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FACE ENUMERATION FOR LINE ARRANGEMENTS IN A 2-TORUS

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A toric arrangement is a finite collection of codimension-1 subtori in a torus. These subtori stratify the ambient torus into faces of various dimensions. Let f_i denote the number of i-dimensional faces; these so-called face numbers satisfy the Euler relation $\sum_i (-1)^i f_i = 0$. However, not all tuples of natural numbers satisfying this relation arise as face numbers of some toric arrangement. In this paper we focus on toric arrangements in a 2-dimensional torus and obtain a characterization of their face numbers. In particular we show that the convex hull of these face numbers is a cone.

Key words : Toric arrangements; face enumerations; f-vector.

1. INTRODUCTION

Counting the number of connected components of a certain geometric set divided by its codimension-1 subsets is a classical problem in combinatorial geometry. The simplest possible case is that of a partitioning of the Euclidean plane by finitely many straight lines. Such a collection determines a stratification of the plane consisting of *vertices* (intersections of lines), *edges* (maximal connected components of the lines not containing any vertex) and *chambers* (maximal connected components of the plane containing neither the edges nor the vertices). The combinatorics that emerges from these intersections is intriguing. This is evident by the number of interesting problems and conjectures described in Grünbaum's exposition [4] (classically, line arrangements are studied in the projective plane). A systematic discussion of combinatorial aspects of hyperplane arrangements (i.e., higher-dimensional analogues of line arrangements) can be found in [3, Chapter 18].

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The first question that one can ask is to count the number of chambers formed by a line arrangement. An easy case is that of n lines in general position; here the number of chambers is $1 + \binom{n}{2}$. The study of this and related problems goes back to the work of early 19th century mathematician Steiner. A formula that counts the number of connected components of the complement of a hyperplane arrangement was discovered by Zaslavsky in [10].

Let f_i denote the number of i-dimensional strata, for $i = 0, 1, 2$, of the projective plane induced by a line arrangement. These are the *face numbers* and the triple (f_0, f_1, f_2) is the f*-vector* of a line arrangement. Many interesting questions arise when one wants to study relations between face numbers. For example, these numbers satisfy the Euler relation $f_0 - f_1 + f_2 = 1$ but not all triples of natural numbers satisfying this relation arise as face numbers of line arrangements. One can find a list of known results and some conjectures in [4, Section 2.2]. In this paper we wish to answer similar questions but in the context of line arrangements in a torus.

Partitioning problems for spaces other than Euclidean and projective spaces were studied by only handful of authors. To our knowledge the first paper that deals with a more general situation is by Zaslavsky [11]. He derives a formula for counting the number of connected components a topological space when dissected by finitely many of its subspaces. He showed that not only the combinatorics of the intersections (which is encoded in the Möbius function of the intersection poset) but also their geometry (as captured by the Euler characteristic) plays a role in determining the number of chambers. In recent years several authors have considered toric arrangements. A toric arrangement is a finite collection of codimension-1 subtori in a torus. The formula for the number of chambers for such arrangements was first discovered by Ehrenborg *et al.* in [2]. The same formula was also independently discovered by Lawrence in [5] and by the second author in [1]. Recently, Shnurnikov has characterized the set of all possible values of top-dimensional faces for arrangements in a 2-torus in [7] and for arrangements in higher-dimensional tori in [6]. See also [8] for arrangements in hyperbolic spaces and the icosahedron and arrangements of immersed circles in surfaces.

The aim of this paper is to give a characterization of the f-vector for toric arrangements in a 2-torus. The paper is organized as follows. In Section 2 we introduce toric arrangements in full generality and fix notations. In Section 3 we prove some properties of f-vectors for toric line arrangements. In particular we show that for toric line arrangements the convex hull of the pairs (f_0, f_2) is a cone in the first quadrant. Conversely, for every pair of integers in this cone there corresponds a toric line arrangement (Theorem 16). We also characterize the face numbers for arrangements containing at most three toric lines. Finally, in Section 4 we outline future research by commenting on arrangements in surfaces of higher genus.

