lambda)$ is given by

$$\mathcal{V}_g(f_\lambda)(x,\omega)=\frac{1}{\lambda^d}\mathcal{V}_{g_{\frac{1}{\lambda}}}(f)(\lambda x,\frac{\omega}{\lambda}). \tag{2.12}$$

Finally given two positive real numbers a,b and let f and g be the gaussian functions defined respectively by $f(x)=(4a)^{\frac{d}{4}}e^{-a|x|^2}$ and $g(x)=(4b)^{\frac{d}{4}}e^{-b|x|^2}$, then by a standard calculus we have $\|f\|_{2,\mathbb{R}^d}=\|g\|_{2,\mathbb{R}^d}=1$ and for every $(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d$

$$\begin{aligned}\mathcal{V}_g(f)(x,\omega) &= (16ab)^{\frac{d}{4}}\int_{\mathbb{R}^d} e^{-a|z|^2}e^{-b|z-x|^2}e^{-i\langle\omega|z\rangle}d\mu_d(z)\\ &= (16ab)^{\frac{d}{4}}e^{-b|x|^2}\int_{\mathbb{R}^d} e^{-(a+b)|z|^2}e^{2b\langle z|x\rangle}e^{-i\langle\omega|z\rangle}d\mu_d(z)\\ &= (16ab)^{\frac{d}{4}}e^{-b|x|^2}e^{\frac{b^2}{a+b}|x|^2}\int_{\mathbb{R}^d} e^{-(a+b)\left(|z|^2-2\langle z|\frac{b}{a+b}x\rangle+|\frac{b}{a+b}x|^2\right)}e^{-i\langle\omega|z\rangle}d\mu_d(z)\\ &= (16ab)^{\frac{d}{4}}e^{\frac{-ab}{a+b}|x|^2}\int_{\mathbb{R}^d} e^{-(a+b)|z-\frac{b}{a+b}x|^2}e^{-i\langle\omega|z\rangle}d\mu_d(z)\\ &= (16ab)^{\frac{d}{4}}e^{\frac{-ab}{a+b}|x|^2}e^{-i\langle\omega|\frac{b}{a+b}x\rangle}\int_{\mathbb{R}^d} e^{-(a+b)|y|^2}e^{-i\langle\omega|y\rangle}d\mu_d(z). \end{aligned}$$

Hence,

$$\mathcal{V}_g(f)(x,\omega)=\frac{(4ab)^{\frac{d}{4}}}{(a+b)^{\frac{d}{2}}}e^{\frac{-ab}{a+b}|x|^2}e^{-i\langle\omega|\frac{b}{a+b}x\rangle}e^{-\frac{|\omega|^2}{4(a+b)}}. \tag{2.13}$$

In particular by relation (2.13) we get

$$|\mathcal{V}_g(f)(x,\omega)|^2 = \frac{(4ab)^{\frac{d}{2}}}{(a+b)^d} e^{\frac{-2ab}{a+b}|x|^2} e^{-\frac{|\omega|^2}{2(a+b)}}. \tag{2.14}$$

3. UNCERTAINTY PRINCIPLES FOR THE STFT

In [2] Beckner used Stein-Weiss and Pitt's inequalities to obtain a logarithmic estimate of the uncertainty, he showed that for every $f \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}^d} \ln|y||\widehat{f}(y)|^2 dy \geqslant \left(\psi\left(\frac{d}{4}\right) + \ln 2\right) \int_{\mathbb{R}^d} |f(x)|^2 dx, \tag{3.1}$$

where $\widehat{f}$ is the classical Fourier transform defined on $\mathbb{R}^d$ by

$$\forall y \in \mathbb{R}^d,\ \widehat{f}(y) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x|y\rangle} d\mu_d(x),$$

and ψ is the logarithmic derivative of Γ function defined by $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$.

The previous inequality is known as Beckner's logarithmic uncertainty principle. In the following we are interested to generalize inequality (3.1) to the radar ambiguity function $\mathcal{A}(f)$ for $f \in \mathcal{S}(\mathbb{R}^d)$.

