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EXTRACTION OF ORTHOGONAL POLYNOMIALS FROM GENERATING FUNCTION FOR RECIPROCAL OF ODD NUMBERS

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In the present paper the orthogonality relations, exhibited by both numerator and denominator polynomials of both even and odd order convergents of a regular C-fraction of a power series with coefficients as reciprocal of odd numbers are described. The two sequences of convergents are nothing but diagonal and upper diagonal Pade approximants for the power series. The two orthogonal polynomials extracted from denominators are shown to be classical orthogonal polynomials and two orthogonal polynomials extracted from numerators are shown to be non-classical orthogonal polynomials..

Key words : Combinatorial enumeration problems; continued fractions and generalizations; continued fractions (function-theoretic results); orthogonal polynomials and functions of hypergeometric type; other special orthogonal polynomials and functions and Pade approximation.

1. INTRODUCTION

Pade approximants and continued fractions have a strong connection. A Pade approximant has a power series expansion which matches with the power series to be approximated as for as possible. This property completely defines the denominator as well as numerator of the Pade approximant under consideration [5]. The numerator and denominator sequences of polynomials of both even and odd order convergents of a C-fraction expansion corresponding to a power series expansions are naturally orthogonal polynomials. The two sequences of convergents are nothing but diagonal and upper diagonal Pade approximants for the power series. Orthogonal polynomials have very useful properties in the solution of mathematical and physical problems. They have relations with moment problems, rational approximation, operator theory, analytic functions, interpolation, quadrature, electrostatics, statistical quantum mechanics, number theory, graph theory, combinatorics, random matrices, stochastic process and etc. [13].

There is a very interesting literature [2, 3] which interprets that $[n-1/n]$ and $[n/n]$ order Pade approximant provides an orthogonality relation between its denominator and numerator polynomials and the power series expansion. They are nothing but even and odd order convergents [2, 3, 9, 15] of a regular C-fraction expansion of the power series expansion. The denominator and numerator polynomials transformed to monic form are orthogonal polynomials with respect to a linear moment functional $L : \mathbb{P} \longrightarrow \mathbb{R}$ from the space of all polynomials over $\mathbb{R}$ into $\mathbb{R}$ which has nth moment same as the coefficient of x^n in a known power series [11, 12].

According to Favard's theorem [4, 7, 8, 10] the necessary and sufficient condition for orthogonality of $P_n(x)$ is to satisfy the following three term recurrence relation:

$$\begin{aligned} P_{-1}(x) &:= 0, \qquad P_0(x) := 1, \\ P_n(x) &:= (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),\ n = 1, 2, 3, 4, \dots, \end{aligned} \tag{1}$$

where c_n 's are real and λ_n 's are non-zero numbers. The orthogonality relation [7, 8, 10] is given by

$$L\{P_m(x)P_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n. \end{cases} \tag{2}$$

Motivated strongly by the above works, in the present paper, the orthogonality relations, exhibited by both numerator and denominator polynomials of both even and odd order convergents of a regular C-fraction of a series with coefficient as reciprocal of odd numbers connected to Pade approximants. In Section two and three, we compute four sequences of orthogonal polynomials. In the last Section, the two orthogonal polynomials extracted from denominators are shown to be classical orthogonal polynomials and two orthogonal polynomials extracted from numerators are shown to be non-classical orthogonal polynomials.

2. Computation of Desired Orthogonal Polynomials

The series with coefficient as reciprocal of odd number is given by

$$T(x) = 1 + \frac{1}{3}\,x + \frac{1}{5}\,x^2 + \frac{1}{7}\,x^3 + \cdots + \frac{1}{2n+1}\,x^n + \cdots\ ,$$

which has the regular C-fraction [2, 9]

$$T(x) = \frac{1}{1}_{+}\ \frac{\frac{-1}{3}\,x}{1}_{+}\ \frac{\frac{-4}{15}\,x}{1}_{+}\ \frac{\frac{-9}{35}\,x}{1}_{+\cdots+}\ \frac{\frac{-n^2}{(2n-1)(2n+1)}\,x}{1}_{+\cdots}\ . \tag{3}$$

It has a a remarkable hypergeometric representation

$$T(x) = {}_2F_1\left(\frac{1}{2},\ 1; \frac{3}{2}\ ; x\right)$$

In the standard notation of hypergeometric series [6, 14] is

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_0 = 1, (a)_n = a(a+1)\cdots(a+n-1)$.

