Indian J. Pure Appl. Math., **48**(2): 285-294, June 2017

DOI: 10.1007/s13226-017-0228-5

HYPERBOLIC DIMENSION AND POINCARÉ CRITICAL EXPONENT OF RATIONAL MAPS

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(*Received* 25 *May* 2016; *after final revision* 8 *October* 2016;
accepted 25 *December* 2016)

We study the Poincaré series of rational maps. By investigating the property of conical Julia set and dissipative measure, we prove that the Poincaré critical exponents are equal to the hyperbolic dimensions for a large class of rational maps.

Key words : Rational map; hyperbolic dimension; Poincaré critical exponent.

1. INTRODUCTION

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree at least 2 on the Riemann sphere $\overline{\mathbb{C}}$, and let $J(f)$ denote the Julia set of f. It is interesting to investigate the relationships among various fractal dimensions of the Julia sets, and some equalities of several fractal dimensions were proved under some weak forms of hyperbolicity, see for example [1,3, 5, 6]. An important tool to study the fractal dimensions of the Julia set is the Patterson-Sullivan conformal measure which can be constructed by using the Poincaré series. The relationship between the Poincaré series of rational maps and the Patterson-Sullivan conformal measures have been studied for a long time, see for example [4].

In this paper, we study the relationship between the Poincaré critical exponent of rational maps and hyperbolic dimension. We now proceed to describe our main results in more details.

1.1 *Statement of Main Results*

Let f be a rational map of degree at least 2, and let $\mathrm{Crit}(f)$ denote the set of critical points of f and set

$$\mathrm{Crit}'(f) := \mathrm{Crit}(f) \cap J(f).$$

[1]The first named was supported by the National Natural Science Foundation of China (Grant No. 11471098) and the China Scholarship Council (CSC No. 201508410033)

We use $\mathrm{PC}(f)$ to denote the union of the forward orbits of all critical points in $J(f)$, that is

$$\mathrm{PC}(f) = \bigcup_{c \in \mathrm{Crit}'(f)} \bigcup_{n=0}^{\infty} \{f^n(c)\}.$$

The *post-critical set P*(f) is the closure of the set $\mathrm{PC}(f)$.

We say that a compact forward invariant subset X (i.e. satisfying property $f(X) \subset X$) is *hyperbolic* if there exist $C > 0$ and $\lambda > 1$ such that for every positive integer n and every point z in X, the following holds:

$$|Df^n(z)| \geq C\lambda^n.$$

Clearly, a hyperbolic set is contained in the Julia set.

For a compact set $X \subset \overline{\mathbb{C}}$, let $\mathrm{HD}(X)$ denote its Hausdorff dimension. The *hyperbolic dimension of* f, denoted by $\mathrm{HD}_{hyp}(f)$, is the supremum of the Hausdorff dimensions of hyperbolic subsets of the Julia set $J(f)$, that is

$$\mathrm{HD}_{hyp}(f) = \sup\{\mathrm{HD}(X) : X \text{ is a hyperbolic subset of } J(f)\}.$$

Clearly, $\mathrm{HD}_{hyp}(f) \leq \mathrm{HD}(J(f))$.

Given $\delta > 0$ and a point z in $\overline{\mathbb{C}} \setminus \mathrm{P}(f)$, the *Poincaré series of* f *at the point* z *with exponent* δ is defined as follows:

$$\Xi_\delta(z) := \sum_{n=0}^{\infty} \sum_{f^n(\zeta)=z} \frac{1}{|Df^n(\zeta)|^\delta},$$

and put

$$\delta_{cr}(f, z) := \inf\{\delta > 0 : \Xi_\delta(z) < \infty\}.$$

The number $\delta_{cr}(f, z)$ is called the *Poincaré exponent of* f *at the point* z. Obviously, for every $\delta > \delta_{cr}(f, z)$ the Poincaré series $\Xi_\delta(z)$ converges. On the other hand, by standard distortion considerations, if **F** is a component of the Fatou set, then for all points z in **F** the Poincaré exponents coincide, so we set

$$\delta_{cr}(\mathbf{F}) := \delta_{cr}(f, z).$$

We define the *Poincaré critical exponent of* f by

$$\delta_{cr}(f) := \max\{\delta_{cr}(\mathbf{F}) : \mathbf{F} \text{ is a component of the Fatou set}\}.$$

For more information on the Poincaré series see, for example [1, 3].