***Theorem* 3.1** — *Let $f \in \mathcal{S}(\mathbb{R}^d)$ be a nonzero function, then $\mathcal{A}(f) \in \mathcal{S}(\mathbb{R}^{2d})$, and we have*

$$\begin{aligned}\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} &\ln|(x,\omega)||\mathcal{A}(f)(x,\omega)|^4 d\mu_{2d}(x,\omega)\\ &\geqslant \frac{1}{2}\left(\psi\left(\frac{d}{2}\right) + \ln 2\right) \int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathcal{A}(f)(x,\omega)|^4 d\mu_{2d}(x,\omega),\end{aligned} \tag{3.2}$$

where ψ denotes the logarithmic derivative of Euler's function Γ.

PROOF : Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, then the function $h(x,z) = f(z)\overline{g(z-x)}$ belongs to $\mathcal{S}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Since $\mathcal{S}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is invariant under partial Fourier transform then by relation (2.5) we deduce that $\mathcal{V}_g(f) \in \mathcal{S}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Let ϕ be the function defined on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ by

$$\phi(x,\omega) = \mathcal{V}_g(f)(x,\omega)\overline{\mathcal{V}_f(g)(x,\omega)},$$

then $\phi \in \mathcal{S}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and according to relations (2.7), (2.10) and (2.11), we have

$$\begin{aligned}
\widehat{\phi}(y,\lambda) &= \int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} \overline{\mathcal{V}_f(g)(x,\omega)}\mathcal{V}_g(f)(x,\omega)e^{-i\langle x|y\rangle}e^{-i\langle \omega|\lambda\rangle}d\mu_{2d}(x,\omega)\\
&= \int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} \overline{\mathcal{V}_f(g)(x,\omega)}\mathcal{V}_g(f)(x,\omega)e^{i(\langle x|-y\rangle-\langle \omega|\lambda\rangle)}d\mu_{2d}(x,\omega)\\
&= \int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} \mathcal{V}_{M_{-y}\mathcal{T}_\lambda g}(M_{-y}\mathcal{T}_\lambda f)(x,\omega)\overline{\mathcal{V}_f(g)(x,\omega)}d\mu_{2d}(x,\omega)\\
&= \langle \mathcal{V}_{M_{-y}\mathcal{T}_\lambda g}(M_{-y}\mathcal{T}_\lambda f)|\mathcal{V}_f(g)\rangle_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\\
&= \langle M_{-y}\mathcal{T}_\lambda f|g\rangle_{\widehat{\mathbb{R}}^d}\overline{\langle M_{-y}\mathcal{T}_\lambda g|f\rangle_{\mathbb{R}^d}}\\
&= \left(\int_{\mathbb{R}^d}\overline{g(z)}f(z-\lambda)e^{-i\langle y|z\rangle}d\mu_d(z)\right)\overline{\left(\int_{\mathbb{R}^d}\overline{f(z)}g(z-\lambda)e^{-i\langle y|z\rangle}d\mu_d(z)\right)}\\
&= \overline{\left(\int_{\mathbb{R}^d}g(z)\overline{f(z-\lambda)}e^{-i\langle -y|z\rangle}d\mu_d(z)\right)}\left(\int_{\mathbb{R}^d}f(z)\overline{g(z-\lambda)}e^{-i\langle -y|z\rangle}d\mu_d(z)\right)\\
&= \mathcal{V}_g(f)(\lambda,-y)\overline{\mathcal{V}_f(g)(\lambda,-y)},
\end{aligned}$$

and therefore

$$\widehat{\phi}(y,\lambda) = \phi(-\lambda, y). \tag{3.3}$$

Now, assume that $\|f\|_{2,\mathbb{R}^d} = \|g\|_{2,\mathbb{R}^d} = 1$, then by combining relations (2.8), (3.1) and (3.3), we deduce that

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|(x,\omega)||\mathcal{V}_g(f)(x,\omega)|^2|\mathcal{V}_f(g)(x,\omega)|^2d\mu_{2d}(x,\omega)\\
&\qquad\geqslant \frac{1}{2}\left(\psi\left(\frac{d}{2}\right)+\ln 2\right)\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{V}_g(f)(x,\omega)|^2|\overline{\mathcal{V}_f(g)(x,\omega)}|^2d\mu_{2d}(x,\omega).
\end{aligned}$$