In the context of Pade table [2, 3], the continued fraction provides a staircase sequence of Pade approximants

$$[0/0]_{T(x)}, [0/1]_{T(x)}, [1/1]_{T(x)}, [1/2]_{T(x)}, [2/2]_{T(x)}, \ldots, [n-1/n]_{T(x)}, [n/n]_{T(x)}, \ldots,$$

which are given by

$$\frac{A_1}{B_1} = \frac{1}{1} = \frac{P_0^{(0,0)}}{Q_0^{(0,0)}},\ \ \frac{A_3}{B_3} = \frac{1-\frac{4}{15}x}{1-\frac{3}{5}x} = \frac{P_1^{(1,1)}}{Q_1^{(1,1)}}, \ldots, \frac{A_{2n+1}}{B_{2n+1}} = \frac{P_n^{(n,n)}}{Q_n^{(n,n)}}$$

and

$$\frac{A_2}{B_2} = \frac{1}{1-\frac{1}{3}x} = \frac{P_0^{(0,1)}}{Q_0^{(0,1)}}, \frac{A_4}{B_4} = \frac{1-\frac{11}{21}x}{1-\frac{6}{7}x+\frac{3}{35}x^2} = \frac{P_1^{(1,2)}}{Q_1^{(1,2)}}, \ldots, \frac{A_{2n+2}}{B_{2n+2}} = \frac{P_n^{(n-1,n)}}{Q_n^{(n-1,n)}}.$$

Now, we compute even and odd order convergents of (3). Let us make use of definitions of even parts of continued fraction as given in [15] is

$$\frac{1}{1+a_2}\,{}_{-}\,\frac{a_2a_3}{1+a_3+a_4}\,{}_{-}\,\frac{a_4a_5}{1+a_5+a_6}\,{}_{-\ \ldots\ -} \tag{4}$$

$[n-1/n]_T(x)$ Pade approximants can be computed using the continued fraction (4):

$$\frac{1}{1+(\frac{-1}{3})x}\,{}_{-}\,\frac{(\frac{-1}{3})(\frac{-4}{15})x^2}{1-\frac{55}{(3)(5)(7)}x}\,{}_{-}\,\frac{(\frac{-9}{35})(\frac{-16}{63})x^2}{1-\frac{351}{(7)(9)(11)}x}\,{}_{-\ldots-}\,\frac{\left(\frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)}\right)x^2}{1-\frac{32n^3+24n^2-1}{(4n-1)(4n+1)(4n+3)}x}\,{}_{+\cdots}. \tag{5}$$

The nth convergent of the continued fraction (5) is

$$\frac{A_{2n+2}(x)}{B_{2n+2}(x)} = \frac{\left(1-\frac{32n^3+24n^2-1}{(4n-1)(4n+1)(4n+3)}x\right)A_{2n}(x) - \left(\frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)}\right)x^2A_{2n-2}(x)}{\left(1-\frac{32n^3+24n^2-1}{(4n-1)(4n+1)(4n+3)}x\right)B_{2n}(x) - \left(\frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)}\right)x^2B_{2n-2}(x)},$$

with

$$\frac{A_2}{B_2} = \frac{1}{1-\frac{1}{3}x}, \qquad \frac{A_4}{B_4} = \frac{1-\frac{11}{21}x}{1-\frac{6}{7}x+\frac{3}{35}x^2},\ n = 2, 3, \ldots\ .$$

Let us make use of definitions of odd parts of continued fraction as given in [15] is

$$1-\frac{a_2}{1+a_2+a_3}\,_{-}\,\frac{a_3a_4}{1+a_4+a_5}\,_{-}\,\frac{a_5a_6}{1+a_6+a_7}\,_{-}\ldots_{-} \tag{6}$$

$[n/n]_T(x)$ Pade approximants can be computed using the continued fraction (6):

$$1-\frac{\frac{-1}{3}x}{1-\frac{9}{15}x}\,_{-}\,\frac{(\frac{4}{15})(\frac{9}{35})3x^2}{1-\frac{161}{(5)(7)(9)}x}\,_{-}\ldots_{-}\,\frac{\left(\frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)}\right)\,x^2}{1-\frac{32n^3+72n^2+48n+9}{(4n+1)(4n+3)(4n+5)}x}\,_{-}\ldots\quad. \tag{7}$$

