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BILOCAL AUTOMORPHISMS OF $\mathcal{T}_\infty(F)$

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We prove that if F is a field such that $|F| > 2$, then every bilocal automorphism of $\mathcal{T}_\infty(F)$ - the algebra of $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over F, is an automorphism.

Key words : Bilocal automorphisms; triangular matrices; infinite matrices.

1. Introduction and Statement of Results

Let $\mathcal{A}$ be an algebra and let $\phi : \mathcal{A} \to \mathcal{A}$ be a linear map. If for every x in $\mathcal{A}$ there exists an automorphism ϕ_x of $\mathcal{A}$ such that

$$\phi(x) = \phi_x(x), \tag{1}$$

then we say that ϕ is a local automorphism of $\mathcal{A}$.

Quite a lot is known about the local automorphisms of $\mathcal{B}(H)$ (see [5, 4]). In [3] the local automorphisms of some matrix algebras are described, in [6] we can read about the local automorphisms of the nest algebras on some Banach spaces, in [10] and [16] the authors investigate such maps on some operator algebras.

Some modifications of condition (1) are also of interest. For instance, if for every pair $x, y \in \mathcal{A}$ we can find an automorphism $\phi_{x,y}$ such that

$$\phi_{x,y}(x) = \phi(x) \quad \text{and} \quad \phi_{x,y}(y) = \phi(y),$$

then we say that ϕ is a 2-local automorphism. We know quite a lot about 2-local automorphisms of various algebras [15, 13, 11, 7, 8, 9]. Hence, we would like to focus on some other modifications of (1). Let us introduce the definition.

Suppose F is a field, V – a linear space over F, $\mathcal{A}$ – an algebra of linear operators on V. Consider a linear map ϕ defined on $\mathcal{A}$. If for all pairs x, v, where $x \in \mathcal{A}$, $v \in V$ there exists an automorphism $\phi_{x,v}$ of $\mathcal{A}$ such that

$$\phi(x)v = \phi_{x,v}(x)v, \tag{2}$$

then ϕ is called a bilocal automorphism. Clearly, every local automorphism is a bilocal automorphism. According to latest knowledge so far we only know the form of bilocal automorphisms of $\mathcal{B}(H)$ and $M_n(F)$ [12, 14].

Let us consider $\mathcal{T}_\infty(F)$ – an algebra consisting of all $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over a field F. We wish to investigate the form of the maps ϕ satsifying (2) in the case when $\mathcal{A} = \mathcal{T}_\infty(F)$. Let us denote by $M_{\infty\times 1}(F)$ the space of all vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}, \quad x_i \in F \quad \text{for all } i \in \mathbb{N}.$$

Because of the infinte sums the action $a \cdot v$ for $a \in \mathcal{T}_\infty(F)$, $v \in M_{\infty\times 1(F)}$ is not well-defined. Hence, we will limit ourselves to $M^{fin}_{\infty\times 1}(F)$ – the subspace of those vectors from $M_{\infty\times 1}(F)$ that have a finite support. Thanks to that, the action $a \cdot v$ will be well-defined and we can consider the maps satisfying (2). However, as we do not like any restrictions, we define one more action.

Let us denote by $M_{1\times\infty}(F)$ the space of all the vectors of the form

$$(x_1, x_2, x_3, \ldots), \quad x_i \in F \quad \text{for all } i \in \mathbb{N}. \tag{3}$$

Since every upper triangular matrix has only a finite number of nonzero entries in each column, for any v given by (3) and any $a \in \mathcal{T}_\infty(F)$ the product $v \cdot a$ exists and we do not have to restrict $M_{1\times\infty}$ to any of its subsets. Hence, we will also consider the maps ϕ on $\mathcal{T}_\infty(F)$ satisfying

$$v\phi(x) = v\phi_{x,v}(x). \tag{4}$$

Our result is the following.

***Theorem* 1.1** — *Assume F is a field of at least three elements. Then $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ satisfies either (2) or (4) if and only if it is an automorphism of $\mathcal{T}_\infty(F)$.*

2. PROOFS

First we introduce some notation. By e_∞ we mean the $\mathbb{N}\times\mathbb{N}$ identity matrix, by e_{nm} the $\mathbb{N}\times\mathbb{N}$ matrix with 1 in the position (n,m) and 0 in every other position.

