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ON THE LENGTH OF THE PERIOD OF A REAL QUADRATIC IRRATIONAL

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We review some known and not so well known results on the length of the period of the continued fraction expansion of a quadratic irrational $\sqrt{D}$ with $D > 0$. We also show that this length is $o((D \log D)^{1/2})$ for almost all D.

Key words : Quadratic irrational; continued fraction expansion; period; length.

1. Introduction and Statements of the Results

Let D be a positive non-square integer. We shall denote by $\omega(D)$ the number of distinct prime divisors of D and $h(D)$ the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$. In the seventies, there have been several articles on the classical problem of estimating $\ell(D)-$ the length of the period of the continued fraction expansion of $\sqrt{D}$. See Cohn [5], Hickerson [9], Hirst [10], Stanton, Sudler & Williams [15], Williams [17] and Lu [12]. From the results in these papers we now know that if D is *square free* then

$$\ell(D) < 0.24 D^{1/2} \log D.$$

In fact using the theory of genera, one can have

$$\ell(D) \ll D^{1/2} \log D / 2^{\omega(D)}.$$

Also under GRH, by a result of Littlewood [11], we obtain

$$\ell(D) \ll D^{1/2} \log \log D.$$

In [17], $\ell(D)$ has been calculated for all $D < 2 \times 10^7$ and these calculations indicate that unconditionally we must have

$$\ell(D) = o(D \log D)^{1/2}. \tag{1}$$

By taking $D = m^2 + 1$, one sees that there is an infinite sequence of values of D for which

$$\ell(D) = 1.$$

For general non-square $D > 0$, write

$$D = D_0 s^2 \tag{2}$$

where D_0 is the square free factor of D. Improving on the result of [10], it is shown in [15] that

$$\ell(D) < 3.76 D^{1/2} \log(D_0).$$

On the other hand, regarding the lower bound for $\ell(D)$ the following result was shown in [15].

Theorem SSW — *Suppose the following hypothesis H holds: There exists an infinite sequence S of square free numbers D such that $h(D) = o(D^\epsilon)$ for D in S and all $\epsilon > 0$. Then*

$$\ell(D) \gg D^{1/2-\epsilon}.$$

In fact, a stronger hypothesis is believed to be true viz., $h(D) = 1$ for infinitely many square free numbers D. All the above said papers have no mention of some earlier papers of Vijayaraghavan [16], Chowla [2], Chowla & Pillai [3] and Pillai [13] which have appeared from 1925 to 1930. These papers have been brought to light in the Collected works of Pillai [6]. It has been shown in [3] (see [6, No. 1, 2010, pp 62-67]) that

$$\sqrt{D} \ll \ell(D) \ll \sqrt{D} \tag{3}$$

for infinitely many values of D. Unfortunately, there are some errors in the proof of the left hand side inequality. For instance, [3, Lemma 1] is wrong and the proof of [3, Lemma 2] unclear but the result is well known. In Lemma 1 of [3], the authors attributed the following result to Lagrange:

If $1 < m < \sqrt{D}, m = u^2 - v^2 D, gcd(u, v) = 1$ then m occurs as a partial quotient in the simple continued fraction for $\sqrt{D}$.

This result is wrong as seen from several examples, for instance

$$D = 47, m = 2, 7^2 - 47 = 2, \sqrt{47} = [6, \overline{1, 5, 1, 12}];$$

$$D = 59, m = 5, 8^2 - 59 = 5, \sqrt{59} = [7, \overline{1, 2, 7, 2, 1, 14}];$$

$$D = 109, m = 5, 21^2 - (109)2^2 = 5,$$

$$\sqrt{109} = [10, \overline{2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2, 20}].$$

They have also claimed that if the class number $h(D) = 1$, then $x^2 - Dy^2 = -1$ is solvable. This is untrue for $D = 7$. Thus the left hand side inequality in (3) remains unproven and so far, the result of [15] in Theorem SSW remains the best known. Here again, Vijayaraghavan [16] had shown that

Theorem V — *We have*

$$\ell(D) \gg D^{1/2-\epsilon}$$

for infinitely many values of square free D.

His proof depends on a result of Schur which we shall state later.