The nth convergent of the continued fraction (7) is

$$\frac{A_{2n+1}(x)}{B_{2n+1}(x)}=\frac{\left(1-\frac{32(n-1)^3+72(n-1)^2+48(n-1)+9}{(4n-3)(4n-1)(4n+1)}x\right)A_{2n-1}(x)-\frac{(2n-2)^2(2n-1)^2}{(4n-5)(4n-3)^2(4n-1)}x^2A_{2n-3}(x)}{\left(1-\frac{32(n-1)^3+72(n-1)^2+48(n-1)+9}{(4n-3)(4n-1)(4n+1)}x\right)B_{2n-1}(x)-\frac{(2n-2)^2(2n-1)^2}{(4n-5)(4n-3)^2(4n-1)}x^2B_{2n-3}(x)},$$

with $\quad \dfrac{A_1}{B_1}=\dfrac{1}{1}, \qquad \dfrac{A_3}{B_3}=\dfrac{1-\frac{4}{15}x}{1-\frac{3}{5}x},\ n=2,3,\ldots\ .$

3. The Orthogonality Results

In this section, we establish the orthogonality for the following four polynomials:

$$\begin{aligned} p_n(x) &= x^nA_{2n+2}\left(\frac{1}{x}\right), \qquad q_n(x)=x^nB_{2n}\left(\frac{1}{x}\right),\\ r_n(x) &= x^nA_{2n+1}\left(\frac{1}{x}\right), \qquad s_n(x)=x^nB_{2n+1}\left(\frac{1}{x}\right),\\ n &= 0,1,2,\ldots,\ \text{where}\ B_0\left(\frac{1}{x}\right):=1. \end{aligned}$$

Orthogonality of $q_n(x)$:

Consider the series

$$T(x)=1+\frac{1}{3}x+\frac{1}{5}x^2+\frac{1}{7}x^3+\cdots+\frac{1}{2n+1}x^n+\cdots\ .$$

The linear moment generating function with respect to $T(x)$, denoted by L_T, has nth moment,

$$L_T\{x^n\}=\frac{1}{2n+1}.$$

The three term recurrence relation of $q_n(x)$ is

$$\begin{aligned} q_{n+1}(x) &= \left(x-\frac{32n^3+24n^2-1}{(4n-1)(4n+1)(4n+3)}\right)q_n(x)-\frac{(2n-1)^2(2n)^2}{(4n-3)(4n-1)^2(4n+1)}q_{n-1}(x),\\ q_0(x) &= 1, \quad q_1(x)=x-\frac{1}{3},\ \ n=1,2,3,\ldots\ . \end{aligned} \tag{8}$$

As a result, application of (1) and (2), the orthogonality of $q_n(x)$ is given by

$$L_T\{q_m(x)q_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{(2k-3)^2(2k-2)^2}{(4k-7)(4k-5)^2(4k-3)},\ k = 2, 3, 4, \ldots, n+1.$

Orthogonality of $s_n(x)$:

Following [2, 3] we obtain the series

$$T_1(x) = \frac{3[T(x)-1]}{x} = 1 + \frac{3}{5}x + \frac{3}{7}x^2 + \cdots + \frac{3}{2n+3}x^n + \cdots \quad .$$

The linear moment generating function with respect to $T_1(x)$, denoted by L_{T_1}, has nth moment

$$L_{T_1}\{x^n\} = \frac{3}{2n+3}.$$

The three term recurrence relation of $s_n(x)$ is

$$\begin{aligned} s_{n+1}(x) &= \left(x - \frac{32n^3+72n^2+48n+9}{(4n+1)(4n+3)(4n+5)}\right)s_n(x) - \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)}s_{n-1}(x), \\ s_0(x) &= 1, \qquad s_1(x) = x - \frac{3}{5},\ n = 1, 2, 3, \ldots \quad . \end{aligned} \tag{9}$$

Invoking (1) and (2), we obtain the orthogonality of $s_n(x)$ is

$$L_{T_1}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{(2k-2)^2(2k-1)^2}{(4k-5)(4k-3)^2(4k-1)}$, $k = 2, 3, \ldots, n+1$.

Orthogonality of $r_n(x)$:

Following [2, 3], we obtain the series

$$\frac{1}{T(x)} = 1 - \frac{1}{3}x - \frac{4}{9\times 5}x^2 - d_3x^3 - d_4x^4 - \cdots - d_nx^n - \cdots$$

and

$$T_2(x) = -3\left(\frac{\frac{1}{T(x)} - 1}{x}\right) = 1 + d_2x + d_3x^2 + d_4x^3 + \cdots + d_{n+1}x^n + \cdots \quad .$$

The linear moment generating function with respect to $T_2(x)$ denoted by L_{T_2} has nth moment $L_{T_2}\{x^n\}$ $= d_{n+1}$.

