Indian J. Pure Appl. Math., **48**(2): 271-283, June 2017

DOI: 10.1007/s13226-017-0225-8

JULIA SET OF $\lambda \exp(z)/z$ WITH REAL PARAMETERS λ

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(*Received* 1 *April* 2016; *after final revision* 30 *September* 2016;
accepted 16 *December* 2016)

In this paper, we investigate the Julia set of the family $\lambda \exp(z)/z$ with real parameters λ. We look for what values of real parameters λ such that the Julia set of $\lambda \exp(z)/z$ does not coincide with the whole plane, and thus gives a complete classification for real parameters, which is similar to Jang's result of a family of transcendental entire functions. Moreover, We also discuss the shape and size of Fatou sets and Julia sets of $\lambda \exp(z)/z$ with real parameters λ when the Julia sets are not the whole plane.

Key words : Julia set; Fatou set; Hausdorff dimension; meromorphic function.

1. INTRODUCTION

In 1926, Fatou [6] conjectured that the Julia set of the exponential map $E: z \mapsto e^z$ is the whole plane, which means the family of forward iteration $\{E^n\}_{n=0}^{\infty}$ is not normal at any point. This conjecture was proved by Misiurewicz [12] in 1980. Since then, the dynamics of exponential maps had been extensively studied and quite well understood by several authors. For instance, McMullen [11] proved that the Julia set of any member of the exponential family λe^z always has Hausdorff dimension two. For more details about the exponential family, one may see e.g. [3, 10, 14, 15, 17, 19].

Let $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the complex plane and punctured plane, respectively. As the exponential family may be considered as the simplest family of transcendental entire functions, the family $F_\lambda : z \mapsto \lambda \exp(z)/z$, defined on the punctured plane $\mathbb{C}^*$, may be viewed as the simplest one of transcendental meromorphic functions. For the map $z \mapsto \exp(z)/z$, it is proved in [20] that for almost every point in $\mathbb{C}^*$, its ω-limit set equals to the set consisting of zero and infinity, which means that the set Λ of all those points in $\mathbb{C}^*$ whose ω-limit set of the map is not the set of zero and infinity has zero measure. Moreover, [23] proved that the set Λ has Hausdorff dimension two, thus Λ is a fractal set. The M-set of F_λ is defined to be the set consisting of all parameters λ for which the Fatou set of F_λ is empty. It is proved in [21] that the M-set of F_λ has infinite area and in [22] that the point 1 is a density point of M-set, which means the probability that F_λ has empty Fatou set approaches 1 as $\lambda \to 1$.

Jang [7] investigated the Julia set of entire functions f_μ defined by $f_\mu(z) = z\exp(z+\mu)$ with parameters μ. More precisely, it is proved in [7, Theorem 1] that there is a constant $\mu_* > 2$ such that if the real parameter μ belongs to the set $(-\infty, 2)\cup(2, \mu_*)$, then the Julia set of $f_\mu :\ z \mapsto z\exp(z+\mu)$ is not the whole complex plane. However, [7, Theorem 2], [8, Proposition 1] and [8, Theorem 2] imply there exist sequences $\{\mu_n\}$ and $\{\mu'_n\}$ of real parameters contained in $(\mu_*, +\infty)$ and converging to infinity such that the Julia set of f_{μ_n} is the whole plane and the Julia set of $f_{\mu'_n}$ is not the whole plane.

Motivated by the results of f_μ as above, we consider the family F_λ with real parameters λ in this paper. We look for what values of real parameters λ such that the Julia sets of F_λ are not the whole plane and thus gives a complete classification for real parameter λ of F_λ, which is similar to [7, Theorem 1]. Moreover, we also discuss the shape and size of Fatou sets and Julia sets of F_λ when the Julia sets are not the whole plane. Before stating our main results, let us give some notations and definitions for convenience.

Notation: As in [18], a $standard$ meromorphic function is referred to be a meromorphic function which is not rational of degree less than two. Let f be a standard meromorphic function and denote by f^n the nth iterate for all $n \in \mathbb{N}$. We denote by $\mathcal{F}(f)$ the $Fatou\ set$ of f, which consists of all points z on $\widehat{\mathbb{C}}$ such that the family $\{f^n\}_{n\in\mathbb{N}}$ is well defined, meromorphic and normal in a neighborhood of z. We denote by $\mathcal{J}(f) = \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$ the $Julia\ set$ of f, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Let $D(z, \rho)$ be an open disk centered at z with radius ρ. Let meas(X) denote the two-dimensional Lebesgue measure for any measurable set $X \subset \mathbb{C}$.

Definition — (i) As in [18], we define $E(f)$ to be the set of exceptional values of f, that is, the points whose inverse orbit

$$O^-(z) = \{w : f^n(w) = z \text{ for some } n \in \mathbb{N}\}$$

is finite.