Remark 1 : In [3], it was proved that for a rational map f of degree at least 2 without Siegel disk, Herman Rings or parabolic points, if the Julia set has zero lebesgue measure, then the upper box dimension $\overline{\mathrm{BD}}(J(f))$ coincides with the Poincaré critical exponent $\delta_{cr}(f)$. In particular, if the Julia set is hyperbolic, then

$$\delta_{cr}(f) = \mathrm{HD}(J(f)).$$

By investigating the property of conical Julia set and dissipative measure, we prove the following theorem.

***Theorem* 1** — *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree at least 2. Assume that for every critical point c in Crit$'(f)$ we have*

$$\limsup_{n\to\infty} |Df^n(f(c))| > 0,$$

and suppose that $J(f)\setminus P(f) \neq \emptyset$. Then the Poincaré critical exponent of f is equal to the hyperbolic dimension of f, that is

$$HD_{hyp}(f) = \delta_{cr}(f).$$

1.2 *Notations and Terminology*

Throughout of the rest of this paper, all the distances, diameters and norms of derivatives are measured using the spherical metric and $B(z,r)$ denotes a ball of radius r which is centered at z.

We shall write $A \asymp B$ if $\frac{1}{C} < \frac{A}{B} < C$ with some constant $C > 0$ independent of particular A and B under consideration. A topological disk U is said to be *a pullback* of an open topological disk V if there exists a positive integer n such that U is a component of $f^{-n}(V)$. We will also denote by $\mathrm{diam}(X)$ the diameter of the set X.

2. Preliminaries

Throughout the rest of this paper we fix a rational map f of degree at least 2.

Given $t > 0$. Recall that a probability measure μ on the Julia set $J(f)$ is said to be *t-conformal* for f if for every Borel set $A \subset J(f)$ such that f is injective on A, we have

$$\mu(f(A)) = \int_A |Df|^t d\mu$$

.

The number t is called the *exponent of the conformal measure*. The *minimum exponent of* f, denoted by $\delta_*(f)$, is the infimum of the exponents of conformal measures on the Julia set $J(f)$, that is

$$\delta_*(f) = \inf\{t : \text{ there is a } t\text{-conformal measure on } J(f)\}.$$

Conformal measures were introduced in holomorphic dynamics by Sullivan [9], who proved the existence of at least one such measure on $J(f)$. Denker, Urbanski and Przytycki (see [2, 8]) proved that for any rational map f of degree at least 2, the hyperbolic dimension is equal to the minimum exponent, i.e.

$$\delta_*(f) = \mathrm{HD}_{hyp}(f) \leq \mathrm{HD}(J(f)).$$

2.1 *Koebe Distortion*

We shall frequently use the following version of the Koebe distortion theorem appeared in [7].

Koebe Distortion Theorem : There exists $r(f) > 0$ depending on f, and for each ϵ in $(0, 1)$ there exists a constant $K(\epsilon) > 1$ such that the following holds. Let x be a point in $J(f)$, n a positive integer and $r \in (0, r(f))$. Suppose that $f^n : W \to B(x, r)$ is a conformal map. Then for every pair of points z_1 and z_2 in W such that both of the points $f^n(z_1)$ and $f^n(z_2)$ are in $B(x, \epsilon r)$, we have

$$\frac{|(f^n)'(z_1)|}{|(f^n)'(z_2)|} \leq K(\epsilon).$$

Moreover, $K(\epsilon) \to 1$ as $\epsilon \to 0$.

