Indian J. Pure Appl. Math., **48**(2): 211-220, June 2017

DOI: 10.1007/s13226-017-0222-y

EXISTENCE OF AN INVARIANT FORM UNDER A LINEAR MAP

Krishnendu Gongopadhyay* and Sudip Mazumder**

*Indian Institute of Science Education and Research (IISER) Mohali, Sector 81,
S.A.S. Nagar, Punjab 140 306, India

**Department of Mathematics, Jadavpur University, Jadavpur,
Kolkata 700 032, India

e-mails: krishnendug@gmail.com, krishnendu@iisermohali.ac.in, topologysudip@gmail.com

(*Received* 9 *May* 2016; *accepted* 26 *October* 2016)

Let $\mathbb{F}$ be a field of characteristic different from 2 and $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $J : \alpha \to \alpha^J$ be a fixed involutory automorphism on $\mathbb{F}$. In this paper we answer the following question: given an invertible linear map $T : \mathbb{V} \to \mathbb{V}$, when does the vector space $\mathbb{V}$ admit a T-invariant non-degenerate J-hermitian, resp. J-skew-hermitian, form?

Key words : Linear map; Hermitian form; isometry.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic different from 2. Let $J : \alpha \to \alpha^J$ be a fixed involutory automorphism on $\mathbb{F}$, i.e. $(\alpha+\beta)^J = \alpha^J + \beta^J$, $(\alpha\beta)^J = \alpha^J\beta^J$, $(\alpha^J)^J = \alpha$. If there exists a non-zero α such that $J(\alpha) = \alpha^J = -\alpha$, we say: "$x^J = -x$ has a solution in $\mathbb{F}$". This is always the case if J is non-trivial. Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$. In this paper we ask the following question.

Question 1 : Given an invertible linear map $T : \mathbb{V} \to \mathbb{V}$, when does the vector space $\mathbb{V}$ over $\mathbb{F}$ admit a T-invariant non-degenerate J-hermitian, resp. J-skew-hermitian, form?

We have answered the question in this note. This generalizes earlier work by Gongopadhyay and Kulkarni [3] where the authors obtained conditions for an invertible linear map to admit an invariant non-degenerate quadratic and symplectic form assuming that the underlying field is of large characteristic. de Seguins Pazzis [1] extended the work of [3] over arbitrary characteristic. The technicalities are slightly different in these works due to the underlying field characteristic. In this work we follow ideas from [3].

Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Let $f(x) = \sum_{i=0}^{d} a_i x^i$, $a_d = 1$, be a monic polynomial of degree d over $\mathbb{F}$ such that -1, 0, 1 are not its roots. The *dual* of $f(x)$ is defined to be the polynomial $f^*(x) = (f(0)^J)^{-1} x^d f^J(x^{-1})$, where $f^J(x) = \sum_{i=0}^{d} a_i^J x^i$. Thus, $f^*(x) = \frac{1}{a_0^J} \Sigma_{i=0}^{d} a_{d-i}^J x^i$. In other words, if α in $\overline{\mathbb{F}}$ is a root of $f(x)$ with multiplicity k, then $(\alpha^J)^{-1}$ is a root of $f^*(x)$ with the same multiplicity. The polynomial $f(x)$ is said to be *self-dual* if $f(x) = f^*(x)$.

Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation. A T-invariant subspace is said to be *indecomposable* with respect to T, or simply *T-indecomposable* if it can not be expressed as a direct sum of two proper T-invariant subspaces. $\mathbb{V}$ can be written as a direct sum $\mathbb{V} = \oplus_{i=1}^{m} \mathbb{V}_i$, where each $\mathbb{V}_i$ is T-indecomposable for $i = 1, 2, \ldots, m$. In general, this decomposition is not canonical. But for each i, $(\mathbb{V}_i, T|_{\mathbb{V}_i})$ is "dynamically equivalent" to $(\mathbb{F}[x]/(p(x)^k), \mu_x)$, where $p(x)$ is an irreducible monic factor of the minimal polynomial of T, and μ_x is the operator $[u(x)] \mapsto [xu(x)]$, for eg. see [5]. Such $p(x)^k$ is an *elementary divisor* of T. If $p(x)^k$ occurs d times in the decomposition, we call d the *multiplicity* of the elementary divisor $p(x)^k$.