If a is a $n\times m$, $n\times\mathbb{N}$, $\mathbb{N}\times m$, $\mathbb{N}\times\mathbb{N}$ matrix we will sometimes write $a_{n\times m}$, $a_{n\times\infty}$, $a_{\infty\times n}$, $a_{\infty\times\infty}$ respectively. In particular, we will write $0_{n\times m}$, $0_{n\times\infty}$, ... for the zero matrices of proper dimensions.

By $\mathcal{R}^{kl}_\infty(F)$, where $k\leq l$, we understand the following subalgebra of $\mathcal{T}_\infty(F)$.

$$\mathcal{R}^{kl}_\infty(F)=\{a\in\mathcal{T}_\infty(F):\ i>k\ \vee\ j<l\ \Rightarrow\ a_{ij}=0\}$$

In the spaces $M_{\infty\times 1}(F)$ and $M_{1\times\infty}$ we distinguish the elements e_n and f_n ($n\in\mathbb{N}$) defined below.

$$(e_n)_k=\begin{cases}1 & \text{if } k=n\\ 0 & \text{otherwise,}\end{cases}\qquad\qquad f_n=e_n^T,$$

where a^T is the transpose of a.

In these two spaces we also introduce two subspaces. Namely,

$$V^n_\infty=\{v\in M_{\infty\times 1}(F):\ i>n\ \Rightarrow\ v_i=0\}=\left\{(v_1,v_2,\ldots,v_n,0,0,0,\ldots)^T\right\},$$

$$U^n_\infty=\{u\in M_{1\times\infty}(F):\ i<n\ \Rightarrow\ u_i=0\}=\{(0,0,\ldots,0,u_n,u_{n+1},u_{n+2},\ldots)\}.$$

Note that we can identify the elements of $V^1_\infty(F)$, $V^2_\infty(F)$, $V^3_\infty(F)$, $\ldots$ with the first, second, third, $\ldots$ columns of upper triangular matrices. Analogously with $U^1_\infty(F)$, $U^2_\infty(F)$, $\ldots$ and the rows.

For any $a,t\in\mathcal{T}_\infty(F)$, if t is invertible, we write a^t for the element $t^{-1}at$.

In the paper we will make use of maxima and minima of some subsets of $\mathbb{N}$ that may be empty. We will assume that $\max\emptyset=-\infty$ and that $\min\emptyset=\infty$.

In our considerations we are going to use the knowledge about automorphisms of $\mathcal{T}_\infty(F)$. In [17] we have investigated $\mathcal{T}_\infty(F)$ as a ring and we have obtained the following.

***Theorem* 2.1** — *(Thm.1.1, [17]). Let F be a field of at least 3 elements. If $\phi:\mathcal{T}_\infty(F)\to\mathcal{T}_\infty(F)$ is a ring epimorphism, then there exist $t\in\mathcal{T}_\infty(F)$, σ - an automorphism of F, $k,n\in\mathbb{N}\{0\}\cup\{0\}$, and a multiplicative map $f:(F^*)^n\to F^*$ such that*

$$\phi(x)=f(x_{k+1,k+1},\ldots,x_{k+n,k+n})\cdot\mathcal{I}nn_t\cdot\overline{\sigma}\cdot\mathcal{U}p_{k+n,k+n}(x).$$

Here $\mathcal{I}nn_t$ is an inner automorphism such that $\mathcal{I}nn_t(x) = t^{-1}xt$, $\overline{\sigma}$ is called a field automorphism and it is defined as follows

$$(\overline{\sigma(x)})_{nm} = \sigma(x_{nm})$$

for σ - automorphism of a field F, and $\mathcal{U}p_k$ is a map such that

$$(\mathcal{U}p_k(x))_{nm} = x_{n+k,m+k}.$$

From Theorem 2.1 we easily get

Corollary 2,1 — Let F be a field of at least 3 elements. If $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is an algebra automorphism, then ϕ is an inner automorphism.

In some works about local automorphisms, local derivations or some other similar types of maps we come across the fact that they may preserve some properties of the elements of our algebra (like in [1, 2, 14]). We will also use a similar argument. Namely, to prove our result we will show that the maps ϕ that satisfy (2) or (4) preserve the rank-one matrices, i.e. if $\mathrm{rank}(x) = 1$, then $\mathrm{rank}(\phi(x)) = 1$. By the rank we mean the row-rank. (Clearly, in our case such rank can be infinite.)