2.2 *Conical Julia sets*

Definition 1 — Let f be a rational map of degree at least 2, the *conical Julia set* of f, denoted by $J_{con}(f)$, is the set of all those points z in $J(f)$ for which there exist $\rho(z) > 0$ and an arbitrarily large positive integer n, such that the pullback of the ball $B(f^n(z), \rho(z))$ to z by f^n is univalent. This set is also called *radial Julia set.*

For $r > 0$, if we set

$$J_{con}(f, r) := \{z \in J_{con}(f) : \rho(z) \geq r\},$$

then

$$J_{con}(f) = \bigcup_{r>0} J_{con}(f, r).$$

Lemma 1 — Let f be a rational map of degree at least 2, and let μ be a δ-conformal measure. If $\delta > \delta_*(f)$, then

$$\mu(J_{con}(f)) = 0.$$

PROOF : Set $\delta_* := \delta_*(f)$ and let μ_* be a δ_*-conformal measure. For every $r > 0$ and every point z in $J_{con}(f, r)$, by the definition of $J_{con}(f, r)$ there exist an arbitrarily large positive integer n and a corresponding topological disk U_n containing z such that the map

$$f^n : U_n \to B(f^n(z), r)$$

is univalent. Let $\widetilde{U}_n$ be the corresponding pullback of the ball $B(f^n(z), \frac{r}{2})$ to z by f^n. By the Koebe distortion theorem and δ-covariance of μ, we have

$$\mu(\widetilde{U}_n) \asymp (\mathrm{diam}(\widetilde{U}_n))^{\delta} = (\mathrm{diam}(\widetilde{U}_n))^{\delta_*}(\mathrm{diam}(\widetilde{U}_n))^{\delta-\delta_*} \asymp \mu_*(\widetilde{U}_n)(\mathrm{diam}(\widetilde{U}_n))^{\delta-\delta_*}.$$

By the Besicovich covering lemma, we have

$$\mu(J_{con}(f, r)) \leq C\mu_{\delta_*}(J(f))(\mathrm{diam}(\widetilde{U}_n))^{\delta-\delta_*},$$

where C is a universal constant.

Because $\mathrm{diam}(\widetilde{U}_n) \to 0$ as $n \to \infty$, it follows that $(\mathrm{diam}(\widetilde{U}_n))^{\delta-\delta_*} \to 0$. Therefore, we have $\mu(J_{con}(f, r)) = 0$. It follows that

$$\mu(J_{con}(f)) = \mu(\bigcup_{r>0} J_{con}(f, r)) = 0.$$

This proves the lemma. □

2.3 *Dissipative Measures*

Definition 2 — A measure μ is said to be *dissipative* if there exists a wandering set X of positive measure, $\mu(X) > 0$, such that for all integer $m > n \geq 0$ we have

$$f^{-m}(X) \cap f^{-n}(X) = \emptyset.$$

Lemma 2 — For any δ-conformal measure μ. If $\mu(J_{con}(f)) = 0$ and $J(f) \setminus \mathrm{P}(f) \neq \emptyset$, then μ is dissipative.

PROOF : By the hypothesis that $J(f) \setminus \mathrm{P}(f) \neq \emptyset$, we can find a set $D = B(z, r)$ for some point z in $J(f)$ such that $D \cap \mathrm{P}(f) = \emptyset$. Since D is an open set, we have that $\mu(D) > 0$ by using the property of conformal measure. Put

$$D_0 := \{z \in D \mid \exists\, k \geq 1 \text{ such that } f^k(z) \in D\}$$

Claim : $\mu(D \setminus D_0) > 0$.

In fact, let $R : D_0 \to D$ be the first return map. Arguing by contradiction we assume that $\mu(D \setminus D_0) = 0$, then $\mu(D_0) = \mu(D) > 0$. It follows that

$$\mu(\bigcap_{n=0}^{\infty} R^{-n}(D_0)) > 0.$$

On the other hand, notice that the set $\bigcap_{n=0}^{\infty} R^{-n}(D_0)$ is a subset of the set $J_{con}(f)$, then we have

$$\mu(J_{con}(f)) \geq \mu(\bigcap_{n=0}^{\infty} R^{-n}(D_0)) > 0.$$

This is in contradiction with $\mu(J_{con}(f)) = 0$. The claim is proved.