Let $\chi_T(x)$ denote the characteristic polynomial of an invertible linear map T. Let

$$\chi_T(x) = (x-1)^e (x+1)^f \chi_{oT}(x),$$

where $e, f \geq 0$, and $\chi_{oT}(x)$ has no root 1, or -1. The vector space $\mathbb{V}$ has a T-invariant decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_o$, where for $\lambda = 1, -1$, $\mathbb{V}_\lambda$ is the generalized eigenspace to λ, i.e.

$$\mathbb{V}_\lambda = \{v \in \mathbb{V} \mid (T - \lambda I)^n v = 0\},$$

and $\mathbb{V}_o = ker\ \chi_{oT}(T)$. Let T_o denote the restriction of T to $\mathbb{V}_o$. Clearly T_o has the characteristic polynomial $\chi_{oT}(x)$ and does not have any eigenvalue 1 or -1.

With the notations as given above, we prove the following theorem that answers the above question. When J is the trivial automorphism, the following theorem descends to Theorem 1.1 of [3] in view of Lemma 2.1 in Section 2.

***Theorem* 1.1** — *Let $\mathbb{F}$ be a field with characteristic different from two. Let $J : \alpha \to \alpha^J$ be a fixed non-trivial involutory automorphism on $\mathbb{F}$. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ of dimension at least 2. Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map. Then $\mathbb{V}$ admits a T-invariant non-degenerate J-hermitian, resp. J-skew-hermitian form if and only if an elementary divisor of T_o is either self-dual, or its dual is also an elementary divisor with the same multiplicity.*

We prove this theorem in the next section.

Suppose $T : \mathbb{V} \to \mathbb{V}$ is a linear map that admits an invariant hermitian or skew-hermitian form H. Then canonical forms for T is known in the literature, for example, see [8]. This provides the necessary condition in the above theorem. The focus of this article is the converse part, i.e. the sufficient conditions for a linear map T to admit a non-degenerate hermitian form.

After finishing this work, we found the papers by Sergeichuk [6, Theorem 5], [7, Theorem 2.2], where the author has obtained canonical forms for the pairs (A, B), where B is a non-degenerate form and A is an isometry of B over a field of characteristic not 2. The work of Sergeichuk not only gives the necessary condition stated above, but the sufficient condition is also implicit there. However, it has not been stated in a precise form as in the above theorem. Sergeichuk's approach involves quivers to derive the results. Our approach is simpler here.

2. Proof of Theorem 1.1

For a matrix $A = (a_{ij})$, let A^J be the matrix $A^J = (a^J_{ij})$. Let H be a J-sesquilinear form on $\mathbb{V}$. The form H is *J-hermitian* or simply *hermitian*, resp. *skew-hermitian* if for all u, v in $\mathbb{V}$, $H(u, v) = H(v, u)^J$, resp. $H(u, v) = -H(v, u)^J$. For a linear map $T : \mathbb{V} \to \mathbb{V}$, we say H is T-invariant or T is an isometry of H if for all $u, v \in \mathbb{V}$, $H(Tu, Tv) = H(u, v)$.

Lemma 2.1 — Let $T : \mathbb{V} \to \mathbb{V}$ be a unipotent linear map with minimal polynomial $(x - 1)^n$, $n \geq 2$. Let $\mathbb{V}$ be T-indecomposable.

(i) If n is even, resp. odd, then $\mathbb{V}$ admits a T-invariant skew-hermitian, resp. hermitian form.

(ii) If n is even, resp. odd, and $x^J = -x$ has a non-zero solution in $\mathbb{F}$, then $\mathbb{V}$ admits a T-invariant hermitian, resp. skew-hermitian form.