(ii) As in [20], let

$$\Lambda(F_\lambda) = \{z \in \mathbb{C}^* :\ \omega_{F_\lambda}(z) \neq \{0, \infty\}\}$$

for each $\lambda \in \mathbb{C}^*$, where $\omega_{F_\lambda}(z)$ is the ω-$limit\ set$ of F_λ at the point z, which consists of all accumulation points of $\bigcup_{n\in\mathbb{N}} \{F_\lambda^n(z)\}$ on $\widehat{\mathbb{C}}$.

We prove:

***Theorem* 1.1** — *Let $\lambda \in \mathbb{R}$, then $\mathcal{J}(F_\lambda)$ does not coincide with the whole plane if and only if $\lambda \in (0, 4e^{-2}]$. Moreover, if $\lambda \in (0, 4e^{-2})$, then $\mathcal{F}(F_\lambda) \cap \mathbb{R} = (a, b)$, where $a \in (0, 1)$ and $b \in (2, +\infty)$ satisfy $F_\lambda(a) = b$ and $F_\lambda(b) = b$, and $\mathcal{F}(F_\lambda)$ consists of infinitely many simply-connected components and contains no completely invariant components. In particular, $\mathcal{J}(F_\lambda)$ is connected on $\widehat{\mathbb{C}}$ for each $\lambda \in (0, 4e^{-2})$.*

Remark : For the exponential maps $E_\lambda(z) = \lambda e^z$ with $0 < \lambda < 1/e$, the immediate basin of the (unique) attracting fixed point of E coincides with $\mathcal{F}(E_\lambda)$ and $\mathcal{J}(E_\lambda)$ is a Cantor bouquet.

***Theorem* 1.2** — *If F_λ has an attracting periodic cycle, then for almost every point $z \in \mathcal{J}(F_\lambda)$, $\omega_{F_\lambda}(z) = \{0, \infty\}$, that is, $\mathrm{meas}\big(\mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda)\big) = 0$. Moreover, $\mathcal{J}(F_\lambda)$ has infinite area for each $\lambda \in (0, 4e^{-2})$.*

Remark : For the exponential family, McMullen [11] proved when λe^z has an attracting periodic cycle (e.g. $0 < \lambda < 1/e$), the area of its Julia set is zero.

2. Preliminary Lemmas

Lemma 2.1 (see [22, Lemma 2.1]) — If $\omega_{F_\lambda}(1) \subset \{0, \infty\}$, then $\mathcal{J}(F_\lambda)$ is the whole plane.

Lemma 2.2 (see [1, Theorem 1]) — If f is a (non-Möbius) analytic map of $\mathbb{C}^*$ to itself, then the components of $\mathcal{F}(f)$ are simply or doubly-connected. There is at most one doubly-connected component, except in the case of $f(z) = kz^n$ with $k \neq 0$, $n \in \mathbb{Z}$ and $n \neq 0, \pm 1$.

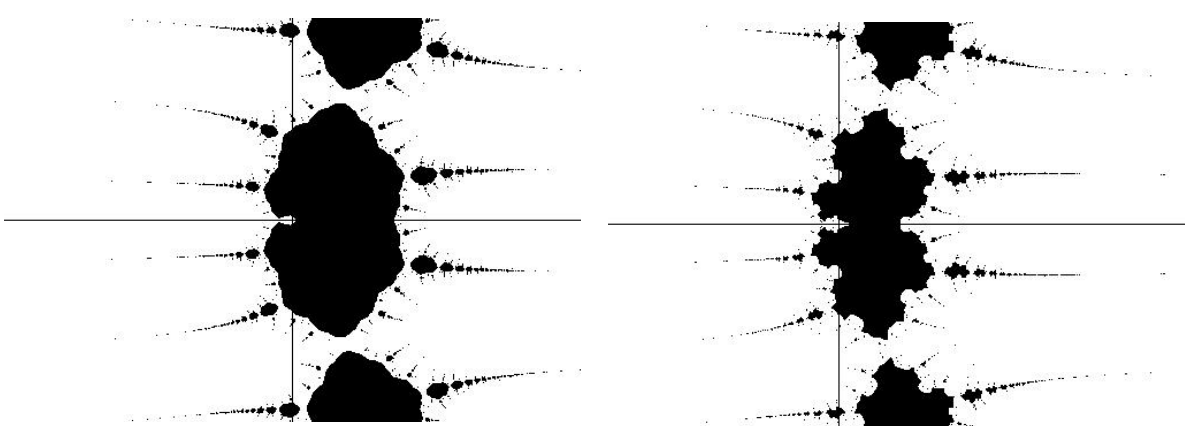

Fig. 1. Fatou sets of two maps in the family F_λ with real parameters. Left: $z \mapsto \frac{1}{4\sqrt{e}} \exp(z)/z$; right: $z \mapsto \frac{1}{e} \exp(z)/z$.