The rank-one preservers on $\mathcal{T}_\infty(F)$ are described below.

***Theorem* 2.2** — *(Thm.1.1, [18]). Assume that F is a field and that ϕ is a linear rank-one preserver. Then either*

1. *the image of ϕ consists only of matrices of rank one and the zero matrix*

   or

2. *there exist matrices a, b such that $\phi(x) = axb$ for $x \in \mathcal{T}_\infty(F)$.*

One can notice that in the above theorem we do not claim that the matrices a and b are triangular. Indeed, they do not have to be so. Obviously, it would be good to know what we have to assume to ensure that $axb \in \mathcal{T}_\infty(F)$ for all $x \in \mathcal{T}_\infty(F)$. Such criterion is given in the following lemma.

Lemma 2.1 — (Cor. 2.2, [18]). Let a be an infinite matrix over a field F such that

1. every column $v(n)$ of a is in the set $V_\infty^{k_n}(F)$ for some $k_n \in \mathbb{N}$,

2. columns of a are linearly independent,

and let b be an infinite matrix over the same field such that

1. every row $u(m)$ of a is in the set $U_\infty^{l_m}(F)$ for some $l_m \in \mathbb{N}$,

2. rows of b are linearly independent.

For all $n \leq m$ consider the set differences $l_m - k_n$. If this set contains a minimal element which is nonnegative, then the map ϕ defined on $\mathcal{T}_\infty(F)$ by the formula $\phi(x) = axb$, is a rank-one preserver on $\mathcal{T}_\infty(F)$.

Now we can move to our proof.

We know that an inner automorphism acts on the algebra by conjugacy. For this operation we have the following.

Lemma 2.2 — Suppose that F is any field and $1 \leq k \leq l$. If $a \in \mathcal{R}_\infty^{kl}(F)$ and $t \in \mathcal{T}_\infty(F)$ is invertible, then $a^t \in \mathcal{R}_\infty^{kl}(F)$.

Moreover, $a_{kl} \neq 0$ if and only if $(a^t)_{kl} \neq 0$.

PROOF : As $a \in \mathcal{R}_\infty^{kl}(F)$, we can write a as below

$$a = \left(\begin{array}{c|c|c} 0_{k\times k} & 0_{k\times l-k-1} & b_{k\times\infty} \\ \hline & 0_{l-k-1\times l-k-1} & 0_{l-k-1\times\infty} \\ \hline & & 0_{\infty\times\infty} \end{array}\right) \quad \text{in the case when } k < l, \tag{5}$$

and

$$a = \left(\begin{array}{c|c|c} 0_{k-1\times k-1} & (c_1)_{k-1\times 1} & (c_2)_{k-1\times\infty} \\ \hline & (c_3)_{1\times 1} & (c_4)_{1\times\infty} \\ \hline & & 0_{\infty\times\infty} \end{array}\right) \quad \text{in the case when } k = l. \tag{6}$$

Let a be of form (5). When we write t as

$$t = \left(\begin{array}{c|c} g_{l-1\times l-1} & g'_{k-1\times\infty} \\ \hline & g''_{\infty\times\infty} \end{array}\right)$$

we get that

$$a^t = \left(\begin{array}{c|c|c} 0_{k\times k} & 0_{k\times l-k-1} & *_{k\times\infty} \\ \hline & 0_{l-k-1\times l-k-1} & *_{l-k-1\times\infty} \\ \hline & & *_{\infty\times\infty} \end{array}\right),$$

so $(a^t)_{ij} = 0$ for all the pairs i, j such that $1 \leq i \leq j < l$.

When we write t as

$$t = \left(\begin{array}{c|c} h_{k\times k} & h'_{k\times\infty} \\ \hline & h''_{\infty\times\infty} \end{array}\right)$$

we obtain that

$$a^t = \left(\begin{array}{c|c} *_{k\times k} & *_{k\times\infty} \\ \hline & 0_{\infty\times\infty} \end{array}\right),$$

so $(a^t)_{ij} = 0$ for $l < i \leq j$.