2. Toric Arrangements

The l-dimensional torus $\mathbb{T}^l$ is the quotient space $\mathbb{R}^l/\mathbb{Z}^l$. When identified with the set $[0,1)^l$ it forms an abelian group with the group structure given by the componentwise addition modulo 1. There is also a 'multiplicative' way of looking at the torus when we consider it as the product of S^1's. The group structure here is the componentwise multiplication of complex numbers of modulus 1. However, throughout this paper, we stick to the additive way of looking at a torus. In this section we define toric arrangements and collect some relevant background material.

We assume the reader's familiarity with basic algebraic topology and combinatorics. The combinatorics of posets and lattices that we need can be found in Stanley's book [9]. The field of toric arrangements is fairly recent; Ehrenborg, Readdy and Slone mainly study the problem of enumerating faces of the induced decomposition of the torus in [2]. On the other hand a number theoretic aspect is explored by Lawrence in [5].

We denote by $\pi\colon \mathbb{R}^l \to \mathbb{T}^l$ the quotient map. Note that π is also the covering map and $\mathbb{R}^l$ is the universal cover of the l-torus which is a compact manifold. We say that a k-subspace V of $\mathbb{R}^l$ is *rational* if it is the kernel of an $n \times l$ matrix A with integer entries. The image $\overline{V} := \pi(V)$ is a closed subgroup of $\mathbb{T}^l$. Topologically $\overline{V}$ is disconnected and each connected component is a k-torus. The connected components are known as *toric subspaces* (or *cosets*) of $\overline{V}$. Let ${}_0\overline{V}$ denote the coset containing $\mathbf{0}$. Then $\overline{V}/{}_0\overline{V}$ is a finite abelian group whose order is the number of cosets of $\overline{V}$. One can check that every closed subgroup of the torus arises in this manner. It is important to note that the subgroup $\overline{V}$ depends only on the free abelian group generated by the row-space of A. Hence one can assume that the rows of A form a basis for the row-space. The subgroup $\overline{V}$ is connected if and only if the greatest common divisor of all the $k \times k$ minors of A is 1. Two $k \times l$ matrices A and A' represent the same subgroup if and only if there exists a $k \times k$ unimodular matrix U such that $A' = AU$.

A *toric hyperplane* is a toric subspace of codimension-1, i.e., it is the projection of an affine hyperplane in $\mathbb{R}^l$. We have the following definition.

Definition 1 — A toric arrangement in $\mathbb{T}^l$ is a finite collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of toric hyperplanes.

A rational, codimension-1 subspace in $\mathbb{R}^l$ is specified by an equation $a_1x_1 + \cdots + a_lx_l = c'$ where each $a_i \in \mathbb{Z}$. Hence we represent a toric hyperplane by a pair $(\mathbf{a}, c)$ where $\mathbf{a}$ is a row vector of integers and $c \in [0,1)$. It is convenient to express a toric arrangement $\mathcal{A}$ as an augmented matrix $[A \mid \mathbf{c}]$ where A is an $n \times l$ matrix of integers such that its each row represents the corresponding toric hyperplane and $\mathbf{c}$ is a vector in $[0,1)^n$ representing intercept of each hyperplane.

To every toric arrangement there is an associated periodic hyperplane arrangement $\tilde{\mathcal{A}}$ in $\mathbb{R}^l$. The inverse image of each H_i under the covering map π is the union of parallel integer translates of a codimension-1 subspace. Recall that a hyperplane arrangement is said to be *essential* if the largest dimension of the subspace spanned by the normals to hyperplanes is l. We say that a toric arrangement is *essential* if the associated hyperplane arrangement $\tilde{\mathcal{A}}$ is essential. Equivalently, it means that the rank of the matrix A is l. Without loss of generality we assume that a toric arrangement is always essential; which forces $n \geq l$. If this is not the case then the enumerative problems that we consider in this paper reduce to equivalent problems in a torus of smaller dimension.