Proof : Let T be an unipotent linear map. Suppose the minimal polynomial of T is $m_T(x) = (x - 1)^n$. Without loss of generality we can assume that T is of the form

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots. 0 & 0 \\ 1 & 1 & 0 & 0 & \dots. 0 & 0 \\ 0 & 1 & 1 & 0 & \dots. 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix} \tag{2.1}$$

Suppose T preserves a J-sesquilinear form H. In matrix form, let $H = (a_{ij})$. Then, $(T^J)^t HT = H$. This gives the following relations: For $1 \leq i \leq n - 1$,

$$a_{i+1,n} = 0 = a_{n,i+1}, \tag{2.2}$$

$$a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} = a_{i,j}, \tag{2.3}$$

$$\text{i.e. } a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} = 0. \tag{2.4}$$

From the above two equations we have, for $1 \le l \le n-3$ and $l+2 \le i \le n-1$,

$$a_{i,n-l} = 0 = a_{n-l,i}. \tag{2.5}$$

This implies that H is a matrix of the form

$$H = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \dots & a_{2,n-2} & a_{2,n-1} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \dots & a_{3,n-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{n,1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

where,

$$a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1} = 0.$$

Choose a basis $e_1, e_2, e_3, ..., e_n$ of V such that T and H has the above forms with respect to the basis. From (2.4) we have

$$a_{l,n-l+1} = (-1)^{n+1-2l}\, a_{n-l+1,l}. \tag{2.7}$$

Note that it follows from (2.4) that $a_{l,n-l+1} = (-1)^{l-1} a_{1,n}$ for $1 \le l \le n$. Hence H is non-singular if and only if $a_{1,n} \neq 0$. Continuing the procedure, all entries of H except $a_{1,1}$ can be expressed as a scalar multiple of $a_{1,n}$.

For H to be hermitian, from (2.7) we must have $a^J_{l,n-l+1} = a_{n-l+1,l} = (-1)^{n+1} a_{l,n-l+1}$. This implies, $a^J_{1,n} = (-1)^{n+1} a_{1,n}$. So, a non-zero choice of $a_{1,n}$ is possible only if either n is odd or, in case n is even, then $x^J = -x$ must have a solution in $\mathbb{F}$, which is always the case. The other case is similar. □

Corollary 2.2 — Let $\mathbb{F}$ be a field of characteristic different from 2 such that $x^J = -x$ has a solution in $\mathbb{F}$. Let $\mathbb{V}$ be an n-dimensional vector space of dimension ≥ 2 over $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be a unipotent linear map such that $\mathbb{V}$ is T-indecomposable. Then $\mathbb{V}$ admits a T-invariant non-degenerate hermitian, as well as skew-hermitian form.

Lemma 2.3 — Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map. If T has no eigenvalue 1 or -1, then there exists a T-invariant non-degenerate hermitian form on $\mathbb{V}$ if and only if there exists a T-invariant non-degenerate skew-hermitian form on $\mathbb{V}$.

PROOF : Assume, H is a T-invariant hermitian form. Define a form H_T on $\mathbb{V}$ as follows:

$$\text{For } u, v \text{ in } \mathbb{V}, \quad H_T(u,v) = H((T - T^{-1})u, v).$$

Note that

$$\begin{aligned} H_T(u,v) &= H((T-T^{-1})u,v) \\ &= H(Tu,v) - H(T^{-1}u,v) \\ &= H(u,T^{-1}v) - H(u,Tv), \text{ since } T \text{ is an isometry.} \\ &= H(u,T^{-1}v - Tv) \\ &= -H(u,(T-T^{-1})v) \\ &= -H_T^J((T-T^{-1})v,u), \text{ since } H \text{ is hermitian.} \\ &= -H_T^J(v,u). \end{aligned}$$

Thus H_T is a T-invariant non-degenerate skew-hermitian form on $\mathbb{V}$. Also it follows by the same construction that corresponding to each T-invariant skew-hermitian form, there is a canonical T-invariant hermitian form. □

Lemma 2.4 — Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map with characteristic polynomial $\chi_T(x) = p(x)^d$, where $p(x) \neq x \pm 1$, is irreducible over $\mathbb{F}$, and is self-dual. Let $\mathbb{V}$ be T-indecomposable. Then there exists a T-invariant non-degenerate hermitian, resp. skew-hermitian form on $\mathbb{V}$.

