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EULERIAN GRAPHS AND AUTOMORPHISMS OF A MAXIMAL GRAPH

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Let R be a commutative ring with identity. Let $\Gamma(R)$ denote the maximal graph corresponding to the non-unit elements of R, i.e., $\Gamma(R)$ is a graph with vertices the non-unit elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. In this paper, we have shown that, for any finite ring R which is not a field, $\Gamma(R)$ is a Euler graph if and only if R has odd cardinality. Moreover, for any finite ring $R \cong R_1 \times R_2 \times \cdots \times R_n$, where the R_i is a local ring of cardinality $p_i^{\alpha_i}$ for all i, and the p_i's are distinct primes, it is shown that $Aut(\Gamma(R))$ is isomorphic to a finite direct product of symmetric groups. We have also proved that $clique(G(R)') = \chi(G(R)')$ for any semi-local ring R, where $G(R)'$ denote the comaximal graph associated to R.

Key words : Maximal graphs; comaximal graphs; graph automorphisms.

1. INTRODUCTION

The main objective of this paper is to study the interplay of ring-theoretic properties of R with graph-theoretic properties of $\Gamma(R)$.

In 1988, Beck [3], first introduced the idea of associating a graph to a commutative ring with unity. Beck considered R as a simple graph whose vertices are the elements of R such that two different elements x and y are adjacent if and only if $xy = 0$.

In 1995, Sharma and Bhatwadekar [7], introduced another graphical structure on R, which later came to be known as comaximal graphs. In their graphical structure, R is a graph whose vertices are elements of R, and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. Later,

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in 2008, Maimani *et al*. [6], further studied the graph structure defined by Sharma and Bhatwadekar, and named such graph structures as “comaximal graphs”.

In 1999, the zero-divisor graph was introduced by Anderson and Livingston [1]. The zero-divisor graph is defined as a graph in which each non-zero zero-divisor of R is a vertex, and two distinct vertices x and y are adjacent if and only if $xy = 0$. This approach is slightly different from that of Beck [3] in the sense that the set of all non-zero zero-divisors are considered as the vertex set instead of considering the whole of R. This study helps illuminate the structure of $Z(R)$, i.e., the set of zero-divisors of R. They had shown that $Aut(\Gamma(\mathbb{Z}_n))$ is a finite direct product of symmetric groups, and thus $\Gamma(\mathbb{Z}_n)$ is highly symmetrical.

As commutative rings were being associated with different graph structures with the aim of understanding the ring-theoretic properties with the help of graph-theoretic properties, the maximal graph $G(R)$ associated to R was introduced by the authors [5] in 2013. The authors considered $G(R)$ as a simple graph whose vertices are elements of R, and two distinct vertices x and y are adjacent if and only if there is a maximal ideal of R containing both. Note that the maximal graph of a ring R can be obtained as a complement of the comaximal graph, and vice versa also.

In this paper, we assume that R is a commutative ring with identity. Let $\Gamma(R)$ denote the simple graph with vertices as non-unit elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. We shall continue to call this graph a maximal graph of R as the units in R are just the isolated vertices in $G(R)$.

For a graph G, the degree of a vertex v is the number of edges incident with v. It is denoted by $d(v)$. Recall that a walk in a graph G is a finite sequence of vertices $u = v_0, v_1, \ldots, v_n = v$ and edges $a_1, a_2, \ldots, a_n$ of G:

$$v_0, a_1, v_1, a_2, \ldots, a_n, v_n,$$

where the endpoints of a_i are v_{i-1} and v_i, for each i. A walk is closed when the first and last vertices, v_0 and v_n, are the same. A path is a walk in which no vertex is repeated. A graph is said to be connected if there is at least one path between every pair of vertices in G. The distance, $d(u, v)$, between connected vertices u and v is the length of a shortest path connecting them, and $d(u, u) = 0$. The diameter of a connected graph is the supremum of the distances between vertices. The eccentricity of a vertex v is denoted by $E(v)$ and is defined as the distance from v to a vertex farthest from v in G, i.e.,

$$E(v) = max\{d(v, u) : \ u \text{ is a vertex in } G\}.$$

The minimum eccentricity value in a graph G is called the radius of that graph and is denoted by $rad(G)$.

