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ON THE IMAGES OF ENTIRE FUNCTIONS UNDER THE LIMIT *q*-BERNSTEIN OPERATOR

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The limit q-Bernstein operator B_q comes out naturally as the limit for the sequence of q-Bernstein operators in the case $0 < q < 1$. Alternatively, it can be viewed as a modification of the Szász-Mirakyan operator related to the Euler distribution. In this paper, a necessary and sufficient condition for a function g to be an image of an entire function under B_q is presented.

Key words : Limit q-Bernstein operator; entire function; divided difference.

1. Introduction

The limit q-Bernstein operator emerges as a limit for the sequence of the q-Bernstein operators in the case $0 < q < 1,$ see [2] and [9]. Later, Wang showed in [11] that the same operator is the limit for the sequence of q-Meyer-König and Zeller operators. The approximation properties of this operator as well as its connections with other disciplines have been studied. See, for example, [3, 5, 7, 8, 10].

To begin with, let us recall some notions related to the q-calculus (see, e.g., [1], Ch. 10). We use the following standard notations:

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1-aq^s), \qquad (a;q)_\infty := \prod_{s=0}^{\infty}(1-aq^s), \;\; a\in\mathbb{C}, \;\; q\in(0,1).$$

The function

$$\psi_q(z) := (z;q)_\infty, \; 0<q<1, \; z\in\mathbb{C} \tag{1.1}$$

is an entire function satisfying the Euler Identities below (cf. [1], Ch. 10, Cor. 10.2.2):

$$\psi_q(z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k}\, z^k, \tag{1.2}$$

$$\frac{1}{\psi_q(z)} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}, \quad |z| < 1. \tag{1.3}$$

Definition 1.1 — Given $q \in (0,1)$, the *limit q-Bernstein operator* $B_q : C[0,1] \to C[0,1]$ is defined by:

$$(B_q f)(x) := \begin{cases} \psi_q(x) \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k} x^k, & \text{if } x \in [0,1), \\ f(1), & \text{if } x = 1. \end{cases} \tag{1.4}$$

For each $f \in C[0,1]$, the function $B_q f$ admits an analytic continuation from $[0,1]$ to the open unit disc $\{z : |z| < 1\}$. In general, it may not have an analytic continuation into a wider disc. The possibility of such a continuation is discussed in [6] in detail, showing that (1.4) can be extended as an entire function whenever f is infinitely differentiable at 1. If f itself admits an analytic continuation as a transcendental entire function, then, by Theorem 4.2 of [6] its image under B_q is an entire function whose growth is strictly slower than that of f; while the image of a polynomial is a polynomial of the same degree. In this article, some elaboration of these results will be presented.

2. Preliminaries

Denote by $E[0,1]$ the set of (complex-valued) functions on $[0,1]$ which admit analytic continuations from $\{1-q^k\}_{k=0}^{\infty}$ as entire functions. Whenever $f \in E[0,1]$, we denote its analytic continuation into the complex plane by $f(z),\ z \in \mathbb{C}$.

Further, it should be mentioned that formula (1.4), along with identity (1.2), leads to an alternative representation of $B_q f$ in the form:

$$(B_q f)(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} f[0; 1-q; \ldots, 1-q^k] x^k, \quad |x| < 1, \tag{2.1}$$

where $f[x_0, \ldots, x_k]$ denotes the k-th order divided difference of f with $k+1$ distinct nodes $x_0, \ldots, x_k$. Notice that the power series representation:

$$(B_q f)(z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} f[0; 1-q; \ldots, 1-q^k] z^k, \tag{2.2}$$

is valid in every disc $\{z : |z| < r\}$ where $(B_q f)(x)$ has an analytic continuation. If f is an analytic function, then the following equality is true:

$$f[x_0; x_1; \ldots; x_k] = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{f(\zeta)\, d\zeta}{(\zeta - x_0)\ldots(\zeta - x_k)}, \tag{2.3}$$

where $\mathcal{L}$ is a contour encircling $x_0, \ldots, x_k$ and f is analytic on and inside of $\mathcal{L}$ (cf., e.g. [4], §2.7, p. 44).