PROOF : Since $\mathbb{V}$ is T-indecomposable, $(\mathbb{V}, T)$ is dynamically equivalent to the pair $\big(\mathbb{F}[x]/(p(x)^d), \mu_x\big)$, where μ_x is the operator $\mu_:[u(x)] \mapsto [xu(x)]$, cf. [5]. Hence without loss of generality we assume $\mathbb{V} = \mathbb{F}[x]/(p(x)^d)$, $T = \mu_x$ and let

$$\mathcal{B} = \{e_1 = 1, e_2 = x, \ldots, e_k = x^{k-1}\}, \ k = d.\deg p(x).$$

be the corresponding basis.

Let $\chi_T(x) = p(x)^d = \sum_{i=0}^{k} c_i x^i$. Since $\chi_T(x)$ is self-dual, we must have $c_i = \frac{c_{k-i}^J}{c_0^J}$, $1 \leq i \leq k-1$, $c_0 c_0^J = 1$. If possible, suppose $H = (h_{ij})$ be a T-invariant sesquilinear form on $\mathbb{V}$. Then

$$h_{ij} = H(e_i, e_j) = H(\mu_x e_i, \mu_x e_j) = h_{i+1,j+1}, \ 1 \leq i \leq k, \ 1 \leq j \leq k.$$

Hence, a possible T-invariant sesquilinear form should be represented necessarily by a matrix of the following form:

$$X = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k \\ \beta_2 & \alpha_1 & \alpha_2 & \dots & \alpha_{k-2} & \alpha_{k-1} \\ \beta_3 & \beta_2 & \alpha_1 & \dots & \alpha_{k-3} & \alpha_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{k-1} & \beta_{k-2} & \beta_{k-3} & \dots & \alpha_1 & \alpha_2 \\ \beta_k & \beta_{k-1} & \beta_{k-2} & \dots & \beta_2 & \alpha_1 \end{pmatrix}. \tag{2.8}$$

Let

$$\mathcal{S} = \{A = (a_{ij}) \in M_k(\mathbb{F}) \mid (T^J)^t AT = A\}.$$

Thus T has an invariant form if and only if $\mathcal{S} \neq \phi$.

Let C be the companion matrix of μ_x given by:

$$C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & 0 & \dots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & -c_{k-1} \end{pmatrix}.$$

Let

$$H_1 = \begin{pmatrix} \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_2 & \beta_1 & 0 & \dots & 0 & 0 \\ \beta_3 & \beta_2 & \beta_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \beta_{k-1} & \beta_{k-2} & \beta_{k-3} & \dots & \beta_1 & 0 \\ \beta_k & \beta_{k-1} & \beta_{k-2} & \dots & \beta_2 & \beta_1 \end{pmatrix}.$$

We consider the equation $(C^J)^t H_1 C = H_1$. Simplifying the left hand side, we get

$$(C^J)^t H_1 C = \begin{pmatrix} \beta_1 & 0 & 0 & 0 & \dots & 0 & a_1 \\ \beta_2 & \beta_1 & 0 & 0 & \dots & 0 & a_2 \\ \beta_3 & \beta_2 & \beta_1 & 0 & \dots & 0 & a_3 \\ \beta_4 & \beta_3 & \beta_2 & \beta_1 & \dots & 0 & a_4 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \beta_{k-1} & \beta_{k-2} & \beta_{k-3} & \beta_{k-4} & \dots & \beta_1 & a_{k-1} \\ b_{k-1} & b_{k-2} & b_{k-3} & b_{k-4} & \dots & b_1 & -\sum_{i=0}^{k-1} c_i b_{k-i} \end{pmatrix},$$

where

$$a_i = -c_0\beta_{i+1} - c_1\beta_i - c_2\beta_{i-1} - \dots - c_{i-1}\beta_1,$$

$$b_i = -c_{k-1}^J\beta_i - c_{k-2}^J\beta_{i-1} - \dots - c_{k-i}^J\beta_1.$$

Comparing both sides of $(C^J)^t H_1 C = H_1$ gives

$$\beta_{i+1} = \frac{1}{c_0}(-c_1\beta_i - c_2\beta_{i-1} - \dots - c_{i-1}\beta_1),\ 1 \le i \le k-2. \tag{2.9}$$