A subset C of the set of all vertices in a graph G is said to be a clique if every pair of distinct vertices in C is adjacent in G. If G contains a clique with n vertices, and every clique in G contains at most n vertices, we say that the clique number of G is n and denote it by $clique(G)$.

Let $\chi(G)$ denote the chromatic number of a graph G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that no two adjacent vertices have the same color.

In Section 2, we prove that $\Gamma(R)$ is a Euler graph if and only if R has odd cardinality. We also show that $Center(\Gamma(R)) = J(R)$, where $J(R)$ is the Jacobson radical of R. In Section 3, we prove that $Aut(\Gamma(R))$ is isomorphic to a finite direct product of symmetric groups, for any finite ring $R \cong R_1 \times R_2 \times \cdots \times R_n$, where the R_i is a local ring of cardinality $p_i^{\alpha_i}$ for all i, and the p_i's are distinct primes. Finally, in Section 4, we have extended the result given by Sharma and Bhatwadekar [7, Theorem 2.3] that $clique(R) = \chi(R)$ for comaximal graphs of finite rings to comaximal graphs of semi-local rings.

2. BASIC PROPERTIES OF A MAXIMAL GRAPH

Throughout this section, R denotes a finite commutative ring with unity that is not a field unless otherwise stated. Clearly, R has finitely many maximal ideals, say, $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. For any ideal I of R, $|I|$ denotes the cardinality of I, i.e., the number of elements in I. Also $J(R)$ denotes the Jacobson radical of R.

Lemma 2.1 — Let a be a non-unit of R and let $\{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } R \ \ and\, a \in \mathfrak{m}\} = \{\mathfrak{m}_{i_1}, \ldots, \mathfrak{m}_{i_k}\}$. Then $d(a) = |\mathfrak{m}_{i_1} \cup \cdots \cup \mathfrak{m}_{i_k}| - 1$ in $\Gamma(R)$.

PROOF : Clearly, an element $b \in R$ is adjacent to a if and only if $b \in \mathfrak{m}_{i_1} \cup \cdots \cup \mathfrak{m}_{i_k}$. As $\Gamma(R)$ is a simple graph, we have $d(a) = |\mathfrak{m}_{i_1} \cup \cdots \cup \mathfrak{m}_{i_k}| - 1$. □

Corollary 2.2 — For all $a \in J(R)$, $d(a) = | \cup_{i=1}^{n} \mathfrak{m}_i| - 1$ in $\Gamma(R)$.

PROOF : This follows immediately from Lemma 2.1. □

Note that for any finite local ring R with maximal ideal $\mathfrak{m}$, $|R| = p^{mn}$, where p is a prime such that $|R/\mathfrak{m}| = p^m$, and n is the length of R. We now have the following:

Proposition 2.3 — Let R be a ring. Then R has odd cardinality if and only if every maximal ideal has odd cardinality.

PROOF : The necessary part is obvious. For the sufficient part, suppose every maximal ideal of R has odd cardinality.

Suppose, if possible, that $|R|$ is even. Then there exists $a \in R, a \neq 0$, such that $2a = 0$. Assume, if possible, that a is a unit in R. Then $2 = 0$ in R, and hence R is a vector space over the field of integers modulo 2. In particular, the cardinality of every non-zero maximal ideal of R is a positive power of 2. Thus (0) is the only maximal ideal of R, i.e., R is a field. A contradiction. Therefore, a is a non-unit in R. Thus there is a maximal ideal $\mathfrak{m}$ of R containing a. Then $|\mathfrak{m}|$ is even. Again, a contradiction. □

We now recall the definition of a Euler graph from [4].

Definition 2.4 — **Euler Graph** : A closed walk running through every edge of the graph G exactly once is called a Euler line and a graph G that contains a Euler line is called a Euler graph.

Recall from [4, Theorem 2.4] that a connected graph G is a Euler graph if and only if all the vertices of G are of even degree. In the next theorem, we give another characterization for a maximal graph $\Gamma(R)$ to be a Euler graph.