In the sequel, the following notation will be used: whenever $f(z)$ is an entire function, we denote:

$$M_f(r) := \max_{|z|\leq r} |f(z)|.$$

It has been proved in [12] that, for $\psi_q(z)$ and some $C_1, C_2 > 0, r > r_0$, the next estimate holds:

$$C_1 \exp\left\{\frac{\ln^2 r}{2\ln(1/q)} + \frac{\ln r}{2}\right\} \leq M(r;\psi_q) \leq C_2 \exp\left\{\frac{\ln^2 r}{2\ln(1/q)} + \frac{\ln r}{2}\right\}. \tag{2.4}$$

This implies immediately that $\psi_q(z)$ is an entire function of order 0. In the forthcoming section, some properties of the image $B_q f$, where $f \in E[0,1]$ will be discussed. In this case, one may also say that $B_q f$ is an *image of an entire function under* B_q.

3. Images of Entire Functions Under B_q

It can be derived from (2.2) that if $f \in E[0,1]$, then $B_q f$ is an entire function. See also [6, Lemma 2.1 and Theorem 4.2]. On the other hand, the following simple statement shows that every entire function can be viewed as an image of a continuous function under B_q.

Lemma 3.1 — If g is an entire function, then $g = B_q f$ for some $f \in C[0,1]$.

PROOF : The Euler Identity (1.3) implies that constant functions are fixed points for B_q, i.e. $B_q c = c$ for all $c \in \mathbb{C}$. Therefore, without loss of generality one may assume that $g(1) = 0$. Since $g(z)$ is entire and $g(1) = 0$, one can observe that

$$h(z) := \frac{g(z)}{1-z} =: \sum_{k=0}^{\infty} a_k z^k$$

is also an entire function, whence

$$\frac{g(z)}{\psi_q(z)} = \frac{h(z)}{\psi_q(qz)} =: \sum_{k=0}^{\infty} b_k z^k \tag{3.1}$$

is analytic in $\{z : |z| < q^{-1}\}$ by virtue of (1.3). Equality (3.1) implies that

$$g(z) = \psi_q(z) \sum_{k=0}^{\infty} b_k z^k,\ |z| < q^{-1}.$$

Since $\{b_k\} \to 0$, it follows that there is $f \in C[0,1]$ satisfying $f(1-q^k)/(q;q)_k = b_k,\ k \in \mathbb{N}_0$. Then $g(z) = (B_q f)(z)$, as stated. □

The next theorem provides a necessary and sufficient condition for g to be an image of an entire function.

***Theorem* 3.2** — *Let $g(z),\ z \in \mathbb{C}$ be entire. Then $g = B_q f$ for an $f \in E[0,1]$ if and only if the following condition holds:*

$$\forall \varepsilon > 0\ \exists C = C_\varepsilon > 0 : M_g(r) \leq C\psi_q(-\varepsilon r),\ r > r_0. \tag{3.2}$$

PROOF : (i) Assume that $g = B_q f,\ f \in E[0,1]$. Then $g(z)$ admits representation (2.2) for all $z \in \mathbb{C}$. To estimate the coefficients in (2.2), notice that by virtue of (2.3), one has:

$$\left| f[0; 1-q; \ldots, 1-q^k] \right| \leq \frac{M_f(r)}{(r-1)^k} \ \text{ for all } \ \mathrm{r} > 1,$$

where, as before, $f(z), z \in \mathbb{C}$ denotes an analytic continuation of f.

Given $\varepsilon > 0$, let us set $r = 1 + 1/\varepsilon$ and obtain:

$$M_g(r) \leq M_f(1 + 1/\varepsilon) \sum_{k=0}^{\infty} q^{k(k-1)/2} (\varepsilon r)^k$$

$$\leq M_f(1 + 1/\varepsilon; f) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(\varepsilon r)^k}{(q;q)_k} =: C_\varepsilon \psi_q(-\varepsilon r),$$

as stated.