Hence by relation (1.3), we get

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|(x,\omega)||\mathcal{A}(f)(x,\omega)|^4d\mu_{2d}(x,\omega)\\
&\qquad\geqslant \frac{1}{2}\left(\psi\left(\frac{d}{2}\right)+\ln 2\right)\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{A}(f)(x,\omega)|^4d\mu_{2d}(x,\omega).
\end{aligned}$$

□

According to Shannon [12], the entropy of a probability density function ρ on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is defined by

$$E(\rho) = -\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\ln(\rho(x,\omega))\rho(x,\omega)d\mu_{2d}(x,\omega),$$

whenever the integral on the right hand side is well defined. The entropy plays an important role in quantum mechanics and in signal theory, for a better understanding of its physical's signification we refer the reader to [3]. Clearly the entropy represents an advantageous way to measure the decay of a function f, so that it was very interesting to localize the entropy of a probability measure and its Fourier transform. In this context, the first estimation has been given by Hirschman [9] and has been improved by Beckner [2] who used the Hausdorff-Young inequality to derive the following uncertainty inequality that is for every $f \in L^2(\mathbb{R}^d)$ with $\|f\|_{2,\mathbb{R}^d} = 1$, we have

$$E(|f|^2) + E(|\widehat{f}|^2) \geqslant d(1 - \ln 2),$$

whenever the left side is well defined. The aim of the following is to generalize the localization of the entropy to the STFT over the time-frequency plane and also to the radar ambiguity function.

***Theorem* 3.2** — *Let g be a window function and $f \in L^2(\mathbb{R}^d)$, then*

$$E(|\mathcal{V}_g(f)|^2) \geqslant \|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}\left(d - \ln\left(\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}\right)\right), \tag{3.4}$$

and the inequality (3.4) is sharp.

PROOF : Assume that $\|f\|_{2,\mathbb{R}^d} = \|g\|_{2,\mathbb{R}^d} = 1$ and following the idea of Lieb [11], then by relation (2.6) we deduce that

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ |\mathcal{V}_g(f)(x,\omega)| \leqslant \|\mathcal{V}_g(f)\|_{\infty,\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \leqslant \|f\|_{2,\mathbb{R}^d}\|g\|_{2,\mathbb{R}^d} = 1. \tag{3.5}$$

In particular $E(\mathcal{V}_g(f)) \geqslant 0$ and therefore if the entropy $E(\mathcal{V}_g(f))$ is infinite then the inequality (3.4) holds trivially. Suppose now that the entropy $E(\mathcal{V}_g(f))$ is finite and let $0 < x < 1$ and h be the function defined on $]2,3]$ by

$$h(p) = \frac{x^p - x^2}{p-2},$$

then

$$\forall p \in]2,3[,\ h'(p) = \frac{(p-2)x^p\ln(x) - (x^p - x^2)}{(p-2)^2} \geqslant 0,$$

and therefore h is increasing on $]2,3]$, in particular

$$\forall p \in]2,3],\ x^2\ln(x) = \lim_{p\to 2^+}\frac{x^p - x^2}{p-2} \leqslant \frac{x^p - x^2}{p-2},$$

hence

$$0 \leqslant \frac{x^2 - x^p}{p-2} \leqslant -x^2\ln(x), \tag{3.6}$$

and by combining relations (3.5) and (3.6) we get, for every $(x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$

$$0 \leqslant \frac{|\mathcal{V}_g(f)(x,\omega)|^2 - |\mathcal{V}_g(f)(x,\omega)|^p}{p-2} \leqslant -|\mathcal{V}_g(f)(x,\omega)|^2\ln(|\mathcal{V}_g(f)(x,\omega)|). \tag{3.7}$$

Let φ be the function defined on $[2,+\infty[$ by

$$\varphi(p)=\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{V}_g(f)(x,\omega)|^p\,d\mu_{2d}(x,\omega)\right)-\left(\frac{2}{p}\right)^d.$$

According to Lieb [11], we know that for every $2\leqslant p<+\infty$ the STFT $\mathcal{V}_g(f)$ belongs to $L^p(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ and we have

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{V}_g(f)(x,\omega)|^p\,d\mu_{2d}(x,\omega)\leqslant\left(\frac{2}{p}\right)^d\|f\|^p_{2,\mathbb{R}^d}\|g\|^p_{2,\mathbb{R}^d}. \tag{3.8}$$