On the other hand Chowla and Pillai have proved that

Theorem CP — *Let D be square free. Then $\ell(D)$ is, on average, of order $\sqrt{D}$.*

We shall reproduce the proofs of Vijayaraghavan and Chowla & Pillai below. All the results on upper bound for $\ell(D)$ quoted above have been proved in the papers [16] and [3] except for the explicit constants. In fact in [13] (see [6, No. 1, 2010, pp 113-121]), it was shown that

$$\ell(D) \leq (1 + o(1))D^{1/2} \log D.$$

Our aim in this note is to publicize the ideas in these papers and prove the following result in support of the conjecture in (1) on the upper bound for $\ell(D)$.

Theorem 1.1 — *Let $\epsilon > 0$. For almost all values of square free D, we have*

$$\ell(D) \leq 6.5 \times 2^{\epsilon} \sqrt{D} (\log D)^{1-\log 2}.$$

Further, for almost all values of D, we have

$$\ell(D) \leq 16 \times 2^{\epsilon} \sqrt{D} (\log D_0)^{1-\log 2}.$$

For large values of D, it is possible to improve the constants. Note that $1 - \log 2 \leq \cdot 307$. Thus (1) is true for almost all D.

2. Preliminaries

2.1 *Continued Fraction*

Let $\ell = \ell(D)$ and

$$\sqrt{D} = [a_0, \overline{a_1, \cdots, a_{\ell-1}, 2a_0}]$$

be the simple continued fraction expansion of $\sqrt{D}$ with $a_0 = [\sqrt{D}]$. Let

$$P_{-2} = 0, P_{-1} = 1, Q_{-2} = 1, Q_{-1} = 0$$

and for $n \geq 0$,

$$P_n = a_n P_{n-1} + P_{n-2};\ Q_n = a_n Q_{n-1} + Q_{n-2}.$$

Then P_n/Q_n is the $n-$th convergent to the continued fraction of $\sqrt{D}$. Setting

$$\xi_0 = \sqrt{D}, m_0 = 0, b_0 = 1$$

define (ξ_i, m_i, b_i) for $i \geq 1$, recursively as follows.

$$a_i = [\xi_i], \xi_i = \frac{m_i + \sqrt{D}}{b_i}, m_{i+1} = a_i b_i - m_i, b_{i+1} = \frac{D - m_{i+1}^2}{b_i}. \tag{4}$$

Then one may check (see [14]) that m_i and b_i are integers, $b_i \neq 0$, with

(i) $0 < m_i < \sqrt{D}$,

(ii) $\sqrt{D} - m_i < b_i < \sqrt{D} + m_i$,

(iii) $b_i | (D - m_i^2)$,

(iv) $a_i \leq 2\sqrt{D}$ for $i \geq 1$.

(v) $b_i = 1$ if and only if $\ell | i$ and $b_i \neq -1$ for any i.

It can be seen that ξ_i for $i \geq 1$ are *reduced* quadratic irrationals. By (i) and (ii), the number of distinct ξ_i's is at most $2([\sqrt{D}])([\sqrt{D}] + 1)$. Since each a_i comes from a different ξ_i, we have

$$\ell \leq (3.5)D.$$

In fact, from (i)-(iv), it is also clear that

$$\ell \leq \sum_{x=1}^{\sqrt{D}} d^{(0)}(D - x^2) \tag{5}$$

where $d^{(0)}(D-x^2)$ denotes the number of divisors of $D-x^2$ between $\sqrt{D}-x$ and $\sqrt{D}+x$.

2.2 *Pell's Equations*

It is well known that $(P_{\ell-1}, Q_{\ell-1})$ is the least positive solution of the Pell's equation

$$x^2 - Dy^2 = \pm 1$$

according as ℓ is even or odd. Let us denote by η the quantity

$$\eta = P_{\ell-1} + \sqrt{D}Q_{\ell-1}. \tag{6}$$

Then η is a unit in $\mathbb{Q}(\sqrt{D})$. Let $\epsilon_0 = (u_0 + v_0\sqrt{D})/2$ be the fundamental unit of $\mathbb{Q}(\sqrt{D})$. Then

$$\eta = \epsilon_0^{\mu} \tag{7}$$

where

$$\mu \equiv 1 \; if \; D \not\equiv 5 (\mathrm{mod}\ 8); \; \mu | 3 \; if \; D \equiv 5 (\mathrm{mod} 8). \tag{8}$$