(ii) Whereas g is entire, Lemma 3.1 guarantees that there exists $f \in C[0,1]$ satisfying $g = B_q f$. Suppose that for every $\varepsilon > 0$, condition (3.2) holds. As the Taylor series of g is given by (2.2), one has by the Cauchy estimates:

$$a_k := q^{k(k-1)/2} f[0; 1-q; \ldots, 1-q^k] \leq \frac{M_g(r)}{r^k} \leq C_\varepsilon \frac{\psi_q(-\varepsilon r)}{r^k} = C_\varepsilon \varepsilon^k \frac{\psi_q(-\varepsilon r)}{(\varepsilon r)^k},\ \ r > 0.$$

This implies that

$$a_k \leq C_{\varepsilon, f}\, \varepsilon^k \min_{t > 0} \frac{\psi_q(-t)}{t^k} \leq C_{\varepsilon, f} \varepsilon^k \frac{\psi_q(-q^{-k})}{q^{-k^2}} = C_{\varepsilon, f} \varepsilon^k q^{k^2} \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{q^{-kj}}{(q;q)_j},$$

whence

$$a_k \leq \frac{C_{\varepsilon, f}}{(q;q)_\infty} \varepsilon^k q^{k^2/2} \sum_{j=0}^{\infty} q^{((j-k)^2 - j)/2}. \tag{3.3}$$

To estimate the sum of the series in the right-hand side, we write:

$$\sum_{j=0}^{\infty} q^{((j-k)^2 - j)/2} = \sum_{j=0}^{k} q^{((j-k)^2 - j)/2} + \sum_{j=k+1}^{\infty} q^{((j-k)^2 - j)/2}$$

$$= \sum_{j=0}^{k} q^{(j^2+j-k)/2} + \sum_{j=1}^{\infty} q^{(j^2-j-k)/2} \leq 2q^{-k/2} \sum_{j=0}^{\infty} q^{(j^2-j)/2} =: 2Cq^{-k/2}.$$

As a result, one obtains:

$$a_k \leq C_\varepsilon \varepsilon^k q^{k(k-1)/2}$$

and, consequently, the following inequality holds:

$$\left| f[0; 1-q; \ldots, 1-q^k] \right| \leq C\varepsilon^k, \quad k \in \mathbb{N}_0. \tag{3.4}$$

Consider the Newton series:

$$\sum_{k=0}^{\infty} f[0; 1-q; \ldots, 1-q^k] z(z-(1-q)) \ldots (z-(1-q^k)). \tag{3.5}$$

It can be derived with the help of (3.4) that $\left| f[0; 1-q; \ldots, 1-q^k] z(z-(1-q)) \ldots (z-(1-q^k)) \right|$ $\leq C[\varepsilon(|z|+1)]^k$, whence (3.5) converges for $\{|z| < 1/\varepsilon - 1\}$. As $\varepsilon > 0$ has been chosen arbitrarily, it follows that (3.4) defines an entire function, say $\tilde{f}(z)$ satisfying $\tilde{f}(1-q^k) = f(1-q^k)$ and, hence, $f \in E[0,1]$. □

Generally speaking, Theorem 3.2 supplies a necessary and sufficient condition for an entire function g to be an image of an entire function f in terms of the growth estimate for g. The following result can be derived as immediate consequence of Theorem 3.2.

Corollary 3.3 — An entire function g is an image of an entire function under B_q if and only if the following estimate holds:

$$\forall \alpha > 0, \; M_g(r) \cdot \exp\left\{ -\frac{\ln^2 r}{2\ln(1/q)} \right\} = O\left(r^{-\alpha}\right), \; r \to +\infty. \tag{3.6}$$

PROOF : It follows from (2.4) that

$$\psi_q(-\varepsilon r) = M_{\psi_q}(\varepsilon r) \leq C \exp\left\{ \frac{\ln^2 r}{2\ln(1/q)} + \left(\frac{\ln \varepsilon}{\ln(1/q)} + \frac{1}{2} \right) \ln r \right\}.$$

Therefore, g is an image of an entire function if and only if

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 : \; M_g(r) \cdot \exp\left\{ -\frac{\ln^2 r}{2\ln(1/q)} \right\} \leq C_\varepsilon \exp\left\{ \left(\frac{\ln \varepsilon}{\ln(1/q)} + \frac{1}{2} \right) \ln r \right\}.$$

Now, let us have an arbitrary $\alpha > 0$. Setting $\varepsilon = q^{\alpha+1/2}$ yields:

$$M_g(r) \cdot \exp\left\{ -\frac{\ln^2 r}{2\ln(1/q)} \right\} \leq C_\alpha r^{-\alpha} \;\; \text{for some} \;\; \mathrm{C}_\alpha > 0. \square$$

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