Now setting $X = D \setminus D_0$, then we can know easily from the definitions of the sets D and D_0 that X is a wandering set. So the set X gives us a wandering set of positive measure. Hence, μ is dissipative. The proof is completed. □

Lemma 3 — Let μ be a δ-conformal measure. If μ is dissipative, then there is an alternative:

- either the Poincaré series $\Xi_\delta(z), z \in \overline{\mathbb{C}} \setminus P(f)$ is convergent;
- or J(f)$\subset P(f)$.

PROOF : Let X be a wandering set of positive measure. Now we consider two cases:

Case 1 : $\mu(X \setminus \mathrm{P}(f)) \neq 0$. We can easily construct a wandering set of positive measure $X' \subset B(z_0, r)$ such that $z_0 \in J(f)$ and $B(z_0, 2r) \cap \mathrm{P}(f) = \emptyset$. By the Koebe distortion theorem and the δ-covariance of the measure μ, we have

$$\mu(X') = \int_{f^{-n}(X')} | (f^n)'(\zeta)) |^\delta \, d\mu(\zeta) \asymp | (f^n)'(\zeta)) |^\delta \, \mu(f^{-n}(X'))$$

where $f^{-n}(X')$ is an appropriate local branch of f^{-n} on X'. Hence

$$\Xi_\delta(z_0) = \sum_{n=0}^{\infty} \sum_{f^n(\zeta)=z_0} \frac{1}{|Df^n(\zeta)|^\delta} \asymp \sum_{n=0}^{\infty} \frac{\mu(f^{-n}(X'))}{\mu(X')} = \frac{1}{\mu(X')} \sum_{n=0}^{\infty} \mu(f^{-n}(X')).$$

Since the set X' is a wandering, then we have

$$\sum_{n=o}^{\infty} \mu(f^{-n}(X')) \leq \mu(J(f)).$$

Hence, $\Xi_\delta(z_0) < \infty$.

Case 2 : $\mu(X \setminus \mathrm{P}(f)) = 0$. By the δ-covariance of the conformal measure μ, for every non-negative integer n we can obtain that

$$\mu(f^{-n}(X) \setminus \mathrm{P}(f)) = 0.$$

Therefore, there exists a wandering set X_1 of positive measure such that for every integer $n \geq 0$ we have

$$f^{-n}(X_1) \subset \mathrm{P}(f).$$

It follows that

$$\bigcup_{n=1}^{\infty} f^{-n}(X_1) \subset \mathrm{P}(f). \tag{1}$$

Notice that the backward orbit of every point in the Julia set is dense in $J(f)$ and the set $\mathrm{P}(f)$ is closed, so combining with (1) we obtain that $J(f) \subset \mathrm{P}(f)$. The lemma is proved completely. □

3. Proof of Theorem 1

In this section, we will first prove the existence of conformal measure by using the idea appeared in [1], and then prove our Theorem 1.

3.1 *Existence of Conformal Measure*

Given a point z in $J(f)$ and $r > 0$, for every positive integer n put

$$S_n^r(z) := \{\eta \mid f^n(\eta) = z \text{ and } \operatorname{dist}(f^k(\eta), \mathrm{Crit}(f)) \geq r,\ k = 0, 1, \ldots, n-1\}.$$

Then *cut-off Poincaré series of f at the point z* is defined as follows:

$$\Xi_\delta^r(z) := \sum_{n=0}^{\infty} \sum_{\eta \in S_n^r(z)} \frac{1}{|Df^n(\eta)|^\delta}.$$

The following lemma is essentially the same as Proposition 3.8 in [1]. The details of its proof are left to the interested reader to check.

Lemma 4 — Let f be a rational map of degree at least 2 such that for every critical point c in $\mathrm{Crit}'(f)$ we have

$$\limsup_{n\to\infty} |Df^n(f(c))| > 0.$$

If for any $\delta > \mathrm{HD}_{hyp}(f)$ there is a critical point c in the Julia set such that

$$\lim_{r\to 0} \Xi_\delta^r(c) = \infty,$$

then there exists a δ-conformal measure μ_δ.