Lemma 2.3 (see [4, Theorem K]) — Let f be a transcendental meromorphic function with at most finitely many poles. Then there is at most one completely invariant component of $\mathcal{F}(f)$. In particular, $\mathcal{F}(f)$ consists of one or infinitely many components.

Lemma 2.4 (see [13, p. 144]) — Let W be a compact subset of $\widehat{\mathbb{C}}$. Then W is connected on $\widehat{\mathbb{C}}$ if and only if all components of $\widehat{\mathbb{C}} \setminus W$ are simply-connected.

Lemma 2.5 (see [5, Koebe Distortion Theorem]) — Suppose $f(z)$ is a univalent function in $D(z_0, \rho)$, $t \in (0,1)$, then

$$\frac{|f'(z_0)|t\rho}{(1+t)^2} \leq |f(z) - f(z_0)| \leq \frac{|f'(z_0)|t\rho}{(1-t)^2}$$

and

$$\frac{1-t}{(1+t)^3} \leq \frac{|f'(z)|}{|f'(z_0)|} \leq \frac{1+t}{(1-t)^3}$$

for any $z \in D(z_0, t\rho)$.

Lebesgue Density Theorem is the main tool in proving the two-dimensional Lebesgue measure of a planar set is zero.

Lemma 2.6 (see [16, Theorem 7.8]) — Let X be a measurable set and $meas(X) > 0$, then

$$\lim_{\rho \to 0} \frac{\mathrm{meas}\big(D(z,\rho) \cap X\big)}{\mathrm{meas}\big(D(z,\rho)\big)} = 1$$

for almost all $z \in X$ and each such z is called a density point of X.

3. Proof of Theorem 1.1

3.1 *A Key Lemma*

Before proving Theorem 1.1, we first need to prove the following key Lemma.

Lemma 3.1 — For each $\lambda \in \mathbb{R} \setminus \{0\}$, the preimage $F_\lambda^{-1}\big((-\infty, 0)\big)$ contains the rays L_k for all odd integers $k \geq 1$ and even integers $k \leq -2$; and the preimage $F_\lambda^{-1}\big((0, +\infty)\big)$ contains the rays L_k for all even integers $k \geq 1$ and odd integers $k \leq -2$, where L_k is the graph of $x(y) = y \cot y$ in $y \in \big(k\pi, (k+1)\pi\big)$ which is strictly decreasing (resp. increasing) on y if $k \geq 1$ (resp. $k \leq -2$) with asymptotes $y = k\pi$ and $y = (k+1)\pi$.

Proof : Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $z = x + iy$, we have

$$F_\lambda(z) = \frac{\lambda e^x}{x^2+y^2}(x\cos y + y \sin y) + i\frac{\lambda e^x}{x^2+y^2}(x \sin y - y \cos y).$$

Then $\operatorname{Re} F_\lambda(z) = 0$ if and only if

$$x \cos y + y \sin y = 0 \tag{3.1}$$

and $\operatorname{Im} F_\lambda(z) = 0$ if and only if

$$x \sin y - y \cos y = 0. \tag{3.2}$$

Firstly, we look for what values of z such that (3.2) holds. It is obvious to see that (3.2) holds if $y = 0$ and that (3.2) does not hold if $y = k\pi$ with $k \in \mathbb{Z}$ and $k \neq 0$. If $y \neq k\pi$ with $k \in \mathbb{Z}$ and $k \neq 0$, then (3.2) is equivalent to

$$x = x(y) = y \cot y.$$

Using simple computation implies $x(y)$ is strictly increasing on $y \in \big(k\pi, (k+1)\pi\big)$ for each integer $k \leq -2$ and strictly decreasing on $y \in \big(k\pi, (k+1)\pi\big)$ for each integer $k \geq 1$.

Let L_k denote the graph of $x(y) = y \cot y$ in $y \in \big(k\pi, (k+1)\pi\big)$ for each integer $k \geq 1$ or $k \leq -2$. Moreover, we have for each integer $k \geq 1$

$$\lim_{y \to k\pi^+} x(y) = +\infty$$

and

$$\lim_{y \to (k\pi+\pi)^-} x(y) = -\infty;$$

and for each integer $k \leq -2$

$$\lim_{y \to k\pi^+} x(y) = -\infty$$

and

$$\lim_{y \to (k\pi+\pi)^-} x(y) = +\infty.$$

This implies L_k has the asymptotes $y = k\pi$ and $y = (k+1)\pi$ and thus disconnects the whole plane for each integer $k \geq 1$ or $k \leq -2$.

Secondly, we look for what values of z such that (3.1) holds. It is obvious to see that (3.1) does not hold if $y = k\pi + \pi/2$ with $k \in \mathbb{Z}$. If $y \neq k\pi + \pi/2$ with $k \in \mathbb{Z}$, then (3.1) is equivalent to

$$x = x(y) = -y \tan y.$$

Using simple computation implies $x(y)$ is strictly increasing on $y \in (k\pi + \pi/2, k\pi + 3\pi/2)$ for each integer $k \leq -2$ and strictly decreasing on $y \in (k\pi + \pi/2, k\pi + 3\pi/2)$ for each integer $k \geq 0$.