Then, relation (3.8) implies that $\varphi(p)\leqslant 0$ for every $p\in[2,+\infty[$ and by Plancherel theorem (2.8) we have $\varphi(2)=0$. Therefore $\left(\frac{d\varphi}{dp}\right)_{p=2^+}\leqslant 0$ whenever this derivative is well defined. However, by using relation (3.7) and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}&\left(\frac{d}{dp}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\mathcal{V}_g(f)(x,\omega)|^p\,d\mu_{2d}(x,\omega)\right)_{p=2^+}\\&=-\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\lim_{p\to 2^+}\frac{|\mathcal{V}_g(f)(x,\omega)|^2-|\mathcal{V}_g(f)(x,\omega)|^p}{p-2}d\mu_{2d}(x,\omega)\\&=-\frac{1}{2}E(|\mathcal{V}_g(f)|^2)\end{aligned}$$

and consequently

$$\left(\frac{d\varphi}{dp}\right)_{p=2^-}=-\frac{1}{2}E(|\mathcal{V}_g(f)|^2)+\frac{d}{2},$$

which gives

$$E(|\mathcal{V}_g(f)|^2)\geqslant d.$$

Let $f(x)=(4a)^{\frac{d}{4}}e^{-a|x|^2}$ and $g(x)=(4b)^{\frac{d}{4}}e^{-b|x|^2}$ with $a,b>0$ then $\|f\|_{2,\mathbb{R}^d}=\|g\|_{2,\mathbb{R}^d}=1$ and according to relation (2.14) we have for every $(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d$

$$|\mathcal{V}_g(f)(x,\omega)|^2=\frac{(4ab)^{\frac{d}{2}}}{(a+b)^d}e^{\frac{-2ab}{a+b}|x|^2}e^{-\frac{|\omega|^2}{2(a+b)}}.$$

Therefore by a standard calculus and using relation (2.14), we get

$$E(|\mathcal{V}_g(f)|^2)=d\left(1-\ln\left(\frac{2\sqrt{ab}}{a+b}\right)\right).$$

In particular $E(|\mathcal{V}_g(f)|^2)=d$ if, and only if $a=b$. □

Corollaire 3.3 — Let $f\in L^2(\mathbb{R}^d)$, then

$$E(|\mathcal{A}(f)|^2)\geqslant\|f\|^4_{2,\mathbb{R}^d}\left(d-\ln\left(\|f\|^4_{2,\mathbb{R}^d}\right)\right), \tag{3.9}$$

and the inequality (3.4) is sharp.

In [1] Bonami, Demange and Jaming showed a Heisenberg type inequality for the radar ambiguity function with respect to the second dispersion and as noticed above the same inequality holds obviously for the STFT. The authors showed that given a window function $g \in L^2(\mathbb{R}^d)$ then for every $f \in L^2(\mathbb{R}^d)$ we have

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |x|^2|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\omega|^2|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \geqslant d^2\|f\|^4_{2,\mathbb{R}^d}\|g\|^4_{2,\mathbb{R}^d}$$

In what follows we shall use Theorem 3.2 to generalize the previous Heisenberg uncertainty principle for generalized dispersions.

***Theorem* 3.4** — *Let p and q be two positive real numbers. Then there exists a nonnegative constant $D_{p,q}$ such that for every window function g and for every function $f \in L^2(\mathbb{R}^d)$ we have*

$$\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |x|^p|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\right)^{\frac{q}{p+q}} \left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\right)^{\frac{p}{p+q}} \geqslant D_{p,q}\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}, \tag{3.10}$$

where $D_{p,q} = \dfrac{d}{p^{\frac{q}{p+q}}q^{\frac{p}{p+q}}} e^{\frac{pq\left(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)\right)}{d(p+q)}-1}$.
Moreover, for $p = q = 2$ inequality (3.10) is sharp.