The hyperplane arrangement $\tilde{\mathcal{A}}$ induces a stratification of $\mathbb{R}^l$ such that these open strata are relative interiors of convex polytopes. A nonempty subset $F \subset \mathbb{T}^l$ is said to be a *face* of the toric arrangement $\mathcal{A}$ if there is a stratum $\tilde{F}$ of $\tilde{\mathcal{A}}$ such that $\pi(\tilde{F}) = F$. The *dimension* of F is the dimension of the support of $\tilde{F}$ and it is denoted by $\dim(F)$. It is important to note that the closure of F in $\mathbb{T}^l$ need not be homeomorphic to a disk. Hence a toric arrangement stratifies the ambient torus; this stratification need not define a regular cell structure but it does have special properties.

Definition 2 — A *polytopal complex* is a cell complex $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ with the following additional data.

1. Every cell e_λ is equipped with a k*-polytopal cell structure* which is a pair $(P_\lambda, \phi_\lambda)$ of a k-convex polytope and a cellular map $\phi_\lambda \colon P_\lambda \to X$ such that $\phi_\lambda(P_\lambda) = \overline{e_\lambda}$ and the restriction of ϕ_λ to the interior of P_λ is a homeomorphism.

2. If $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ then $e_\mu \subset \overline{e_\lambda}$.

3. For every face P' of P_λ, there exists a cell e_μ in X and a map $b \colon Q_\mu \to \partial P_\lambda$ such that $b(\operatorname{Int} Q_\mu) = P'$ and $\phi_\lambda \circ b = \phi_\mu$.

The following lemma is a straightforward application of the fact that the covering map π is stratification preserving.

Lemma 3 — If $\mathcal{A}$ is a toric arrangement in $\mathbb{T}^l$ then the induced stratification is a polytopal complex.

Definition 4 — The *intersection poset* $L(\mathcal{A})$ of a toric arrangement $\mathcal{A}$ is defined to be the set of all connected components arising from all possible intersections of the toric hyperplanes ordered by reverse inclusion.

By convention, the ambient torus corresponds to the empty intersection. The intersection poset is graded by the codimensions of the intersections. Before proceeding further let us look at a couple of examples.

Example 5 : Let $\mathcal{A}$ be the toric arrangement in $\mathbb{T}^2$ obtained by projecting the lines $x = -2y$ and $y = -2x$. These toric hyperplanes intersect in three points $p_1 = (0, 0), p_2 = (1/3, 1/3)$ and $p_3 = (2/3, 2/3)$. The arrangement stratifies the torus into three 0-faces, six 1-faces and three 2-faces. This is not a regular subdivision of the torus since the closure of every 2-face is a cylinder. Figure 1 shows the arrangement together with the associated intersection poset.

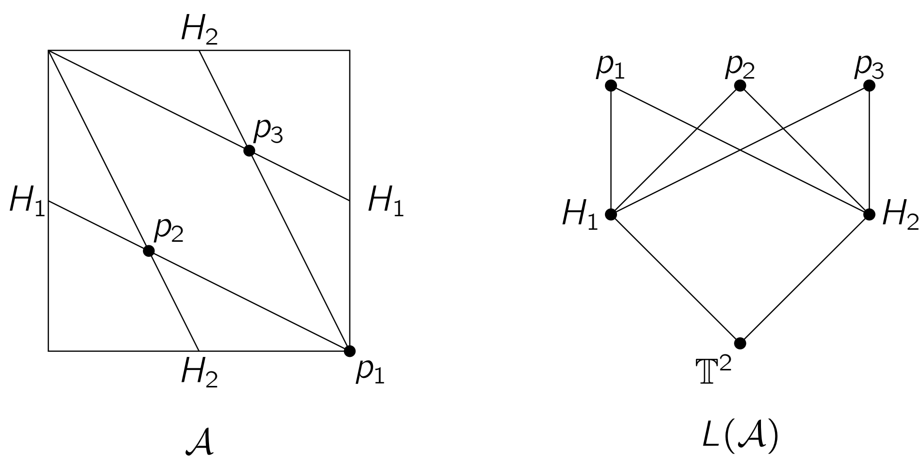

Figure 1: A toric arrangement in $\mathbb{T}^2$.