***Theorem* 2.5** — *Let R be a ring. Then $\Gamma(R)$ is a Euler graph if and only if R has odd cardinality.*

PROOF : First suppose $\Gamma(R)$ is a Euler graph. Then all the vertices of $\Gamma(R)$ would be of even degree. Since by [2, Proposition 1.11], $\mathfrak{m}_i \nsubseteq \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j$, choose $a_i \in \mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j$ for all $i = 1, 2, \ldots, n$. Now, by Lemma 2.1, $d(a_i) = |\mathfrak{m}_i| - 1$, i.e., $|\mathfrak{m}_i| = d(a_i) + 1$, and hence $|\mathfrak{m}_i|$ is odd for all $i = 1, 2, \ldots, n$. Therefore, by Proposition 2.3, R has odd cardinality.

Conversely, suppose R has odd cardinality. Now, by Proposition 2.3, all the maximal ideals of R would be of odd cardinality. Let $a \in \mathfrak{m}_{i_1} \cap \mathfrak{m}_{i_2} \cap \cdots \cap \mathfrak{m}_{i_k}$. Then by Lemma 2.1, $d(a) = |\cup_{j=1}^{k} \mathfrak{m}_{i_j}| - 1$. Now, as the cardinality of all the terms in the expansion of $|\cup_{j=1}^{k} \mathfrak{m}_{i_j}|$ is odd, and the number of terms in the expansion is $2^k - 1$, we conclude that $|\cup_{j=1}^{k} \mathfrak{m}_{i_j}|$ is odd, and hence $d(a)$ is even. Therefore, every vertex of $\Gamma(R)$ has even degree, and hence $\Gamma(R)$ is a Euler graph. □

Proposition 2.6 — Let R be a ring. Then $rad(\Gamma(R)) \leqslant diam(\Gamma(R)) \leqslant 2$. In particular, $rad(\Gamma(R)) = diam(\Gamma(R)) = 0$ if R is a field, and $rad(\Gamma(R)) = diam(\Gamma(R)) = 1$ if R is local other than a field.

PROOF : Since 0 belongs to every maximal ideal, 0 is adjacent to every vertex of $\Gamma(R)$, and hence $E(0) = 1$. Now, as $\Gamma(R)$ is a simple graph, we have

$$rad(\Gamma(R)) = 1.$$

Now, if R is a local ring, then $d(a,b)=1$ for every pair of vertices a,b $(a\neq b)$ of $\Gamma(R)$, and hence

$$diam(\Gamma(R)) = max\{d(a,b): \text{ a and b are vertices of } \Gamma(R)\} = 1.$$

If R is not local, then $diam(\Gamma(R))=2$ as for any non-zero, non-unit elements a and b, belonging to distinct maximal ideals of R, $\{a,0,b\}$ is a path joining a to b. Therefore,

$$rad(\Gamma(R)) \leq diam(\Gamma(R)) \leq 2. \qquad \square$$

We now recall the following definition from [4].

Definition 2.7 — **Center:** The set of vertices with minimum eccentricity of a graph G is called the center of G and denoted by $Center(G)$.

Proposition 2.8 — Let R be a ring. Then $Center(\Gamma(R)) = J(R)$.

PROOF : Let $a \in J(R)$. Then $d(a,b)=1$ for every vertex b $(\neq a)$ of $\Gamma(R)$. Therefore, $a \in Center(\Gamma(R))$.

Conversely, suppose $a \in Center(\Gamma(R))$. Then $E(a)$ would be minimum. Now, by Proposition 2.6, $rad(\Gamma(R))=1$, and hence $E(a)=1$. Since $\Gamma(R)$ is connected, we conclude that $d(a,b)=1$ for every vertex b $(\neq a)$ in $\Gamma(R)$. Therefore, $a \in J(R)$. Hence $Center(\Gamma(R)) = J(R)$. $\square$

3. AUTOMORPHISM GROUP OF $\Gamma(R)$

In this section also, we continue to let R be a finite commutative ring with unity. We begin this section with the following result in the form of a remark which is well known, but is given for the completeness.

Remark 3.1 : If R is a ring, then $R \cong \prod_{i=1}^{n} R_i$, where R_i is a finite local ring with maximal ideal, say $\mathfrak{n}_i$, for all i. Also, $|R_i| = p_i^{m_i\alpha_i}$ for some prime p_i, where m_i is the length of R_i and $|R_i/\mathfrak{n}_i| = p_i^{\alpha_i}$ for all i. Also, if $\mathfrak{m}_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{n}_i \times R_{i+1} \times \cdots \times R_n$, then

$$|\mathfrak{m}_i| = p_i^{(m_i-1)\alpha_i} \prod_{\substack{j=1\\ j\neq i}}^{n} p_j^{m_j\alpha_j} = p_i^{-\alpha_i}|R|$$

for all i. Note that if the p_i's are distinct for all i, then $|\mathfrak{m}_i| \neq |\mathfrak{m}_j|, \mathfrak{m}_i \neq \mathfrak{m}_j$, whenever $i \neq j$.