PROOF : Assume that $\|f\|_{2,\mathbb{R}^d} = \|g\|_{2,\mathbb{R}^d} = 1$ and let $\xi_{t,p,q}$ be the function defined on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ by $\xi_{t,p,q}(x,\omega) = \dfrac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\dfrac{e^{-\frac{|x|^p+|\omega|^q}{t}}}{t^{\frac{d}{p}+\frac{d}{q}}}$, so by basic calculus we see that

$$\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \xi_{t,p,q}(x,\omega)d\mu_{2d}(x,\omega) = 1,$$

in particular $d\sigma_{t,p,q}(x,\omega) = \xi_{t,p,q}(x,\omega)d\mu_{2d}(x,\omega)$ is a probability measure on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Since the function $\varphi(t) = t\ln(t)$ is convex over $]0,+\infty[$, hence according to Jensen's inequality

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \frac{|\mathcal{V}_g(f)(x,\omega)|^2}{\xi_{t,p,q}(x,\omega)} \ln\left(\frac{|\mathcal{V}_g(f)(x,\omega)|^2}{\xi_{t,p,q}(x,\omega)}\right) d\sigma_{t,p,q}(x,\omega) \geqslant 0,$$

which implies in terms of entropy that for every positive real number t

$$E(|\mathcal{V}_g(f)|^2)+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right) \leqslant \ln(t^{\frac{d}{p}+\frac{d}{q}})+\frac{1}{t}\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}(|x|^p+|\omega|^q)|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega),$$

and by means of Theorem 3.2

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}(|x|^p+|\omega|^q)|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\geqslant t\left(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)-\ln(t^{\frac{d}{p}+\frac{d}{q}})\right).$$

However the expression $t(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)-\ln(t^{\frac{d}{p}+\frac{d}{q}}))$ attains its upper bound at $t_0=e^{\frac{pq\left(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)\right)}{d(p+q)}-1}$, and consequently

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}(|x|^p+|\omega|^q)|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)\geqslant C_{p,q},$$

where

$$C_{p,q}=\frac{d(p+q)}{pq}e^{\frac{pq\left(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)\right)}{d(p+q)}-1}.$$

Therefore for every window function g and for every function $f\in L^2(\mathbb{R}^d)$, we get

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)+\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)$$
$$\geqslant C_{p,q}\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}. \tag{3.11}$$

Now for every positive real number λ the dilates f_λ and g_λ belong to $L^2(\mathbb{R}^d)$ and g_λ is nonzero, then by relation (3.11), we have

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_{g_\lambda}(f_\lambda)(x,\omega)|^2 d\mu_{2d}(x,\omega)+\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_{g_\lambda}(f_\lambda)(x,\omega)|^2 d\mu_{2d}(x,\omega)$$
$$\geqslant C_{p,q}\|f_\lambda\|^2_{2,\mathbb{R}^d}\|g_\lambda\|^2_{2,\mathbb{R}^d}$$

hence for every positive real number λ

$$\lambda^{-p}\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)+\lambda^q\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)$$
$$\geqslant C_{p,q}\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}.$$

In particular, the inequality holds at the critical point

$$\lambda=\left(\frac{p\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)}{q\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega)}\right)^{\frac{1}{p+q}},$$

which implies that

$$\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega)\right)^{\frac{q}{p+q}}\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega)\right)^{\frac{p}{p+q}} \geqslant D_{p,q}\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d},$$

where

$$D_{p,q}=C_{p,q}\frac{p^{\frac{p}{p+q}}q^{\frac{q}{p+q}}}{p+q}=\frac{d}{p^{\frac{q}{p+q}}q^{\frac{p}{p+q}}}e^{\frac{pq\left(d+\ln\left(\frac{2^{d-2}pq\Gamma(\frac{d}{2})^2}{\Gamma(\frac{d}{p})\Gamma(\frac{d}{q})}\right)\right)}{d(p+q)}-1}.$$

In the particular case when $p=q=2$, we get

$$\|x\mathcal{V}_g(f)(x,\omega)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\|\omega\mathcal{V}_g(f)(x,\omega)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\geqslant d\|f\|^2_{2,\mathbb{R}^d}\|g\|^2_{2,\mathbb{R}^d}. \tag{3.12}$$

We will show now that relation (3.12) is sharp, indeed let $f(x)=2^{\frac{d}{4}}e^{-\frac{|x|^2}{2}}$ then, according to relation (2.14) we have