If a is of form (6), then the calculations are analogous, so we omit them.

Let us now focus on a_{kl}. We have

$$(a^t)_{kl} = \sum_{k\leq i\leq j\leq l} (t^{-1})_{ki} a_{ij} t_{jl} = (t^{-1})_{kk} a_{kl} t_{ll} = t_{kk}^{-1} a_{kl} t_{ll},$$

so indeed $(a^t)_{kl} \neq 0$ if and only if $a_{kl} \neq 0$. □

Now we will consider the ranks of the matrices of the forms av and va, where $a \in \mathcal{T}_\infty(F)$ and v is either in $M^{fin}_{\infty\times 1}(F)$ or in $M_{1\times\infty}(F)$.

Lemma 2.3 — Let $a \in \mathcal{T}_\infty(F) \setminus \{0\}$. Then $\operatorname{rank}(a) = 1$ if and only if

- for every $v \in M^{fin}_{\infty\times 1}(F)$ the number

$$m(a, v) = \max\{i \in \mathbb{N} : \ (av)_i \neq 0\}$$

  is equal either to some n that is fixed for a, or to $-\infty$;

- for every $v \in M_{1\times\infty}(F)$ the number

$$m^T(a, v) = \min\{i \in \mathbb{N} : \ (va)_i \neq 0\}$$

  is equal either to some m that is fixed for a, or to ∞.

PROOF : We start with the first claim.

Suppose first that $\operatorname{rank}(a) = 1$. Then there exists a vector $r \in M_{1\times\infty}(F)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, $\alpha_n \neq 0$ (for some $n \in \mathbb{N}$) such that

$$a = (\alpha_1 r \ \alpha_2 r \ \ldots \ \alpha_n r \ 0_{1\times\infty} \ 0_{1\times\infty} \ 0_{1\times\infty} \ \ldots)^T.$$

Consider an arbitrary $v \in M^{fin}_{\infty\times 1}(F)$. If $rv = 0$, then $av = 0$ and $m(a, v) = -\infty$.

If $rv = \beta \neq 0$, then

$$av = (\alpha_1\beta \ \alpha_2\beta \ \ldots \ \alpha_n\beta \ 0_{1\times\infty} \ 0_{1\times\infty} \ 0_{1\times\infty} \ \ldots)^T$$

and $\alpha_i\beta \neq 0$ if and only if $\alpha_i \neq 0$. Thus, since $\alpha_n \neq 0$ and $\beta \neq 0$, $m(a,v) = n$. Concluding, $m(a,v)$ is equal to either n or $-\infty$.

Assume now that $\operatorname{rank}(a) > 1$.

Consider first the situation when this rank is finite. Then there exists $n \in \mathbb{N}$, $n \geq 2$, such that for all $i > n$ and all $j \in \mathbb{N}$ we have $a_{ij} = 0$. Since $\operatorname{rank}(a) > 1$, there exist linearly independent vectors $r_1, r_2 \in M_{1\times\infty}(F)$ and $\alpha, \beta \in F^*$, $\alpha', \ldots, \alpha'' \in F$ such that

$$a = (\gamma r_k \ \ldots \ \underbrace{\beta r_2}_{m} \ \alpha'' r_1 \ \ldots \ \alpha' r_1 \ \underbrace{\alpha r_1}_{n} \ 0_{1\times\infty}\, 0_{1\times\infty}\, 0_{1\times\infty} \ \ldots)^T.$$

Suppose that s is a minimal number such that $(r_1)_s \neq 0$. Then if we take $v = e_s$, then we have

$$av = (* \ * \ \ldots \ \underbrace{(r_1)_s}_{n} \ 0_{1\times\infty}\, 0_{1\times\infty}\, 0_{1\times\infty} \ \ldots)^T,$$

so $m(a,v) = n$.

Now notice that since r_1, r_2 are linearly independent, there exists $v \in M^{fin}_{\infty\times1}(F)$ such that $r_1v = 0$ and $r_2v \neq 0$. For this v we have

$$\begin{aligned} av = & \ \ (* \ * \ \ldots \ \underbrace{r_2v}_{m} \ \ldots \ \underbrace{r_1v}_{n} \ 0_{1\times\infty}\, 0_{1\times\infty}\, 0_{1\times\infty} \ \ldots)^T \\ = & \ \ (* \ * \ \ldots \ \underbrace{r_2v}_{m} \ 0_{1\times\infty}\, 0_{1\times\infty}\, 0_{1\times\infty} \ \ldots)^T, \end{aligned}$$

so $m(a,v) = m < n$ and we are done.