Denote M_k by the graph of $x(y) = -y \tan y$ in $y \in (k\pi + \pi/2, k\pi + 3\pi/2)$ for each integer $k \geq 0$ or $k \leq -2$. Moreover, we have for each integer $k \geq 0$

$$\lim_{y \to (k\pi + \pi/2)^+} x(y) = +\infty$$

and

$$\lim_{y \to (k\pi + 3\pi/2)^-} x(y) = -\infty;$$

and for each integer $k \leq -2$

$$\lim_{y \to (k\pi + \pi/2)^+} x(y) = -\infty$$

and

$$\lim_{y \to (k\pi + 3\pi/2)^-} x(y) = +\infty.$$

This implies M_k has the asymptotes $y = k\pi + \pi/2$ and $y = k\pi + 3\pi/2$ and thus disconnects the whole plane for each integer $k \geq 0$ or $k \leq -2$.

Furthermore, let T_k be the domain bounded by M_k and M_{k+1} for each integer $k \geq 0$ or $k \leq -3$. By continuity of $\mathrm{Re}\, F_\lambda(z)$, we have

$$\mathrm{Re}\, F_\lambda(z) < 0$$

if $z \in T_k$ for each even integer $k \geq 0$ or odd integer $k \leq -3$; and

$$\mathrm{Re}\, F_\lambda(z) > 0$$

if $z \in T_k$ for each odd integer $k \geq 0$ or even integer $k \leq -3$.

This, together with the facts that $F_\lambda(L_k) \subset \mathbb{R}$ and that $L_k \subset T_{k-1}$ for each integer $k \geq 1$ or $k \leq -2$, implies

$$F_\lambda(L_k) \subset (-\infty, 0)$$

for each odd integer $k \geq 1$ or even integer $k \leq -2$; and

$$F_\lambda(L_k) \subset (0, +\infty)$$

for each even integer $k \geq 1$ or odd integer $k \leq -2$.

Therefore, the preimage $F_\lambda^{-1}\big((-\infty, 0)\big)$ contains the rays L_k for all odd integers $k \geq 1$ and even integers $k \leq -2$; and the preimage $F_\lambda^{-1}\big((0, +\infty)\big)$ contains the rays L_k for all even integers $k \geq 1$ and odd integers $k \leq -2$, where L_k is the graph of $x(y) = y \cot y$ in $y \in \big(k\pi, (k+1)\pi\big)$ which is strictly decreasing (resp. increasing) on y if $k \geq 1$ (resp. $k \leq -2$) with asymptotes $y = k\pi$ and $y = (k+1)\pi$. □

3.2 *Proof of Theorem 1.1*

PROOF : Let $\lambda < 0$ and $z > 0$, then

$$F_\lambda(z) = \lambda \exp(z)/z < 0$$

and

$$F_\lambda^2(z) = z \exp\big(F_\lambda(z) - z\big) \in (0, z),$$

which implies that the sequence $\{F_\lambda^{2n}(z)\}$ is strictly decreasing with respect to n for fixed λ and z. So $\lim_{n\to\infty} F_\lambda^{2n}(z) = 0$ and $\lim_{n\to\infty} F_\lambda^{2n+1}(z) = -\infty$. In particular, $\omega_{F_\lambda}(1) = \{0, \infty\}$ and thus by Lemma 2.1, $\mathcal{J}(F_\lambda)$ is the whole plane.

Let now $\lambda > 0$, then F_λ is strictly decreasing on $z \in (0,1)$ and strictly increasing on $z \in (1,\infty)$. This, together with $\lim_{z\to 0^+} F_\lambda(z) = +\infty$ and $\lim_{z\to+\infty} F_\lambda(z) = +\infty$, implies $F_\lambda\big((0,+\infty)\big) \subset (0,+\infty)$. Observe that $F_\lambda(z) = z$ for some $z > 0$ if and only if $\ln\lambda = 2\ln z - z$ for some $z > 0$. Hence, if $\ln\lambda > \max_{z>0}(2\ln z - z)$, that is, $\lambda > 4e^{-2}$, then by continuity, $F_\lambda(z) > z$ and thus $\lim_{n\to\infty} F_\lambda^n(z) = +\infty$ for each $z > 0$. In particular, $\omega_{F_\lambda}(1) = \{\infty\}$ and thus by Lemma 2.1, $\mathcal{J}(F_\lambda)$ is the whole plane.

If $\lambda = 4e^{-2}$, then $F_\lambda(2) = 2$ and $F_\lambda'(2) = 1$. So the point $z = 2$ is a parabolic fixed point of F_λ and thus $\mathcal{J}(F_\lambda)$ is not the whole plane.