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^2|\mathcal{V}_f(f)(x,\omega)|^2d\mu_{2d}(x,\omega)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|x|^pe^{\frac{-2ab}{a+b}|x|^2}dx\int_{\widehat{\mathbb{R}}^d}e^{-\frac{|\omega|^2}{2(a+b)}}d\omega$$
$$=d$$

and

$$\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^2|\mathcal{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{\frac{-2ab}{a+b}|x|^2}dx\int_{\widehat{\mathbb{R}}^d}|\omega|^qe^{-\frac{|\omega|^2}{2(a+b)}}d\omega$$
$$=d$$

knowing that $\|f\|_{2,\mathbb{R}^d}=1$, we deduce that

$$\|x\mathcal{V}_f(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\|\omega\mathcal{V}_f(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}=d\|f\|^4_{2,\mathbb{R}^d}.\square$$

Corollary 3.5 — Let p and q be two positive real numbers then there exists a nonnegative constant $D_{p,q}$ such that for every function $f\in L^2(\mathbb{R}^d)$, we have

$$\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|x|^p|\mathcal{A}(f)(x,\omega)|^2d\mu_{2d}(x,\omega)\right)^{\frac{q}{p+q}}\left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^q|\mathcal{A}(f)(x,\omega)|^2d\mu_{2d}(x,\omega)\right)^{\frac{p}{p+q}} \geqslant D_{p,q}\|f\|^4_{2,\mathbb{R}^d}. \tag{3.13}$$

Moreover, for $p=q=2$ inequality (3.10) is sharp.

The Heisenberg uncertainty principle proved above says that the STFT and the radar ambiguity function cannot be concentrated near the origin in the time-frequency plane but it does not claim whether the result remains true near several given points or more generally into a subset of the time-frequency plane with finite measure. In the following, we will prove through the local Price's inequality, that in fact the so called property remains true.

***Theorem* 3.6** — *Let ξ, p be two positive real numbers such that $0 < \xi < d$ and $p \geqslant 1$, then there is a nonnegative constant $M_{\xi,p}$ such that for every window function g, for every function $f \in L^2(\mathbb{R}^d)$ and for every finite measurable subset Σ of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, we have*

$$\begin{aligned}\int\int_{\Sigma} &|\mathcal{V}_g(f)(x,\omega)|^p d\mu_{2d}(x,\omega)\\ &\leqslant M_{\xi,p}\left(\mu_{2d}(\Sigma)\right)^{\frac{1}{(p+1)}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{2pd}{(d+\xi)(p+1)}} \|f\|_{2,\mathbb{R}^d}^{p-\frac{2pd}{(d+\xi)(p+1)}} \|g\|_{2,\mathbb{R}^d}^{p-\frac{2pd}{(d+\xi)(p+1)}}.\end{aligned}$$

PROOF : Without loss of generality we can assume that $\|f\|_{2,\mathbb{R}^d} = \|g\|_{2,\mathbb{R}^d} = 1$, then for every positive real number s, we have

$$\|\mathcal{V}_g(f)\|_{p,\Sigma} \leqslant \|\mathcal{V}_g(f)\mathbf{1}_{B_s}\|_{p,\Sigma} + \|\mathcal{V}_g(f)\mathbf{1}_{B_s^c}\|_{p,\Sigma},$$

where B_s denotes the ball of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ of radius s. However, by Hölder's inequality and relation (2.6) we get for every $0 < \xi < d$

$$\begin{aligned}\|\mathcal{V}_g(f)\mathbf{1}_{B_s}\|_{p,\Sigma} &= \left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathcal{V}_g(f)(x,\omega)|^p \mathbf{1}_{B_s}(x,\omega)\mathbf{1}_{\Sigma}(x,\omega)d\mu_{2d}(x,\omega)\right)^{\frac{1}{p}}\\ &\leqslant \|\mathcal{V}_g(f)\|_{\infty,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{p}{p+1}} \left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathcal{V}_g(f)(x,\omega)|^{\frac{p}{p+1}} \mathbf{1}_{B_s}(x,\omega)\mathbf{1}_{\Sigma}(x,\omega)d\mu_{2d}(x,\omega)\right)^{\frac{1}{p}}\\ &\leqslant \mu_{2d}(\Sigma)^{\frac{1}{p(p+1)}} \|\mathcal{V}_g(f)\mathbf{1}_{B_s}\|_{1,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}}\\ &\leqslant \mu_{2d}(\Sigma)^{\frac{1}{p(p+1)}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}} \||(x,\omega)|^{-\xi}\mathbf{1}_{B_s}\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}}\\ &\leqslant \frac{\mu_{2d}(\Sigma)^{\frac{1}{p(p+1)}}}{(2^d\Gamma(d)(d-\xi))^{\frac{1}{2(p+1)}}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}} s^{\frac{d-\xi}{p+1}}.\end{aligned}$$