***Theorem* 3.2** — *Let $X = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ be the set of all maximal ideals of R. Assume that $|\mathfrak{m}_i| = p_i^{-\alpha_i}|R|$, where the p_i's are distinct primes. Let $I_1, \ldots, I_r, J_1, \ldots, J_s \in X$ such that the I_j's* *(respectively, J_k's) are pairwise distinct, and $|\cup_{j=1}^{r} I_j| = |\cup_{k=1}^{s} J_k|$. Then $r = s$, and there exists an r-permutation σ such that $I_j = J_{\sigma(j)}$ for all $j = 1, \ldots, r$.*

PROOF : For any $1 \le t \le n$ and $i_1 < i_2 < \cdots < i_t$, by [2, Proposition 1.10], we have

$$|\mathfrak{m}_{i_1} \cap \cdots \cap \mathfrak{m}_{i_t}| = p_{i_1}^{-\alpha_{i_1}} \cdots p_{i_t}^{-\alpha_{i_t}} |R|. \tag{i}$$

Let $|I_j| = |R|/m_j$ (respectively, $|J_k| = |R|/l_k$), where $m_j, l_k \in \{p_1^{\alpha_1}, \ldots, p_n^{\alpha_n}\}$. Clearly, the m_j's (respectively, l_k's) are pairwise distinct.

Let Y denote the power set of R. Define $\phi\colon Y \longrightarrow [0,1]$ by $\phi(A) = |A|/|R|$. Clearly, ϕ is a probability function. By (i), $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are mutually independent with respect to ϕ. Therefore, $R \setminus \mathfrak{m}_1, \ldots, R \setminus \mathfrak{m}_n$ are also mutually independent with respect to ϕ. Thus we have

$\phi(R \setminus \cup_{j=1}^{r} I_j) = \phi(\cap_{j=1}^{r}(R \setminus I_j)) = \prod_{j=1}^{r} \phi(R \setminus I_j) = \prod_{j=1}^{r}(1 - m_j^{-1})$ and
$\phi(R \setminus \cup_{k=1}^{r} J_k) = \phi(\cap_{k=1}^{s}(R \setminus J_k)) = \prod_{k=1}^{s} \phi(R \setminus J_k) = \prod_{k=1}^{s}(1 - l_k^{-1})$,

i.e.,

$$l_1 \cdots l_s \prod_{j=1}^{r}(m_j - 1) = m_1 \cdots m_r \prod_{k=1}^{s}(l_k - 1) \tag{ii}$$

We now claim that $\{l_1, \ldots, l_s\} \cap \{m_1, \ldots, m_r\} \neq \emptyset$. Assume to the contrary. Then $l_1, \ldots, l_s$, $m_1, \ldots, m_r$ are pairwise relatively prime. Thus $m_1 \cdots m_r$ divides $\prod_{j=1}^{r}(m_j - 1)$, which is absurd. Therefore, without loss of generality, we may assume that $m_r = l_s$. Now we have

$$l_1 \cdots l_{s-1} \prod_{j=1}^{r-1}(m_j - 1) = m_1 \cdots m_{r-1} \prod_{k=1}^{s-1}(l_k - 1) \tag{iii}$$

Proceeding as above, we get $r = s$, and $\{l_1, \ldots, l_s\} = \{m_1, \ldots, m_r\}$. Now, by Remark 3.1, we get $\{J_1, \ldots, J_s\} = \{I_1, \ldots, I_r\}$. □

Note that the condition, the p_i's are distinct primes, in the above theorem is necessary as we have the following example:

Example 3.3 : Let $R \cong F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are fields with $|F_1| = 2^4$, $|F_2| = 2^2$, and $|F_3| = 5$. Then $\mathfrak{m}_1 = \{0\} \times F_2 \times F_3$, $\mathfrak{m}_2 = F_1 \times \{0\} \times F_3$, and $\mathfrak{m}_3 = F_1 \times F_2 \times \{0\}$. Therefore, $|\mathfrak{m}_1| = 2^2 \cdot 5 = 20$, $|\mathfrak{m}_2| = 2^4 \cdot 5 = 80$, and $|\mathfrak{m}_3| = 2^4 \cdot 2^2 = 64$. Also, $\mathfrak{m}_1 \cap \mathfrak{m}_3 = \{0\} \times F_2 \times \{0\}$; so that $|\mathfrak{m}_1 \cap \mathfrak{m}_3| = 4$. Therefore,

$$|\mathfrak{m}_1 \cup \mathfrak{m}_3| = 20 + 64 - 4 = 80 = |\mathfrak{m}_2|.$$

We now recall the following definition from [4].

Definition 3.4 — **Graph Automorphism** : A graph automorphism $f : G \to G$ of a graph G is a bijection on the vertex set of G which preserves adjacencies.

The set $Aut(G)$ of all graph automorphism of a graph G forms a group under composition of functions. If G has n vertices, then in an obvious way, $Aut(G)$ is isomorphic to a subgroup of S_n, and clearly $Aut(K^n) \cong S_n$, where K^n is a complete graph with n vertices. In fact, for a graph G with n vertices, $Aut(G) \cong S_n$ if and only if $G = K^n$.

To prove the main result of this section, we first establish some notation. Let

$$V_{(1,0,\ldots,0)} = \{x \in R : x \in \mathfrak{m}_1 \ only\},$$

$$V_{(0,1,\ldots,0)} = \{x \in R : x \in \mathfrak{m}_2 \ only\},$$

$$\ldots$$

$$V_{(0,0,\ldots,1)} = \{x \in R : x \in \mathfrak{m}_n \ only\},$$

$$V_{(1,1,\ldots,0)} = \{x \in R : x \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \ only\},$$

$$\ldots$$

$$V_{(1,1,\ldots,1)} = \{x \in R : x \in \cap_{i=1}^{n} \mathfrak{m}_i\}.$$

Also, let $|V_d| = n_d$, where $d \in X = \{(i_1, i_2, \ldots, i_n) : i_1, i_2, \ldots, i_n \in \{0,1\}\}$.

Lemma 3.5 — Let R be a ring as in Theorem 3.2. Then, for any $x, y \in R$, $d(x) = d(y)$ in $\Gamma(R)$ if and only if x, y belong to the same V_d.

PROOF : The sufficiency follows from Corollary 2.2. For necessity, assume $d(x) = d(y)$ for some $x, y \in R$. Suppose $\mathfrak{m}_{i_1}, \mathfrak{m}_{i_2}, \ldots, \mathfrak{m}_{i_t}$ are the only maximal ideals containing x, and $\mathfrak{m}_{j_1}, \mathfrak{m}_{j_2}, \ldots, \mathfrak{m}_{j_l}$ are the only maximal ideals containing y. Then, by Corollary 2.2, $d(x) = |\cup_{r=1}^{t} \mathfrak{m}_{i_r}| - 1$ and $d(y) = |\cup_{k=1}^{l} \mathfrak{m}_{j_k}| - 1$. Since $d(x) = d(y)$, we have $|\cup_{r=1}^{t} \mathfrak{m}_{i_r}| = |\cup_{k=1}^{l} \mathfrak{m}_{j_k}|$. Now, the result follows from Theorem 3.2. □

Now, we are ready to give the main result of this section.

***Theorem* 3.6** — *Let R be a ring as in Theorem 3.2. Then $Aut(\Gamma(R))$ is isomorphic to a finite direct product of symmetric groups. Specifically,*

$$Aut(\Gamma(R)) \cong \prod \{S_{n_d} : d \in X\}.$$

PROOF : Since a graph automorphism preserves degree, and by Lemma 3.5, two vertices of $\Gamma(R)$ have the same degree if and only if they are in the same V_d, we have $f(V_d) = V_d$ for each $f \in Aut(\Gamma(R))$ and $d \in X$. Also, observe that $\Gamma(V_d)$ is a complete graph with n_d vertices. Therefore, $Aut(\Gamma(V_d)) \cong S_{n_d}$ for all $d \in X$.