Let $\lambda \in (0, 4e^{-2})$. Note that $F_\lambda(z) = z$ for some $z \in \mathbb{R}$ if and only if $\lambda = z^2e^{-z}$ for some $z \in \mathbb{R}$, so we consider the function $g(t) = t^2e^{-t}$ for all $t \in \mathbb{R}$. It is easy to see that $g(t)$ is strictly decreasing on $(-\infty, 0)$ and $(2, +\infty)$, and strictly increasing on $(0, 2)$. By $g(0) = 0$ and $g(2) = 4e^{-2}$ for each $\lambda \in (0, 4e^{-2})$, the equation $g(t) = \lambda$ has three real roots, say $t_1 < 0$, $t_2 \in (0, 2)$ and $t_3 > 2$, among which only t_2 is an attracting fixed point since $|F_\lambda'(t_2)| = |t_2 - 1| < 1$. Thus $\mathcal{J}(F_\lambda)$ is not the whole plane. This proves the first half of Theorem 1.1.

Next we shall use Lemma 3.1 to prove the second half of Theorem 1.1. Let $\lambda \in (0, 4e^{-2})$. We consider the function

$$H_\lambda(x) = \lambda e^x/x^2$$

with $x > 0$. By computation,

$$H_\lambda'(x) = \lambda e^x(x^2 - 2x)/x^4.$$

Then H_λ is strictly decreasing on $(0, 2)$ and strictly increasing on $(2, +\infty)$. Since $H_\lambda(2) < 1$, there is a unique point in $(2, +\infty)$, say b, such that $H_\lambda(b) = 1$, that is, $F_\lambda(b) = b$. Since F_λ has a unique attracting fixed point, say $t \in (0, 2)$, this implies $z \in \mathcal{F}(F_\lambda)$ if and only if $\lim_{n\to\infty} F_\lambda^n(z) = t$. Note that there is a unique point $a \in (0, 1)$ such that $F_\lambda(a) = b$. Since $F_\lambda^n(x) \geq b$ for each $x \geq b$

and $F_\lambda^n(x) < 0$ for each $x < 0$, using the complete invariance of $\mathcal{J}(F_\lambda)$ implies $(-\infty, a] \cup [b, +\infty) \subset \mathcal{J}(F_\lambda)$.

If $\lambda = e^{-1}$, then $t = 1$ and for each $x \in [1, b)$, $F_\lambda^n(x) \geq 1$ such that $\{F_\lambda^n(x)\}$ is strictly decreasing on n, thus $\lim\limits_{n\to\infty} F_\lambda^n(x) = 1$. This, together with $F_\lambda\big((a, 1]\big) = [1, b)$, implies $\mathcal{F}(F_\lambda) \cap \mathbb{R} = (a, b)$.

If $\lambda \in (e^{-1}, 4e^{-2})$, then $a < 1 < t < b$. Since $t < F_\lambda(x) < x$ for each $x \in (t, b)$, by induction, we have $t < F_\lambda^{n+1}(x) < F_\lambda^n(x)$ for each $n \geq 1$, then $\lim\limits_{n\to\infty} F_\lambda^n(x) = t$. From $1 < F_\lambda(1) < t$, we have $1 < F_\lambda^n(1) < F_\lambda^{n+1}(1) < t$ and $F_\lambda^n\big((a, t)\big) = \big(F_\lambda^n(1), b\big)$. Also note that $\lim\limits_{n\to\infty} F_\lambda^n(1) = t$, this implies $\lim\limits_{n\to\infty} F_\lambda^n(x) = t$ for each $x \in (a, b)$ and $\mathcal{F}(F_\lambda) \cap \mathbb{R} = (a, b)$.

If $\lambda \in (0, e^{-1})$, then $a < t < 1 < b$. Note that $F_\lambda(x)$ is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, +\infty)$ with respect to x. Since $t > F_\lambda(1)$, there is a unique $t' > 1$ such that $F_\lambda(t') = t$. We claim that $[t, t'] \subset \mathcal{F}(F_\lambda)$.

Since $F_\lambda([t, t']) = [F_\lambda(1), t]$ and $t < 1$, $F_\lambda^2([t, t']) = [t, F_\lambda^2(1)]$. Also note that $F_\lambda^2(1) < 1$, then $F_\lambda^3([t, t']) = [F_\lambda^3(1), t]$ and $F_\lambda^4([t, t']) = [t, F_\lambda^4(1)]$. From

$$F_\lambda^4(1) = F_\lambda^2(1) \exp\big(F_\lambda^3(1) - F_\lambda^2(1)\big) < F_\lambda^2(1),$$

by induction for each integer $k \geq 1$, we have

$$F_\lambda^{2k+1}([t, t']) = [F_\lambda^{2k+1}(1), t], \;\; F_\lambda^{2k}([t, t']) = [t, F_\lambda^{2k}(1)].$$

This, together with $\lim\limits_{n\to\infty} F_\lambda^n(1) = t$, implies for each $x \in [t, t']$, $\lim\limits_{n\to\infty} F_\lambda^n(x) = t$. So $[t, t'] \subset \mathcal{F}(F_\lambda)$.