Note that

$$\begin{aligned} P_{\ell-1} &\leq (a_{\ell-1}+1)(a_{\ell-2}+1)\cdots(a_2+1)(a_1P_0+P_{-1}) \\ &\leq (2\sqrt{D}+1)^{\ell} \end{aligned}$$

and

$$\begin{aligned} Q_{\ell-1} &\leq (a_{\ell-1}+1)(a_{\ell-2}+1)\cdots(a_2+1)(a_1Q_0+Q_{-1}) \\ &\leq (2\sqrt{D}+1)^{\ell-1}. \end{aligned}$$

Thus

$$P_{\ell-1} + \sqrt{D}Q_{\ell-1} \leq 2(2\sqrt{D}+1)^{\ell}. \tag{9}$$

Further it can be seen easily that

$$P_{\ell-1} > ((\sqrt{5}+1)/2)^{\ell-1}; Q_{\ell-1} > ((\sqrt{5}+1)/2)^{\ell-2}$$

giving

$$P_{\ell-1} + \sqrt{D}Q_{\ell-1} \geq ((\sqrt{5}+1)/2)^{\ell}; \tag{10}$$

We state and prove two lemmas on purely periodic irrationals which will be used in the proof of Theorem V.

Lemma 2.1 — Suppose $\theta > 0$ and $\theta = [\overline{a_1, \cdots, a_n}]$. Then

$$\theta = \frac{p_n - q_{n-1} + \sqrt{(p_n+q_{n-1})^2 - 4(-1)^n}}{2q_n}.$$

PROOF : We have

$$\theta = [a_1, \cdots, a_n, \theta] = \frac{\theta p_n + p_{n-1}}{\theta q_n + q_{n-1}}.$$

Thus

$$q_n\theta^2 + \theta(q_{n-1} - p_n) - p_{n-1} = 0.$$

Solving this quadratic equation, we have

$$\theta = \frac{p_n - q_{n-1} \pm \sqrt{(p_n - q_{n-1})^2 + 4p_{n-1}q_n}}{2q_n}$$

$$= \frac{p_n - q_{n-1} \pm \sqrt{(p_n + q_{n-1})^2 - 4(p_nq_{n-1} - p_{n-1}q_n)}}{2q_n}$$

which gives the result since $p_nq_{n-1} - p_{n-1}q_n = (-1)^n$ and $\theta > 0$. □

Lemma 2.2 — Suppose $\frac{P+\sqrt{D}}{Q} = [\overline{a_1, \cdots, a_{2n}}]$. Then there exists a solution (X, Y) of $x^2 - Dy^2 = 1$ with

$$X = \frac{p_{2n} + q_{2n-1}}{2} \text{ if } p_{2n} + q_{2n-1} \text{ is even}$$

and if $p_{2n} + q_{2n-1}$ is odd, then

$$X = 4\left(\frac{p_{2n} + q_{2n-1} - 1}{2}\right)^3 + 6\left(\frac{p_{2n} + q_{2n-1} - 1}{2}\right)^2 - 1.$$

PROOF : By Lemma 2.1,

$$\frac{P + \sqrt{D}}{Q} = \frac{p_{2n} - q_{2n-1} + \sqrt{(p_{2n} + q_{2n-1})^2 - 4}}{2q_{2n}}.$$

Hence there exists an integer S such that

$$(p_{2n} + q_{2n-1})^2 - DS^2 = 4. \tag{11}$$

Suppose $p_{2n} + q_{2n-1}$ is even. Then (11) implies that S is even. Thus in this case we may take

$$X = \frac{p_{2n} + q_{2n-1}}{2}.$$

Suppose $p_{2n} + q_{2n-1}$ is odd, say $2m + 1$. Then we use the identity

$$((p_{2n} + q_{2n-1})^2 - 4)(2m^2 + 2m)^2 = (4m^3 + 6m^2 - 1)^2 - 1$$

to obtain the result. □

2.3 *Class Number Formula*

Let $D > 1$ be square free and let Δ be the discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$. Then

$$\Delta = \begin{cases} D \; if \; D \equiv 1 (\text{mod } 4) \\ 4D \text{ otherwise} \end{cases} \tag{12}$$