Define $\Phi : Aut(\Gamma(R)) \to \prod\{S_{n_d} : d \in X\}$ by $\Phi(f) = (f|_{V_d})_{d\in X}$, where $f|_{V_d}$ is the restriction of f on V_d, and can be viewed in the natural way as an element of S_{n_d} for all $d \in X$. Obviously, Φ is a one-one group homomorphism.

To show that Φ is onto, it is enough to show that for each $d \in X$, and permutation $\sigma \in S_{n_d}$ there is an $f \in Aut(\Gamma(R))$ with $f|_{V_d} = \sigma$ and $f|_{V_{d'}} = I_{V_{d'}}$ for all $d' \neq d$ in X. Since $Aut(\Gamma(V_d)) \cong S_{n_d}$, σ is an automorphism on V_d. Now extend σ to an automorphism τ on $\Gamma(R)$ by defining $\tau(a) = a$ for all $a \notin V_d$. Clearly $\Phi(\tau)|_{V_d} = \sigma$ and $\Phi(\tau)|_{V_{d'}} = I_{V_{d'}}$ for all $d' \neq d$ in X. □

From the above theorem, to determine $Aut(\Gamma(R))$, it is enough to compute n_d, i.e., the cardinality of V_d for all $d \in X$. Now, we compute the exact value of the n_d's. First, we assume that R is a reduced ring. Then, each R_i is a field, and hence $|R_i| = p_i^{\alpha_i}$, since the length of R_i is one for all $i = 1, 2, \ldots, n$. Therefore,

$$|V_{(1,0,\ldots,0)}| = |\mathfrak{m}_1 \setminus (\mathfrak{m}_2 \cup \mathfrak{m}_3 \cup \cdots \cup \mathfrak{m}_n)|$$
$$= (p_2^{\alpha_2} - 1)(p_3^{\alpha_3} - 1) \cdots (p_n^{\alpha_n} - 1).$$

Similarly,

$$|V_{(0,1,\ldots,0)}| = (p_1^{\alpha_1} - 1)(p_3^{\alpha_3} - 1) \cdots (p_n^{\alpha_n} - 1),$$
$$\cdots$$
$$|V_{(0,0,\ldots,1)}| = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_{n-1}^{\alpha_{n-1}} - 1),$$
$$|V_{(1,1,\ldots,0)}| = (p_3^{\alpha_3} - 1)(p_4^{\alpha_4} - 1) \cdots (p_n^{\alpha_n} - 1),$$
$$\cdots$$
$$|V_{(1,1,\ldots,1,0)}| = (p_n^{\alpha_n} - 1),$$
$$\cdots$$
$$|V_{(1,1,\ldots,1)}| = 1.$$

We now assume that R is not reduced. Then,

$$|V_{(1,0,\ldots,0)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n} (p_2^{\alpha_2} - 1)(p_3^{\alpha_3} - 1) \cdots (p_n^{\alpha_n} - 1),$$

$$|V_{(0,1,\ldots,0)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n} (p_1^{\alpha_1} - 1)(p_3^{\alpha_3} - 1) \cdots (p_n^{\alpha_n} - 1),$$

$$\cdots$$

$$|V_{(0,0,\ldots,1)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n} (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_{n-1}^{\alpha_{n-1}} - 1),$$

$$|V_{(1,1,\ldots,0)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n} (p_3^{\alpha_3} - 1)(p_4^{\alpha_4} - 1) \cdots (p_n^{\alpha_n} - 1),$$

$$\cdots$$

$$|V_{(1,1,\ldots,1,0)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n} (p_n^{\alpha_n} - 1),$$

$$\cdots$$

$$|V_{(1,1,\ldots,1)}| = p_1^{(m_1-1)\alpha_1} p_2^{(m_2-1)\alpha_2} \cdots p_n^{(m_n-1)\alpha_n}.$$

Corollary 3.7 — Let $R = \mathbb{Z}_n$. Then

(a) $Aut(\Gamma(\mathbb{Z}_n))$ is trivial if and only if $\mathbb{Z}_n$ is a field;

(b) $Aut(\Gamma(\mathbb{Z}_n)) \cong S_2$,which is abelian, if and only if $n = 4$ or 6.