Let now $\operatorname{rank}(a) = \infty$.

Let s be the number of the first nonzero column of a. As a is upper triangular, there exists the largest number n such that $a_{ns} \neq 0$. We put $v = e_s$ and get $m(a,v) = n$.

Since $\operatorname{rank}(a) = \infty$, there exists a number m that is larger that n and such that for some r we have $a_{mr} \neq 0$. Let us consider the least m with this property and the least r for this m. If we put $v = e_r$, then $m(a,v) = m > n$, so our first claim follows.

Now we focus on the second claim.

Suppose that $\operatorname{rank}(a) = 1$. Then there exists a nonzero $c \in M^{fin}_{\infty\times1}(F)$, $m \in \mathbb{N}$, and α_m, α_{m+1}, $\alpha_{m+1}, \ldots \in F$, $\alpha_m \neq 0$ such that

$$a = (\underbrace{0_{\infty\times1}\, 0_{\infty\times1} \ \ldots \ 0_{\infty\times1}}_{m-1} \ \alpha_m c \ \alpha_{m+1} c \ \alpha_{m+2} c \ \ldots).$$

Consider an arbitrary $v \in M_{1\times\infty}(F)$. If $vc = 0$, then $va = 0$ and $m^T(v, a) = \infty$. If $vc = \beta \neq 0$, then

$$va = (\underbrace{0_{\infty\times 1}\ 0_{\infty\times 1}\ \ldots 0_{\infty\times 1}}_{m-1}\ \alpha_m\beta\ \alpha_{m+1}\beta\ \alpha_{m+2}\beta\ \ldots),$$

so since $\alpha_m \neq 0$ we have $\alpha_m\beta \neq 0$ and $m^T(a, v) = m$.

Suppose now that $1 < \operatorname{rank}(a) < \infty$. Then there exists n such that for all $i > n$ and for all j we have $a_{ij} = 0$. Moreover, there exists linearly independent vectors c_1, $c_2 \in M^{fin}_{\infty\times 1}(F)$ and α, α', …, α'', $\beta \in F$, α, $\beta \neq 0$ such that

$$a = (\underbrace{0_{\infty\times 1}\ 0_{\infty\times 1}\ \ldots\ 0_{\infty\times 1}}_{m-1}\ \alpha c_1\ \alpha' c_1\ \ldots\ \alpha'' c_1\ \beta c_2\ \ldots).$$

If we put $v = f_n$, then $m^T(a, v) = m$.

Now note that since c_1, c_2 are linearly independent, there exists $v \in M^{fin}_{\infty\times 1}(F)$ such that $vc_1 = 0$ and $vc_2 \neq 0$. For this v we have

$$va = (\underbrace{0\ 0\ \ldots\ 0}_{m-1}\ \underbrace{\alpha vc_1}_{m}\ \alpha' vc_1\ \ldots\ \alpha'' c_1\ \underbrace{\beta c_2}_{k}\ \ldots) = (\underbrace{0\ 0\ \ldots 0}_{k-1}\ \underbrace{\beta c_2}_{k}\ \ldots),$$

so $m^T(a, v) = k > m$.

Let now $\operatorname{rank}(a) = \infty$. There exists a minimal n such that $a_{ns} \neq 0$ for some $s \in \mathbb{N}$. Consider this n and choose any s for which $a_{ns} \neq 0$. If we take $v = f_s$, then $m^T(a, v) = n$. As $\operatorname{rank}(a) = \infty$, there exists $m > n$ such that for some r we have $a_{mr} \neq 0$. If we put $v = f_r$, then $m^T(a, v) = m > n$.

The next lemma tells about the behaviour of the numbers $m(a, v)$, $m^T(a, v)$ when we consider the conjugacies of a.