Then the well known class number formula for $\mathbb{Q}(\sqrt{D})$ is

$$\log \epsilon_0 = \sqrt{\Delta} L(1, \chi_D)/2h \tag{13}$$

where h is the class number of $\mathbb{Q}(\sqrt{D})$ and $L(1, \chi_D)$ is the Dirichlet $L-$ series

$$\sum_{n=1}^{\infty} \frac{\chi_D(n)}{n}$$

with $\chi_D(n)$ denoting the quadratic real character given by the Kronecker symbol $(\frac{n}{D})$. It is a well known fact that

$$2^{\omega(D)} | h. \tag{14}$$

We combine (9) and (10) with (7) and (13) to get

$$\ell(D) \geq \frac{\mu \log \epsilon_0 - \log 2}{\log(2\sqrt{D} + 1)} = \frac{\mu \sqrt{\Delta} L(1, \chi_D) - 2h \log 2}{2h \log(2\sqrt{D} + 1)} \tag{15}$$

and

$$\ell(D) \leq \frac{\mu \log \epsilon_0}{\log \alpha} \leq \frac{\mu \sqrt{\Delta} L(1, \chi_D)}{2h \log(\frac{\sqrt{5}+1}{2})}. \tag{16}$$

3. More Lemmas

From the calculations in [17] we get

Lemma 3.1 — Let D be a squre free integer with $2 < D < 2 \times 10^7$. Then

$$\ell(D) \leq (1.51 + \epsilon)\sqrt{D} \log \log D.$$

Under GRH, for any square free integer D, we have

$$\ell(D) \leq (e^{\gamma}/\log \alpha + \epsilon)\sqrt{D} \log \log D \leq (3.71 + \epsilon)\sqrt{D} \log \log D$$

where γ is Euler's constant and $\alpha = \frac{1+\sqrt{5}}{2}$.

For the next lemma, see Ayoub [1, p. 338].

Lemma 3.2 — We have, $0 < L(1, \chi_D) < 3 \log D$. In fact,

$$0 < L(1, \chi) < \log D + \frac{\varphi(D)}{D}$$

where φ is the Euler's Totient function.

The following lemma on the normal order of $\omega(Q)$ for any integer $Q > 1$ is well known. See [8, p. 356].

Lemma 3.3 — Let $Q > 1$ be any integer. The normal order of $\omega(Q)$ is $\log \log Q$.

As a consequence, we find that for any given $\epsilon > 0$,

$$\log \log Q - \epsilon \leq \omega(Q) \leq \log \log Q + \epsilon \tag{17}$$

for almost all Q in the sense of density as given in [8, p. 8].

4. Proofs of the Theorems

Proof of Theorem V : Let $(P, Q) = (m_i, b_i)$ for some i chosen as in (4). Then $\frac{P+\sqrt{D}}{Q}$ is purely periodic. Let $2n$ be the smallest even period of $\frac{P+\sqrt{D}}{Q}$. Then $\ell(D) \geq 2n$ and by Lemma 2.2 and (9), the least positive solution of $x^2 - Dy^2 = 1$ satisfies

$$x \leq (p_{2n} + q_{2n-1})^3 \leq 8(2\sqrt{D} + 1)^{3n}.$$

Thus

$$n \geq \frac{\log x - \log 8}{3 \log(2\sqrt{D} + 1)}. \tag{18}$$

Vijayaraghavan uses a result of Schur, that there exist infinitely many values of D for which

$$\log x \geq D^{1/2-\epsilon/2}, \epsilon > 0.$$

Hence by taking D large, we get

$$\ell(D) \geq 2n \gg D^{1/2-\epsilon}$$

for infinitely many values of D. □

Proof of Theorem SSW : By (15)

$$\ell(D) \geq \frac{\mu \log \epsilon_0 - \log 2}{\log(2\sqrt{D} + 1)} = \frac{\mu h \log \epsilon_0 - h \log 2}{h \log(2\sqrt{D} + 1)}. \tag{19}$$