PROOF : (a) is obvious. For (b), let $n = 4$. Then $\mathbb{Z}_4$ is a local ring with maximal ideal of cardinality two. Therefore, $|V_1| = 2$, and hence $Aut(\Gamma(\mathbb{Z}_4)) \cong S_2$. Now let $n = 6$. Then $\mathbb{Z}_6$ is a ring with two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, where $\mathfrak{m}_1 = \{0, 3\}$ and $\mathfrak{m}_2 = \{0, 2, 4\}$. Thus $|V_{(1,0)}| = 1, |V_{(0,1)}| = 2$, and $|V_{(1,1)}| = 1$. Therefore, $Aut(\Gamma(\mathbb{Z}_6)) \cong S_1 \times S_2 \times S_1 \cong S_2$.

Conversely, suppose that $Aut(\Gamma(\mathbb{Z}_n)) \cong S_2$. Since the number of V_d's is $2^k - 1$, where k is the number of maximal ideals in $\mathbb{Z}_n$, we have

$$Aut(\Gamma(\mathbb{Z}_n)) \cong S_2 \cong S_1 \times \cdots \times S_1 (m \text{ times}) \times S_2,$$

where $m = 2^k - 2$. Note that the cardinality of any maximal ideal, say $\mathfrak{m}_i$, in $\mathbb{Z}_n$ is the sum of the cardinalities of 2^{k-1} V_d's. Also note that the cardinality of all the V_d's is 1 except the one which is of cardinality two. Therefore, if $k \geq 3$, then $\mathbb{Z}_n$ will have at least two maximal ideals of the same cardinality, which is absurd in $\mathbb{Z}_n$. Now, if $k = 1$, then $\mathbb{Z}_n$ is a local ring with maximal ideal of cardinality two, and hence $n = 4$. If $k = 2$, then $|\mathfrak{m}_1| = |V_{(1,0)}| + |V_{(1,1)}|$ and $|\mathfrak{m}_2| = |V_{(0,1)}| + |V_{(1,1)}|$. Since $\mathbb{Z}_n$ is a reduced ring, we have either $|V_{(1,0)}| = 1, |V_{(0,1)}| = 2$, and $|V_{(1,1)}| = 1$, or $|V_{(1,0)}| = 2, |V_{(0,1)}| = 1$, and $|V_{(1,1)}| = 1$. Therefore, the only possibility for n is 6. □

4. Complement of Maximal Graphs

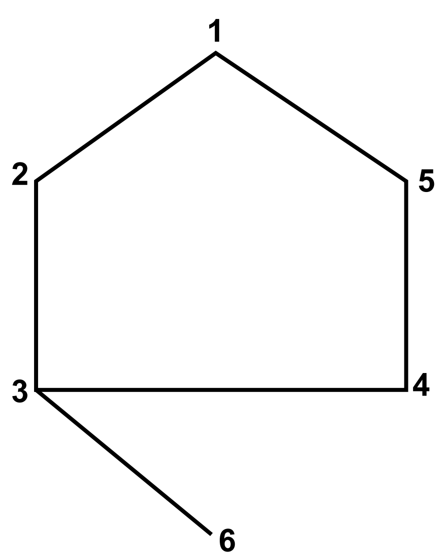
Throughout this section, we will denote the maximal graph [5] associated to a ring R by $G(R)$, i.e., the vertices of $G(R)$ are all the elements of R, and two distinct vertices x, y are adjacent in $G(R)$ if and only if there is a maximal ideal in R containing both x, y. We begin this section with the following definition.

Definition 4.1 — The complement G' of a graph G is a graph with the same vertex set as G, and with the property that two vertices are adjacent in G' if and only if they are not adjacent in G.

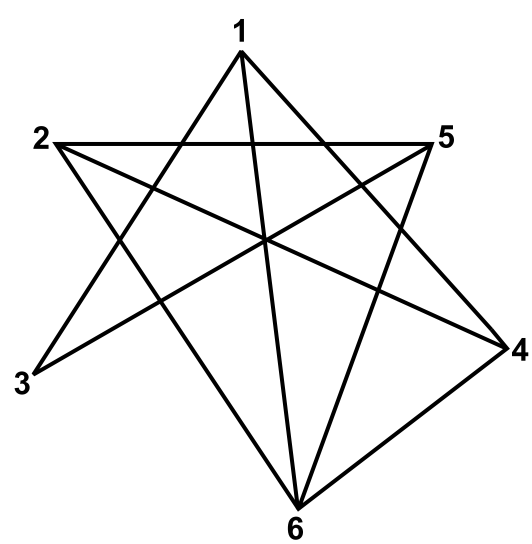
Throughout this section, $G(R)'$ denotes the complement of maximal graph $G(R)$ associated to R. Note that the graph $G(R)'$ is precisely the comaximal graph associated to R as defined in [7] and vice versa also, i.e., the maximal and comaximal graph associated to R are just the complement of each other. In general, for any graph G, the following need not be true:

$$clique(G) = \chi(G) \text{ if and only if } clique(G') = \chi(G'). \quad \text{(iv)}$$

For example, consider the following graph G and its complement G'.