Note that there exist $t_1 > t'$ such that $F_\lambda(t_1) = t'$, so $F_\lambda([t', t_1]) = [t, t']$. Then there is a $t_2 > t_1$ such that $F_\lambda(t_2) = t_1$, so $F_\lambda([t_1, t_2]) = [t', t_1]$. Inductively, we can get a positive and increasing sequence $\{t_n\}_{n\geq 2}$ such that for each $n \geq 2$, $F_\lambda([t_n, t_{n+1}]) = [t_{n-1}, t_n]$. Since $[b, +\infty) \subset \mathcal{J}(F_\lambda)$, $\{t_n\}_{n\geq 1}$ is bounded and convergent. So $\lim\limits_{n\to\infty} t_n = b$. From $[t, t'] \subset \mathcal{F}(F_\lambda)$ and complete invariance of $\mathcal{F}(F_\lambda)$, we have $(t', b) \subset \mathcal{F}(F_\lambda)$ and so $[t, b) \subset \mathcal{F}(F_\lambda)$. This, together with $F_\lambda\big((a, t]\big) = [t, b)$, implies $\mathcal{F}(F_\lambda) \cap \mathbb{R} = (a, b)$.

By Lemma 2.2, each component of $\mathcal{F}(F_\lambda)$ is simply or doubly connected, and there is at most one doubly connected component of $\mathcal{F}(F_\lambda)$. Suppose $\mathcal{F}(F_\lambda)$ has a doubly connected component, say U_0. Note that $\mathcal{F}(F_\lambda)$ is symmetric with respect to the real axis, then U_0 must intersect the real axis such that $U_0 \cap \mathbb{R}$ consists of two open intervals, but this contradicts $\mathcal{F}(F_\lambda) \cap \mathbb{R} = (a, b)$. So the components of $\mathcal{F}(F_\lambda)$ are simply connected.

Let D be the immediate basin of the (unique) attracting fixed point t of F_λ. Then $D \cap \mathbb{R} = (a, b)$ and so D is located between rays L_1 and L_{-2}. Since $(a, b) \subset (0, +\infty)$, using Lemma 3.1 again implies $F_\lambda^{-1}(D \cap \mathbb{R})$ intersects the rays L_k for all even integers $k \geq 1$ and odd integers $k \leq -2$. Therefore, there is at least one component of $F_\lambda^{-1}(D)$ located between rays L_k and L_{k+2} for all odd integers $k \geq 1$ and even integers $k \leq -4$.

Hence $F_\lambda^{-1}(D)$ has infinitely many components and D is not completely invariant. Also note that $F_\lambda^{-1}(D) \subset \mathcal{F}(F_\lambda)$ and the forward orbits of all points in $\mathcal{F}(F_\lambda)$ must approach the (unique) attracting fixed point in D. So $\mathcal{F}(F_\lambda)$ consists of infinitely many simply connected components and contains no completely invariant components. In particular, Lemma 2.4 implies $\mathcal{J}(F_\lambda)$ is connected on $\widehat{\mathbb{C}}$. This proves the second half of Theorem 1.1. □

4. Proof of Theorem 1.2

Proof : Note that for each $\lambda \in \mathbb{C}^*$, F_λ has exact one critical point $z = 1$ and no non-zero finite asymptotic values, let $P_{F_\lambda} = \overline{\bigcup_{n \geq 0} \{F_\lambda^n(1)\}}$. If F_λ has an attracting cycle, then (see [2])

$$\rho = \frac{1}{4} \cdot \operatorname{dist}\big(\mathcal{J}(F_\lambda), P_{F_\lambda}\big) > 0, \tag{4.1}$$

where "dist" denotes the Euclidean distance between $\mathcal{J}(F_\lambda)$ and P_{F_λ}.

Let z_0 be an arbitrary point of $\mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda)$ and denote $z_n = F_\lambda^n(z_0)$ for each $n \in \mathbb{N}$. The assumption $z_0 \in \mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda)$ implies $\omega_{F_\lambda}(z_0) \subsetneqq \{0, \infty\}$ or $\omega_{F_\lambda}(z_0) \not\subset \{0, \infty\}$. Note that $\omega_{F_\lambda}(z_0) \subsetneqq \{0, \infty\}$ means $\omega_{F_\lambda}(z_0) = \emptyset$ or $\omega_{F_\lambda}(z_0) = \{0\}$ or $\omega_{F_\lambda}(z_0) = \{\infty\}$. Since 0 is a pole of F_λ, $\omega_{F_\lambda}(z_0) \neq \{0\}$. It follows, therefore, that $\omega_{F_\lambda}(z_0) = \emptyset$ or $\omega_{F_\lambda}(z_0) = \{\infty\}$ or $\omega_{F_\lambda}(z_0) \not\subset \{0, \infty\}$.