Now, by back substitution it is easy to see that all β_i can be expressed as a multiple of β_1 by an expression in $\frac{c_1}{c_0}, \dots, \frac{c_{k-1}}{c_0}$. Next, consider

$$H_2 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k \\ 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{k-2} & \alpha_{k-1} \\ 0 & 0 & \alpha_1 & \dots & \alpha_{k-3} & \alpha_{k-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 & \dots & 0 & \alpha_1 \end{pmatrix}.$$

Comparing both sides of the equation $(C^J)^t H_2 C = H_2$ we get

$$c_0^J\alpha_{i+1} = -c_1^J\alpha_i - -c_2^J\alpha_{i-1} - -c_3^J\alpha_{i-2} - \dots - c_{i-1}^J\alpha_1,\ \ 1 \le i \le k-1. \tag{2.10}$$

By back substitution it is easy to see that each α_i can be expressed as a multiple of α_1 by an expression in $\frac{c_1^J}{c_0^J}, \dots, \frac{c_{k-1}^J}{c_0^J}$.

Thus H_1 and H_2 are elements from the set $\mathcal{S}$ and for $\beta_1 \neq 0$, resp. $\alpha_1 \neq 0$, they give non-degenerate sesquilinear forms. We also see that $H = H_1 + H_2$ is an element in the set $\mathcal{S}$ and is of the form (2.8). If we choose $\beta_1 = \alpha_1^J$, it follows from (2.9) and (2.10) that $\alpha_{i+1} = \beta_{i+1}^J$, $1 \le i \le k-1$, and hence the form H is hermitian. If we choose $\beta_1 = -\alpha_1^J$, it follows H is skew-hermitian. It can be seen that H may be chosen to be non-degenerate. □

Remark 2.5 : We would like to clarify a small inaccuracy in the statement of [3, Lemma 3.2(i)]. It has been stated there that for T unipotent and $(\mathbb{V}, B)$ a T-indecomposable bilinear space, the bilinear form B degenerate implies $B = 0$. This statement needs slight modification. For example, if $\mathbb{V}$ is of dimension 2,

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $\mathbb{V}$ is T-indecomposable, preserves B which is degenerate, but $B \neq 0$. Following similar computations as in the proof of Lemma 2.1, we can modify the statement as follows: with the hypothesis as in Lemma 3.2 of [3], B degenerate implies that the radical of B is a co-dimension one subspace of $\mathbb{V}$. What had been used in the relevant parts of [3], especially in the proof of Theorem 1.3., is actually the following lemma.

Lemma — Let $\mathbb{V}$ be a vector space equipped with a non-degenerate symmetric or skew-symmetric bilinear form B over a field $\mathbb{F}$ of large characteristic. Suppose $T : \mathbb{V} \to \mathbb{V}$ is a unipotent isometry. Let $\mathbb{W}$ be a T-indecomposable subspace of $\mathbb{V}$. Then either $B|_{\mathbb{W}} = 0$ or, $B|_{\mathbb{W}}$ is non-degenerate.

The proof of the lemma is a slight modification of the proof of Lemma 2.2(i) in [4].

2.1 *Proof of Theorem 1.1*

PROOF : Suppose that the linear map T admits an invariant non-degenerate hermitian, resp skew-hermitian, form H. Then the necessary condition follows from existing literatures, for example see [6-8].

Conversely, let $\mathbb{V}$ be a vector space of dim $n \geq 2$ over the field $\mathbb{F}$ and $T : \mathbb{V} \longrightarrow \mathbb{V}$ an invertible map such that an elementary divisor of T_o is either self-dual or, its dual is also an elementary divisior. For an elementary divisor $h(x)$, let $\mathbb{V}_h$ denote the T-indecomposable subspace isomorphic to $\mathbb{F}[x]/(h(x))$. From the structure theory of linear maps, for eg. see [2, 5], it follows that $\mathbb{V}$ has a primary decomposition of the form