G.



G'.

In the above graph, $clique(G) = 2 \neq \chi(G) = 3$; however,

$$clique(G') = \chi(G') = 3.$$

But for a maximal graph $G(R)$ of a finite ring R, both the equalities in (iv) holds independently. Note that, for any finite ring R, $clique(G(R)) = \chi(G(R))$, by [5, Theorem 3.5]. Since the graph $G(R)'$ is nothing but the comaximal graph associated to R as defined in [7], by [7, Theorem 2.3], we have $clique(G(R)') = \chi(G(R)')$.

Sharma and Bhatwadekar [7], proved that, for any finite ring R,

$$clique(G(R)') = \chi(G(R)') = m + n,$$

where m is the number of units and n is the number of maximal ideals in R.

In the next theorem we are extending the same result to semi-local rings and thereby establishing Beck's conjecture [3] for comaximal graph of semi-local rings. Recall that for any two sets A, B, $|A| \leq |B|$ if there is an injective map from A to B.

***Theorem* 4.2** — *Let R be a semi-local ring. Then*

$$clique(G(R)') = \chi(G(R)') = n + |U(R)|,$$

where $U(R)$ is the set of all units in R and n is the number of maximal ideals in R.

PROOF : First assume $U(R) = \{u_1, u_2, \ldots, u_m\}$ is a finite set. Since the u_i's are just isolated vertices in $G(R)$, $\{u_1, u_2, \ldots, u_m\}$ forms a clique in $G(R)'$. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be the maximal ideals of R. Choose $a_i \in \mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j$ for all $i = 1, 2, \ldots, n$. Now, for any i, a_i cannot be adjacent to u_j for all j, and there cannot be any edge between $a_1, \ldots, a_n$ in $G(R)$; so we conclude that $\{u_1, \ldots, u_m, a_1, a_2, \ldots, a_n\}$ forms a clique in $G(R)'$. Note that this is a maximal clique in $G(R)'$ as any clique in $G(R)'$ can contain at most one element of every maximal ideal. Therefore, $clique(G(R)') = n + m$.

Since $clique(G(R)') = m + n \leq \chi(G(R)')$, we need at least $n + m$ colors to color $G(R)'$. We now produce a coloring with exactly $m + n$ colors. First we assign the $m + n$ distinct colors to the elements $u_1, \ldots, u_m, a_1, a_2, \ldots, a_n$. As no pair of elements in $\mathfrak{m}_1$ is adjacent to each other in $G(R)'$, we assign the same color to every element of $\mathfrak{m}_1$ as that of a_1. Similarly, no pair of elements in $\mathfrak{m}_2$ is adjacent to each other in $G(R)'$, we assign the color of a_2 to all the elements of $\mathfrak{m}_2 \setminus \mathfrak{m}_1$. Proceeding in the same way, eventually we assign the color of a_n to all the elements of $\mathfrak{m}_n \setminus \cup_{i=1}^{n-1} \mathfrak{m}_i$. Thus we can color $G(R)'$ with $m + n$ colors, i.e., $\chi(G(R)') \leq m + n$. Therefore, $clique(G(R)') = \chi(G(R)') = n + |U(R)|$.

Now assume $U(R)$ is an infinite set. Then, by the same argument as above, the elements of $U(R)$ together with $\{a_1, a_2, \ldots, a_n\}$ forms a maximal clique in $G(R)'$. Therefore, $clique(G(R)') = n + |U(R)|$.

Also, as all the non-units in R can be colored with exactly n colors as shown above, we conclude that $\chi(G(R)') \leq n + |U(R)|$. Therefore,

$$clique(G(R)') = \chi(G(R)') = n + |U(R)|. \qquad \square$$

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