If $\omega_{F_\lambda}(z_0) = \emptyset$, then the forward orbit $\{F_\lambda^n(z_0)\}_{n \in \mathbb{N}}$ is bounded and contained in a compact subset of $\mathbb{C}$, and thus consists of finitely many points, so z_0 must be a periodic or preperiodic point of F_λ; if $\omega_{F_\lambda}(z_0) = \{\infty\}$, then z_0 is an escaping point of F_λ, that is, $F_\lambda^n(z_0) \to \infty$ as $n \to \infty$. Obviously, the set of all periodic and preperiodic points of F_λ has zero area, and [9, Proposition 2.3] implies that the set of all escaping points of F_λ has also zero area. This implies the intersection $\mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda) \cap \{z : \omega_{F_\lambda}(z) \subsetneqq \{0, \infty\}\}$ has zero measure.

Denote

$$\Delta(F_\lambda) := \mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda) \cap \{z : \omega_{F_\lambda}(z) \not\subset \{0, \infty\}\}.$$

Then $\operatorname{meas}\big(\mathcal{J}(F_\lambda) \cap \Lambda(F_\lambda)\big) = 0$ if and only if $\operatorname{meas}\big(\Delta(F_\lambda)\big) = 0$.

We shall prove $\operatorname{meas}\big(\Delta(F_\lambda)\big) = 0$ by contradiction. Suppose this is not true. Let z_0 be an arbitrary density point in $\Delta(F_\lambda)$. The assumption $\omega_{F_\lambda}(z_0) \not\subset \{0, \infty\}$ implies that the forward orbit

$\{z_n\}$ must have at least one accumulation point, say ν, in $\mathcal{J}(F_\lambda) \setminus \{0,\infty\}$. Also note that (4.1) implies $P_{F_\lambda} \subset \mathcal{F}(F_\lambda)$, and thus there must exist a compact subset $X \subset \mathcal{J}(F_\lambda) \cap \mathbb{C}^*$ such that X contains a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ with $\lim\limits_{k\to\infty} z_{n_k} = \nu$. By (4.1) again, we have

$$D(z_{n_k}, 2\rho) \cap (P_{F_\lambda} \cup \{0,\infty\}) = \emptyset,$$

so there is a single-valued branch g_k of the inverse of $F_\lambda^{n_k}$ on $D(z_{n_k}, 2\rho)$ such that $g_k(z_{n_k}) = z_0$.

Since g_k is univalent on $D(z_{n_k}, 2\rho)$, by Lemma 2.5 with $t = 1/2$ we have

$$\frac{4}{9}\rho|g_k'(z_{n_k})| \le |g_k(z) - g_k(z_{n_k})| \le 4\rho|g_k'(z_{n_k})| \tag{4.2}$$

and

$$\frac{4}{27} \le \frac{|g_k'(z)|}{|g_k'(z_{n_k})|} \le 12 \tag{4.3}$$

for any $z \in D(z_{n_k}, \rho)$.

Let $B_k = g_k\big(D(z_{n_k}, \rho)\big)$ and $r_k = 4\rho|g_k'(z_{n_k})|/9$. Then (4.2) implies

$$D(z_0, r_k) \subset B_k \subset D(z_0, 9r_k). \tag{4.4}$$

By (4.1), the post-singular set P_{F_λ} is bounded away from the Julia set $\mathcal{J}(F_\lambda)$, so the map F_λ is expanding on $\mathcal{J}(F_\lambda)$, which implies that $\lim\limits_{k\to\infty} r_{n_k} = 0$.

Without loss of generality, we may assume $\lim\limits_{k\to\infty} r_k = 0$. By (4.4), we have

$$\operatorname{meas}(B_k) \ge \frac{1}{3^4} \operatorname{meas}\big(D(z_0, 9r_k)\big).$$

This, together with the assumption that z_0 is a density point of $\Delta(F_\lambda)$, implies

$$\lim_{k\to\infty} \frac{\operatorname{meas}\big(B_k \setminus \Delta(F_\lambda)\big)}{\operatorname{meas}(B_k)} \le 3^4 \cdot \lim_{k\to\infty} \frac{\operatorname{meas}\big(D(z_0, 9r_k) \setminus \Delta(F_\lambda)\big)}{\operatorname{meas}\big(D(z_0, 9r_k)\big)} = 0. \tag{4.5}$$