Lemma 2.4 — Let F be a field and let $a \in \mathcal{T}_\infty(F)$. If rank of a is equal to 1, then

- for any vector $v \in M^{fin}_{\infty\times 1}(F)$ and any invertible $t_v \in \mathcal{T}_\infty(F)$ the number

  $$m(a^{t_v}, v) = \max\left\{i \in \mathbb{N} :\ (a^{t_v} v)_i \neq 0\right\}$$

  is equal to either some n that is fixed for a, or to $-\infty$.

  Moreover, for at least one v this maximum is equal to n.

- for any vector $v \in M_{1\times\infty}(F)$ and any invertible $t_v \in \mathcal{T}_\infty(F)$ the number

  $$m^T(a^{t_v}, v) = \min\left\{i \in \mathbb{N} :\ (v a^{t_v})_i \neq 0\right\}$$

is equal to either some m that is fixed for a, or to ∞.

Moreover, for at least one v this minimum is equal to m.

PROOF : We prove only the first point. The proof of the second one can be led analogously.

If $\operatorname{rank}(a) = 1$, then $a \in \mathcal{R}^{kl}_\infty(F)$ for some $k \leq l$ and $a_{kl} \neq 0$. From Lemma 2.2 we know that for all $t \in \mathcal{T}_\infty(F)$ we have $a^t \in \mathcal{R}^{kl}_\infty(F)$ and $(a^t)_{kl} \neq 0$. Clearly, $\operatorname{rank}(a^t) = 1$. Thus, from Lemma 2.3 we get that for any v either $a^{t_v}v = 0$ or $m(a^{t_v}, v) = k$.

Let us now prove the second claim. From Lemma 2.4 we know that if $a \in \mathcal{R}^{kl}_\infty(F)$ and $a_{kl} \neq 0$, then for any t, in particular for any $t_v \in \mathcal{T}_\infty(F)$ the matrix a^{t_v} is in $\mathcal{R}^{kl}_\infty(F)$ and $(a^{t_v})_{kl} \neq 0$. Thus, it suffices to put $v = e_l$ and then $(a^{t_{e_l}}, e_l)_k \neq 0$. □

The last lemma we prove is a bit different from those above.

Lemma 2.5 — For any field F and any map ϕ satisfying either (2) or (4) we have $\phi(e_\infty) = e_\infty$.

PROOF : Since every $\phi_{x,v}$ is an automorphism we have $\phi_{x,v}(e_\infty) = e_\infty$. Hence for every v either in $M^{fin}_{\infty\times 1}(F)$ or in $M_{1\times\infty}(F)$ we have

$$\phi_{e_\infty,v}(e_\infty)v = v \quad \text{or} \quad v\phi_{e_\infty,v}(e_\infty) = v \quad \text{respectively.}$$

Thus, from (2) or (4) we get $\phi(e_\infty)v = v$ or $v\phi(e_\infty) = v$ respectively. As it holds for any v, we must have $\phi(e_\infty) = e_\infty$.

Now, using all the above lemmas we will prove the following.

Proposition 2.1 — Suppose that F is a field and that $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ satisfies either (2) or (4). Then ϕ is a rank-one preserver.

PROOF : Again we will prove the claim in the case when ϕ satisfies (2).

Suppose that there exists $x \in \mathcal{T}_\infty(F)$ such that $\operatorname{rank}(x) = 1$, but $\operatorname{rank}(\phi(x)) \neq 1$.

Assume first that $\phi(x) = 0$. Then for all $v \in M^{fin}_{\infty\times 1}(F)$ we have $\phi(x)v = 0$. Hence, we should also have $0 = \phi_{x,v}(x)v = (t_{x,v})^{-1}xt_{x,v}v$. However, from Lemma 2.4 we know that the number $m(a^{t_v}, v)$ is distinct from 0 for at least one v – a contradiction.

Assume then that $\operatorname{rank}(\phi(x)) > 1$. Since for all v $\operatorname{rank}(\phi_{x,v}(x))$ is equal to 1 and $\phi_{x,v}(x) \in \mathcal{R}^{kl}_\infty(F)$ the numbers $m(a^{t_v}, v)$ can be equal to either some fixed n or to $-\infty$. However, as $\operatorname{rank}(\phi(x)) > 1$, by Lemma 2.3, the numbers $m(a^{t_v}, v)$ can also take some other values – a contradiction again.