By Siegel's theorem on the size of $L(1,\chi)$, (see [7, p. 130]) there exists $D_0(\epsilon)$ such that for $D > D_0(\epsilon)$, we have

$$h \log \epsilon_0 > D^{1/2-\epsilon/2}.$$

Thus if the hypothesis H holds, then (19) implies that

$$\ell(D) \gg D^{1/2-\epsilon}$$

for infinitely many values of D. □

Remark : As seen above Theorem V depends on a deep result of Schur, which I am not able to access. Theorem SSW depends on the unproven hypothesis H. It will be interesting to access Schur's result and see if it can be avoided in the proof of Theorem V.

PROOF OF THEOREM CP : By (5) we have

$$\sum_{D\le x} \ell(D) \le \sum_{D\le x} \sum_{y\le\sqrt{D}} d^{(0)}(D-y^2)$$

$$\le \sum_{y\le\sqrt{x}} \sum_{D=y^2+1}^{x} d^{(0)}(D-y^2) \le \sum_{y\le\sqrt{x}} \sum_{m=1}^{y+\sqrt{x}} \theta(m) \tag{20}$$

where $\theta(m)$ is the number of values of D such that

$$D - y^2 \equiv 0 (\mathrm{mod}\ m) \tag{21}$$

with

$$\sqrt{D} - y < m < \sqrt{D} + y.$$

The preceding condition implies that

$$m^2 + y^2 - 2my < D < m^2 + y^2 + 2my.$$

Hence D takes $4my$ consecutive values of which (21) will be satisfied for $[(4my-1)/m] = 4y-1$ values. Hence

$$\theta(m) = 4y - 1.$$

Hence from (20), we get

$$\sum_{D\le x} \ell(D) \le \sum_{y\le\sqrt{x}} (y+\sqrt{x})(4y-1)$$

$$\le \sqrt{x}(\sqrt(x)+1)(10\sqrt{x}+2)/3 \le 6.5x^{3/2}.$$

This proves the assertion of the theorem. □

PROOF OF THEOREM 1.1 : Let D be a square free integer. By Lemma 3.1, if $D < 2 \times 10^7$, then

$$\ell(D) \leq 4.5\sqrt{D}.$$

Hence we may assume that $D \geq 2 \times 10^7$. By (16), (14) and Lemma 3.2 we have

$$\ell(D) \leq 6.24\sqrt{D}(\log D + 1)/2^{\omega(D)}.$$

Using Lemma 3.3 here, we find that for almost all D,

$$\ell(D) \leq (6.5)2^{\epsilon}\sqrt{D}(\log D)^{1-\log 2}$$

as desired.

Now we prove the result for any $D = D_0 s^2$. Suppose (a_s, b_s) be the least positive solution of $x^2 - Dy^2 = x^2 - D_0 s^2 y^2 = \pm 1$. Then it is well known that

$$\eta_s = a_s + b_s\sqrt{D} = a_s + sb_s\sqrt{D_0} = \eta_1^{e(s)}$$

with $\eta_1 = a_1 + b_1\sqrt{D}\, and\, e(s) \leq 2s$. See [15]. Now by (7), with $\epsilon_0 = \epsilon_1$ where ϵ_1 is the fundamental unit of $\mathbb{Q}(\sqrt{D_0})$, we get

$$\ell(D) \leq \frac{\log \eta_s}{\log \alpha} < \frac{\mu e(s) \log \epsilon_1}{\log \alpha} < \frac{2\mu s \log \epsilon_1}{\log \alpha}.$$

Thus

$$\ell(D) \leq \frac{\mu s\sqrt{D_0} L(1, \chi_{D_0})}{h \log \alpha}.$$

This gives

$$\ell(D) \leq \frac{16 \times \sqrt{D} \log D_0}{2^{\omega(D_0)}}.$$

Arguing as before, we find that

$$\ell(D) \leq 16 \times 2^{\epsilon}\sqrt{D}(\log D_0)^{1-\log 2}$$

for almost all D_0 and hence for almost all D. □

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