The three term recurrence relation of $r_n(x)$ is

$$\begin{aligned} r_{n+1}(x) &= \left(x - \frac{32n^3+72n^2+48n+9}{(4n+1)(4n+3)(4n+5)}\right) r_n(x) - \frac{(2n)^2(2n+1)^2}{(4n-1)(4n+1)^2(4n+3)} r_{n-1}(x), \\ r_0(x) &= 1, \quad r_1(x) = x - \frac{4}{5}, \;\; n = 1,2,3,\dots . \end{aligned} \tag{10}$$

Invoking (1) and (2), we obtain the orthogonality of $r_n(x)$ is

$$L_{T_2}\{r_m(x)r_n(x)\} = \begin{cases} 0, & m \neq n, \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{(2k-2)^2(2k-1)^2}{(4k-5)(4k-3)^2(4k-1)}$, $k = 2,3,\dots,n+1$.
Suppose $r_n(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0$. Since $L_{T_2}\{r_0(x)r_n(x)\} = 0$, we can compute d_n using

$$d_n = -[r_{n-1}d_{n-1} + \cdots + r_1d_1 + r_0], \quad d_0 = 1, \quad n = 1,2,\dots .$$

Orthogonality of $p_n(x)$:

Following [2, 3], we obtain the series

$$T_3(x) = \frac{9\times 5}{4}\left[\frac{1 - \frac{1}{3}x - \frac{1}{T(x)}}{x^2}\right] = 1 + \frac{11}{21}x + \frac{107}{315}x^2 + e_3x^3 + \cdots + e_nx^n + \cdots .$$

The linear moment generating function with respect to $T_3(x)$ denoted by L_{T_3} has nth moment

$$L_{T_3}\{x^n\} = e_n.$$

The three term recurrence relation of $p_n(x)$ is

$$\begin{aligned} p_{n+1}(x) &= \left(x - \frac{32(n+1)^3+24(n+1)^2-1}{(4n+3)(4n+5)(4n+7)}\right) p_n(x) - \frac{(2n+1)^2(2n+2)^2}{(4n+1)(4n+3)^2(4n+5)} p_{n-1}(x), \\ p_0(x) &= 1, \quad p_1(x) = x - \frac{11}{21}, \;\; n = 1,2,3,\dots, \qquad . \end{aligned} \tag{11}$$

Invoking (1) and (2), we obtain the orthogonality of $p_n(x)$ is

$$L_{T_3}\{p_m(x)p_n(x)\} = \begin{cases} 0, & m \neq n, \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{(2k-1)^2(2k)^2}{(4k-3)(4k-1)^2(4k+1)}$, $k = 2,3,\dots,n+1$.

Suppose $p_n(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0$. Since $L_{T_3}\{p_0(x)p_n(x)\} = 0$, we can compute e_n using

$$e_n = -[p_{n-1}e_{n-1} + \cdots + p_1e_1 + p_0], \quad e_0 = 1, \;\; n = 1,2,\dots .$$

4. CLASSICAL ORTHOGONAL POLYNOMIALS

The following theorem [1, 4], gives necessary and sufficient conditions for classical orthogonality of polynomials:

***Theorem* 4.1** — *$\left\{P_n(x), \frac{d}{dx}\left(\frac{P_{n+1}(x)}{n+1}\right)\right\}$ is a pair of classical orthogonal polynomials if and only if*

1. *$P_n(x)$ satisfies*

$$\begin{aligned} P_{n+1}(x) &= (x-\beta_n)P_n - \gamma_n P_{n-1}, \quad n = 1, 2, 3, \dots, \\ P_0(x) &= 1, \qquad\qquad P_1(x) = x - \beta_0. \end{aligned}$$

2. $P_n(x) = \frac{d}{dx}\left(\frac{P_{n+1}(x)}{n+1}\right) + a_{n,n}\frac{d}{dx}\left(\frac{P_n(x)}{n}\right) + a_{n,n-1}\frac{d}{dx}\left(\frac{P_{n-1}(x)}{n-1}\right),$

   with $a_{n,n-1} \neq (n-1)\gamma_n$, for $n \geq 2$.