Summing up, if $\operatorname{rank}(x) = 1$, then $\operatorname{rank}(\phi(x)) = 1$. □

Now we will prove the main theorem.

PROOF OF THEOREM 1.1 : The ‘if’ part of the proof is obvious, so we focus on ‘and only if’.

Let ϕ satisfy (2) or (4). Then, by Proposition 2.1, ϕ preserves rank-one matrices. Such preservers are described in Theorem 2.2. From this theorem we know that the rank-one preservers can be of two types. Suppose that ϕ is of the first type, i.e. $\phi(\mathcal{T}_\infty(F))$ is a subspace that contains only rank-one matrices and the zero matrix. From Lemma 2.5 we know that $\phi(e_\infty) = e_\infty$, so we get a contradiction. Therefore ϕ must be given by a formula $\phi(x) = axb$. Moreover, since $\phi(e_\infty) = e_\infty$, the matrices a, b are such that $ab = e_\infty$.

Now we want to get more information about a and b. Some of them are given in Lemma 2.1.

Observe that from Lemma 2.2 it follows that $\phi(e_{nn}) \in \mathcal{R}^{nn}_\infty(F)$. Hence, as $\phi(e_{nn}) = ae_{nn}b = v(n)u(n)$ and $l_n - k_n \geq 0$, we must have $l_n = k_n$ for every $n \in \mathbb{N}$. Moreover, since $0 \leq l_m - k_n = k_m - k_n$ for $m \geq n$, the sequence $(k_n)_{n\in\mathbb{N}} = (l_n)_{n\in\mathbb{N}}$ is nondecreasing. We wish to show that $k_n = n$ for all n.

If $k_1 > 1$, then $(ab)_{11} = 0$. Since $ab = e_\infty$, the latter is a contradiction.

Suppose then that for some n we have $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $\ldots$, $k_n = n$, but $k_{n+1} \neq n + 1$.

First assume that $k_{n+1} < n + 1$. Then clearly $k_{n+1} = n$. Suppose that

$$k_{n+1} = k_{n+2} = \cdots = k_{n+m} = n \quad \text{and} \quad k_{n+m+1} > n.$$

Then we can write a and b in the following block form.

$$a = \left(\begin{array}{c|c|c} (a_1)_{n\times n} & (a_2)_{n\times m} & (a_3)_{n\times\infty} \\ \hline 0 & (a_4)_{m\times m} = 0_{m\times m} & (a_5)_{m\times\infty} \\ \hline 0 & (a_6)_{\infty\times m} = 0_{\infty\times m} & (a_7)_{\infty\times\infty} \end{array}\right),$$

$$b = \left(\begin{array}{c|c|c} (b_1)_{n\times n} & (b_2)_{n\times m} & (b_3)_{n\times\infty} \\ \hline 0 & (b_4)_{m\times m} & (b_5)_{m\times\infty} \\ \hline 0 & (b_6)_{\infty\times m} & (b_7)_{\infty\times\infty} \end{array}\right), \qquad \text{where } (b_6)_{i1} = 0 \ \text{ for all } i \in \mathbb{N}.$$

Consider the element $(ab)_{n+1,n+1}$. Since $a_4 = 0$ and $(b_6)_{i1} = 0$, this element is equal to 0 – this contradicts the fact that $ab = e_\infty$.

Suppose now that $k_{n+1} > n+1$. Then we write a and b in the block form as below.

$$a = \left(\begin{array}{c|c} (a_1')_{n\times n} & (a_2')_{n\times\infty} \\ \hline 0 & (a_3')_{\infty\times\infty} \end{array}\right),$$
$$b = \left(\begin{array}{c|c} (b_1')_{n\times n} & (b_2')_{n\times\infty} \\ \hline 0 & (b_3')_{\infty\times\infty} \end{array}\right), \quad \text{where } (b_3')_{i1} = 0 \text{ for all } i \in \mathbb{N}.$$

Since $ab = e_\infty$, we also have $a_3'b_3' = e_\infty$. However, from $(b_3')_{i1} = 0$, we obtain that $(a_3'b_3')_{11} = 0$ – a contradiction.

Summing up $k_n = n$ for all $n \in \mathbb{N}$. This means that a, b are both upper triangular, and consequently $a = b^{-1}$. Thus $\phi(x) = b^{-1}xb$. □

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