Note that $\Delta(F_\lambda)$ is completely invariant and $B_k = g_k\big(D(z_{n_k}, \rho)\big)$. From (4.3), we have

$$\begin{aligned}
&\frac{\operatorname{meas}\big(B_k \setminus \Delta(F_\lambda)\big)}{\operatorname{meas}(B_k)} \\
&= \frac{\operatorname{meas}\Big(g_k\big(D(z_{n_k}, \rho) \setminus \Delta(F_\lambda)\big)\Big)}{\operatorname{meas}\Big(g_k\big(D(z_{n_k}, \rho)\big)\Big)} \\
&\ge \frac{\big(\inf\limits_{z\in D(z_{n_k},\rho)} |g_k'(z)|^2\big) \cdot \operatorname{meas}\big(D(z_{n_k}, \rho) \setminus \Delta(F_\lambda)\big)}{\big(\sup\limits_{z\in D(z_{n_k},\rho)} |g_k'(z)|^2\big) \cdot \operatorname{meas}\big(D(z_{n_k}, \rho)\big)} \\
&\ge \frac{1}{3^8} \cdot \frac{\operatorname{meas}\big(D(z_{n_k}, \rho) \setminus \Delta(F_\lambda)\big)}{\operatorname{meas}\big(D(z_{n_k}, \rho)\big)}.
\end{aligned}$$

It follows from (4.5) that

$$\lim_{k\to\infty} \frac{\operatorname{meas}\big(D(z_{n_k},\rho)\setminus\Delta(F_\lambda)\big)}{\operatorname{meas}\big(D(z_{n_k},\rho)\big)} \le 3^8 \cdot \lim_{k\to\infty} \frac{\operatorname{meas}\big(B_k\setminus\Delta(F_\lambda)\big)}{\operatorname{meas}(B_k)} = 0.$$

Consequently,

$$\lim_{k\to\infty} \frac{\operatorname{meas}\big(D(z_{n_k},\rho)\cap\Delta(F_\lambda)\big)}{\operatorname{meas}\big(D(z_{n_k},\rho)\big)} = 1.$$

Recall that $\lim\limits_{k\to\infty} z_{n_k} = \nu \in \mathcal{J}(F_\lambda)\setminus\{0,\infty\}$. This implies

$$\operatorname{meas}\big(D(\nu,\rho)\big) = \operatorname{meas}\big(D(\nu,\rho)\cap\Delta(F_\lambda)\big),$$

thus

$$D(\nu,\rho) \subset \Delta(F_\lambda) \subset \mathcal{J}(F_\lambda).$$

So ν is an inner point of $\mathcal{J}(F_\lambda)$, and thus $\mathcal{J}(F_\lambda) = \widehat{\mathbb{C}}$, a contradiction to assumption that F_λ has an attracting cycle. Hence, $\Delta(F_\lambda)$ has no density points, and thus Lemma 2.6 implies

$$\operatorname{meas}\big(\Delta(F_\lambda)\big) = 0,$$

which is equivalent to

$$\operatorname{meas}\big(\mathcal{J}(F_\lambda)\cap\Lambda(F_\lambda)\big) = 0.$$

For each $\lambda \in (0, 4e^{-2})$, it is proved in the first half of Theorem 1.1 that the map F_λ has a (unique) attracting fixed point, say t. This implies F_λ has no Siegel disks, Herman rings, or parabolic periodic components. Also note that F_λ has exactly two singular values 0 and λe, then F_λ has neither Baker domains nor wandering domains (see [2]). Let D be the immediate basin of the (unique) attracting fixed point t of F_λ, then

$$\mathcal{F}(F_\lambda) = \bigcup_{n=0}^{\infty} F_\lambda^{-n}(D) = \{z \in \mathbb{C}^* :\ F_\lambda^n(z) \to t \text{ as } n \to \infty\}. \tag{4.6}$$

Let $\Lambda^c(F_\lambda) = \widehat{\mathbb{C}} \setminus \Lambda(F_\lambda)$. Then (4.6) implies $\Lambda^c(F_\lambda) \cap \mathcal{F}(F_\lambda) = \emptyset$ and $\Lambda^c(F_\lambda) \subset \mathcal{J}(F_\lambda)$. For each integer $m \ge b + 100$, let

$$S_m = \{z \in \mathbb{C} :\ m \le x \le m+1,\ -1/2 \le y \le 1/2\}.$$

As in the proof of [20, Lemma 3.3], we can apply McMullen's criteria [11, Proposition 2.1] for each square S_m to construct a nested intersection of dynamically defined subsets of S_m, which is a subset of S_m consisting of the points whose ω-limit sets are $\{0, \infty\}$, and show that the area of each such nested intersection has a uniform positive lower bound $\epsilon > 0$, that is, $\mathrm{meas}\big(S_m \cap \Lambda^c(F_\lambda)\big) \geq \epsilon$.

Therefore, $\mathrm{meas}\big(S_m \cap \mathcal{J}(F_\lambda)\big) \geq \epsilon$ and $\mathcal{J}(F_\lambda)$ has infinite area for each $\lambda \in (0, 4e^{-2})$, which completes the proof of Theorem 1.2. □

Acknowledgement

I would like to express my deep thanks to the anonymous referee(s) for very detailed report and helpful suggestions which help me improve this paper. This work was supported by National Natural Science Foundation of China grants 11601481 and 11526185.

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