On the other hand, and again by Hölder's inequality and relation (2.6), we deduce that

$$\begin{aligned}\|\mathcal{V}_g(f)\mathbf{1}_{B_s^c}\|_{p,\Sigma} &\leqslant \|\mathcal{V}_g(f)\|_{\infty,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{p-1}{p+1}} \left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathcal{V}_g(f)(x,\omega)|^{\frac{2p}{p+1}} \mathbf{1}_{B_s^c}(x,\omega)\mathbf{1}_{\Sigma}(x,\omega)d\mu_{2d}(x,\omega)\right)^{\frac{1}{p}}\\ &\leqslant (\mu_{2d}(\Sigma))^{\frac{1}{p(p+1)}} \left(\int\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathcal{V}_g(f)(x,\omega)|^2 \mathbf{1}_{B_s^c}(x,\omega)d\mu_{2d}(x,\omega)\right)^{\frac{1}{p+1}}\\ &\leqslant (\mu_{2d}(\Sigma))^{\frac{1}{p(p+1)}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{2}{p+1}} s^{-\frac{2\xi}{p+1}}\end{aligned}$$

Hence,

$$\left(\int\int_{\Sigma}|\mathcal{V}_g(f)(x,\omega)|^p d\mu_{2d}(x,\omega)\right)^{\frac{1}{p}} \leqslant (\mu_{2d}(\Sigma))^{\frac{1}{p(p+1)}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}}$$
$$\times\left(\frac{s^{\frac{d-\xi}{p+1}}}{(2^d\Gamma(d)(d-\xi))^{\frac{1}{2(p+1)}}} + \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}} s^{-\frac{2\xi}{p+1}}\right)$$

In particular the inequality holds for

$$s_0 = \left(\frac{2\xi\||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{1}{p+1}}(2^d\Gamma(d)(d-\xi))^{\frac{1}{2(p+1)}}}{d-\xi}\right)^{\frac{p+1}{d+\xi}},$$

and therefore

$$\left(\int\int_{\Sigma}|\mathcal{V}_g(f)(x,\omega)|^p d\mu_{2d}(x,\omega)\right)^{\frac{1}{p}} \leqslant (\mu_{2d}(\Sigma))^{\frac{1}{p(p+1)}} \||(x,\omega)|^{\xi}\mathcal{V}_g(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{2d}{(d+\xi)(p+1)}}$$
$$\times\left(\frac{d+\xi}{2^{\frac{\xi(d+2p+2)}{(d+\xi)(p+1)}}\xi^{\frac{2\xi}{d+\xi}}\Gamma(d)^{\frac{\xi}{(d+\xi)(p+1)}}(d-\xi)^{\frac{d-\xi}{d+\xi}+\frac{\xi}{(d+\xi)(p+1)}}}\right)$$

The proof is complete by applying the previous inequality to $\dfrac{f}{\|f\|_{2,\mathbb{R}^d}}$ and $\dfrac{g}{\|g\|_{2,\mathbb{R}^d}}$ for every nonzero functions $f, g \in L^2(\mathbb{R}^d)$. □

Corollary 3.7 — Let ξ, p be two positive real numbers such that $0 < \xi < d$ and $p \geqslant 1$, then there is a nonnegative constant $M_{\xi,p}$ such that for every function $f \in L^2(\mathbb{R}^d)$ and for every finite measurable subset Σ of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, we have

$$\int\int_{\Sigma}|\mathcal{A}(f)(x,\omega)|^p d\mu_{2d}(x,\omega)$$
$$\leqslant M_{\xi,p}\,(\mu_{2d}(\Sigma))^{\frac{1}{(p+1)}} \||(x,\omega)|^{\xi}\mathcal{A}(f)\|_{2,\mathbb{R}^d\times\widehat{\mathbb{R}}^d}^{\frac{2pd}{(d+\xi)(p+1)}} \|f\|_{2,\mathbb{R}^d}^{2p-\frac{4pd}{(d+\xi)(p+1)}}.$$

ACKNOWLEDGEMENT

The author wishes to thank the referee for its valuable comments.

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