Proposition 2 — Let f be a rational map of degree at least 2. Assume that for every critical point c in $\mathrm{Crit}'(f)$ we have

$$\limsup_{n\to\infty} |Df^n(f(c))| > 0.$$

Then for any $\delta > \mathrm{HD}_{hyp}(f)$ there exists a δ-conformal measure μ_δ.

PROOF : By Lemma 4, it suffices to prove that if $\Xi_\delta(c) < \infty$, then the following measure

$$\mu_\delta := \frac{1}{\Xi_\delta(c)} \sum_{n=0}^{\infty} \sum_{f^n(\zeta)=c} \frac{\delta_\zeta}{|Df^n(\zeta)|^\delta}$$

is a δ-conformal measure.

In fact, let $U \subset J(f)$ be a measurable set such that f is injective on U. By the definition of μ_δ we have

$$\mu_\delta(f(U)) = \frac{1}{\Xi_\delta(c)} \sum_{n=0}^{\infty} \sum_{f^n(\zeta)=c} \frac{\delta_\zeta(f(U))}{|Df^n(\zeta)|^\delta} = \frac{1}{\Xi_\delta(c)} \sum_{n=0}^{\infty} \sum_{\substack{f^n(\zeta)=c,\\ \zeta\in f(U)}} \frac{1}{|Df^n(\zeta)|^\delta}$$

$$= \frac{1}{\Xi_\delta(c)} \sum_{n=1}^{\infty} \sum_{\substack{f^n(\zeta)=c,\\ \zeta\in U}} \frac{|Df(\zeta)|^\delta}{|Df^n(\zeta)|^\delta}.$$

and

$$\int_U |Df(w)|^\delta d\mu(w) = \frac{1}{\Xi_\delta(c)} \sum_{n=0}^{\infty} \sum_{f^n(\zeta)=c} \frac{1}{|Df^n(\zeta)|^\delta} \int_U |Df(w)|^\delta d\delta_\zeta(w)$$

$$= \frac{1}{\Xi_\delta(c)} \sum_{n=0}^{\infty} \sum_{\substack{f^n(\zeta)=c,\\ \zeta\in U}} \frac{|Df(\zeta)|^\delta}{|Df^n(\zeta)|^\delta}.$$

Hence, when U does not contain the point c we have

$$\int_U |Df(w)|^\delta d\mu_\delta(w) - \mu_\delta(f(U)) = 0. \tag{2}$$

On the other hand, for every positive integer n we have

$$\mu_\delta(f^{n+1}(c)) = \int_{\{f^n(c)\}} |Df(w)|^\delta d\mu_\delta(w) = |Df^n(f(c))|^\delta \mu_\delta(f(c)).$$

It follows that

$$\mu_\delta(f(c)) \sum_{n=1}^{\infty} |Df^n(f(c))|^\delta = \sum_{n=1}^{\infty} \mu_\delta(f^{n+1}(c)) < \infty.$$

By the hypothesis that

$$\limsup_{n\to\infty} |Df^n(f(c))| > 0,$$

we have $\sum_{n=1}^{\infty} |Df^n(f(c))|^\delta = \infty$.

It follows that

$$\mu_\delta(f(c)) = 0.$$

Notice that $\int_{\{c\}} |Df(w)|^\delta d\mu_\delta(w) = 0$, we obtain that

$$\mu_\delta(f(c)) = \int_{\{c\}} |Df(w)|^\delta d\mu_\delta(w) = 0.$$

Combining with (2), for any measurable set $U \subset J(f)$ such that f is injective on U, we have that

$$\mu_\delta(f(U)) = \int_U |Df(w)|^\delta d\mu_\delta(w).$$

This completes the proof. □

3.2 Proof of Main Theorem

Proof of Theorem 1 : For any $\delta > \text{HD}_{hyp}(f)$, by Proposition 2 there exists a δ-conformal measure μ. By Lemmas 1 and 2, we know that μ is dissipative. Moreover, by Lemma 3 we obtain that $\delta_{cr}(f) \le \delta$. It follows that

$$\delta_{cr}(f) = \delta_* = \text{HD}_{hyp}(f),$$

and the Theorem 1 is proved. □

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for valuable comments which led to a revision of this paper.

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