***Theorem* 4.2** — *The polynomials $q_n(x)$ and $s_n(x)$ are classical orthogonal polynomials.*

PROOF : Using (8) and (9), we directly obtain the result that $q_n(x)$ and $s_n(x)$ are orthogonal polynomial with respect to L_T and L_{T_1} respectively. Now, we observe that

$$\begin{aligned} q_n(x) &= x^n\, {}_2F_1\left(-n, -n+\frac{1}{2}; -2n+\frac{1}{2}; \frac{1}{x}\right) \\ &= \sum_{r=0}^{n}(-1)^r \frac{\binom{n}{r}\binom{n-\frac{1}{2}}{r}}{\binom{2n-\frac{1}{2}}{r}} x^{n-r}, \; n = 0, 1, 2, \dots\;, \end{aligned} \tag{12}$$

$$\begin{aligned} s_n(x) &= x^n\, {}_2F_1\left(-n, -n-\frac{1}{2}; -2n-\frac{1}{2}; \frac{1}{x}\right) \\ &= \sum_{r=0}^{n}(-1)^r \frac{\binom{n}{r}\binom{n+\frac{1}{2}}{r}}{\binom{2n+\frac{1}{2}}{r}} x^{n-r}, \; n = 0, 1, 2, \dots\;. \end{aligned} \tag{13}$$

Using (12), we obtain the relation

$$\begin{aligned} q_n(x) &= \frac{d}{dx}\left(\frac{q_{n+1}(x)}{n+1}\right) + \frac{2n}{(4n-1)(4n+3)}\frac{d}{dx}\left(\frac{q_n(x)}{n}\right) \\ &\quad - \frac{(2n-2)(2n-1)(2n)^2}{(4n-3)(4n-1)^2(4n+1)}\frac{d}{dx}\left(\frac{q_{n-1}(x)}{n-1}\right), \quad n = 2, 3, \dots\;. \end{aligned} \tag{14}$$

Using (8), (14) and Theorem 4.1, $q_n(x)$ is classical orthogonal polynomials.

Using (13), we obtain the relation

$$\begin{aligned} s_n(x) &= \frac{d}{dx}\left(\frac{s_{n+1}(x)}{n+1}\right) - \frac{2n}{(4n+1)(4n+5)}\frac{d}{dx}\left(\frac{s_n(x)}{n}\right) \\ &\quad - \frac{(2n-2)(2n+1)(2n)^2}{(4n-1)(4n+1)^2(4n+3)}\frac{d}{dx}\left(\frac{s_{n-1}(x)}{n-1}\right), \; n = 2, 3, \dots\;. \end{aligned} \tag{15}$$

Using (9), (15) and Theorem 4.1, $s_n(x)$ is classical orthogonal polynomials.

***Theorem* 4.3** — *The polynomials $r_n(x)$ and $p_n(x)$ are non-classical orthogonal polynomials.*

PROOF : Using (10) and (11), we directly obtain the result that $r_n(x)$ and $p_n(x)$ are orthogonal polynomial with respect to L_{T_2} and L_{T_3} respectively. Now, we observe that $r_n(x)$ and $p_n(x)$ do not satisfy the condition 2 of Theorem 4.1, because

$$\begin{aligned} r_3(x) &= \frac{d}{dx}\left(\frac{r_4(x)}{4}\right) + \frac{149}{13\times 17\times 12}\frac{d}{dx}\left(\frac{r_3(x)}{3}\right) - \frac{22291}{(13)^2\times 11\times 15\times 18}\frac{d}{dx}\left(\frac{r_2(x)}{2}\right) \\ &\quad + \frac{8777941}{(13)^2\times 15\times 17\times 11\times 9^2\times 7\times 4}\frac{d}{dx}\left(\frac{r_1(x)}{1}\right). \end{aligned}$$

$$\begin{aligned} p_3(x) &= \frac{d}{dx}\left(\frac{p_4(x)}{4}\right) - \frac{2}{19\times 15}\frac{d}{dx}\left(\frac{p_3(x)}{3}\right) - \frac{52778}{15\times 13\times 17\times 19\times 9\times 5}\frac{d}{dx}\left(\frac{p_2(x)}{2}\right) \\ &\quad + \frac{164708}{19\times 15\times 13\times 11\times 9\times 7\times 5\times 3}\frac{d}{dx}\left(\frac{p_1(x)}{1}\right). \end{aligned}$$

Hence $r_n(x)$ and $p_n(x)$ are non-classical orthogonal polynomials.

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