$$\mathbb{V} = \bigoplus_{i=1}^{m_1} \mathbb{V}_{f_i} \oplus \bigoplus_{j=1}^{m_2} (\mathbb{V}_{g_i} \oplus \mathbb{V}_{g_i^*}), \tag{2.11}$$

where for each $i = 1, 2, ..., m_1$, $f_i(x)$ is either self-dual, or one of $(x+1)^k$ and $(x-1)^k$, for each $j = 1, 2, ..., m_2$, $g_i(x)$, $g_i^*(x)$ are dual to each other and $g_i(x) \neq g_i^*(x)$. To prove the theorem it is sufficient to induce a T-invariant hermitian (resp. skew-hermitian) form on each of the summands.

Let $\mathbb{W}$ be an T-indecomposable summand in the above decomposition and let $p(x)^k$ be the corresponding elementary divisor. Suppose $p(x)^k$ is self-dual. It follows from Lemma 2.4 that there exists a T-invariant non-degenerate hermitian, as well as skew-hermitian form on $\mathbb{W}$.

Suppose $p(x)^k$ is not self-dual. Then there is a dual elementary divisor $p^*(x)^k$. $\mathbb{W}_p = \ker p(T)^k$, $\mathbb{W}_{p^*} = \ker p^*(T)^k$. Then $\mathbb{W}_{p^*}$ can be considered as dual to $\mathbb{W}_p$ and the dual pairing gives a T-invariant non-degenerate form h on $\mathbb{W}_p \oplus \mathbb{W}_{p^*}$, where $h|_{\mathbb{W}_p} = 0 = h|_{\mathbb{W}_{p^*}}$.

Suppose, $p(x)^k = (x-1)^k$. Then the respective forms are obtained from Lemma 2.1. Suppose $p(x)^k = (x+1)^k$. Let T_w denote the restriction of T to $\mathbb{W}$. Then the minimal polynomial of T_w is $(x+1)^k$. Thus the minimal polynomial of $-T_w$ is $(x-1)^k$. Further T_w preserves a hermitian (resp. skew-hermitian) form if and only if $-T_w$ also preserves it. Thus this case reduces to the previous case and the existence of the required forms are clear.

This completes the proof. □

ACKNOWLEDGEMENT

The second-named author gratefully acknowledges a UGC non-NET fellowship from Jadavpur University. He is highly grateful to his Ph.D advisor Professor Sujit Kumar Sardar for constant guidance, support, kindness and encouragements. He is also grateful to Professor K. C. Chattopadhyay for everything in life that has transformed him into a student of Mathematics. He is also thankful to Dr. Pralay Chatterjee and Dr. R. N. Mukherjee for their encouragements and help at various stages of life.

REFERENCES

1. Clément de Seguins Pazzis, When does a linear map belong to at least one orthogonal or symplectic group?, *Linear Algebra Appl.*, **436**(5) (2012), 1385-1405.
2. Nathan Jacobson, *Lectures in abstract algebra*, Springer-Verlag, New York-Berlin, 1975, Volume II: Linear algebra, Reprint of the 1953 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 31.
3. Krishnendu Gongopadhyay and Ravi S. Kulkarni, On the existence of an invariant non-degenerate bilinear form under a linear map, *Linear Algebra Appl.*, **434**(1) (2011), 89-103.
4. Krishnendu Gongopadhyay, On the conjugacy classes in the orthogonal and symplectic groups over algebraically closed fields, *Expo. Math.*, **28**(4) (2010), 351-356.
5. Ravi S. Kulkarni, Dynamics of linear and affine maps, *Asian J. Math.*, **12**(3) (2008), 321-343.

6. V. V. Sergeĭchuk, Classification problems for systems of forms and linear mappings, *Izv. Akad. Nauk SSSR Ser. Mat.*, **51**(6) (1987), 1170-1190.

7. Vladimir V. Sergeichuk, Canonical matrices of isometric operators on indefinite inner product spaces, *Linear Algebra Appl.*, **428**(1) (2008), 154-192.

8. G. E. Wall, On the conjugacy classes in the unitary, symplectic and orthogonal groups, *J. Austral. Math. Soc.*, **3** (1963), 1-62.