Example 6 : Now consider the arrangement formed by including the projection of the line $y = x$ in the previous arrangement. They intersect in the same three points as above. However, there are nine 1-faces and six 2-faces. The induced stratification is a regular Δ-complex. Figure 2 shows the arrangement and the associated intersection poset.

Since our focus is on counting the number of various-dimensional faces of a toric arrangement we now turn to the combinatorics aspect. The idea that captures the combinatorics of the intersections is the Möbius function of the arrangement which we now define.

Definition 7 — The Möbius function of a toric arrangement is the function $\mu\colon L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ defined recursively as follows:

$$\mu(X, Y) = \begin{cases} 0, & \text{if } Y < X, \\ 1, & \text{if } X = Y, \\ -\sum_{X \leq Z < Y} \mu(X, Z), & \text{if } X < Y. \end{cases}$$

The Möbius function plays an important role in counting the number of faces of an arrangement. The following theorem has appeared in [2, Corollary 3.12], [5, Theorem 3] and [1, Example 5.5].

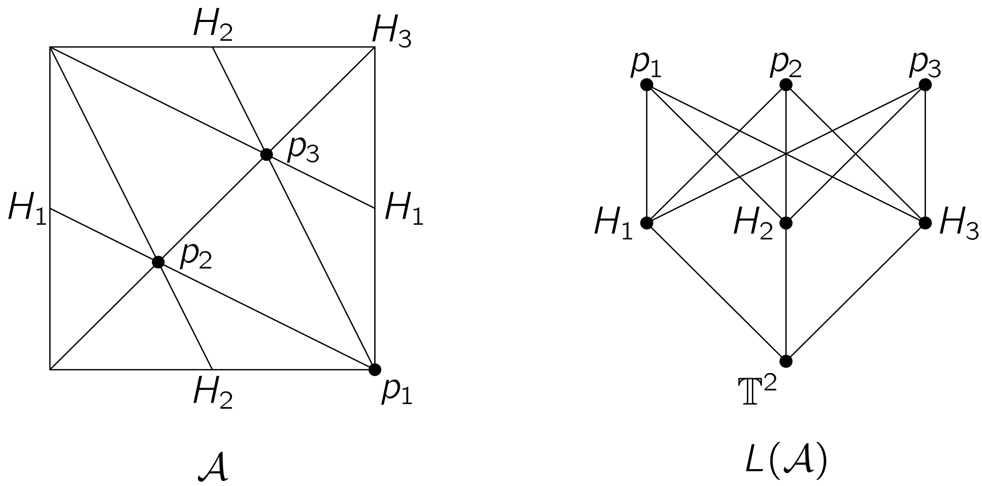

Figure 2: A toric arrangement with regular cell decomposition.

***Theorem* 8** — *Let f_k denote the number of k-dimensional faces of a toric arrangement $\mathcal{A}$. Then we have*

$$f_k = \sum_{\substack{\dim Y = k \\ \dim Z = 0 \\ Y \leq Z}} |\mu(Y, Z)|.$$

In particular the number of top-dimensional faces is determined by the values the Möbius function takes at the points of intersections. The generating function for the face numbers is known as the f-*polynomial* and defined as $f_{\mathcal{A}}(x) = \sum_{k=0}^{l} f_k x^{l-k}$. Using Theorem 8 above we get a particularly nice form for the f-polynomial

$$f_{\mathcal{A}}(x) = \sum_{\substack{\dim Y = k \\ \dim Z = 0 \\ Y \leq Z}} |\mu(Y, Z)| x^{l - \dim Y}.$$

We say that the toric hyperplanes of an arrangement are in *general position* if the intersection of any i of the subtori, $i \geq 1$, is either empty or $(l - i)$-dimensional. A toric arrangement is called *simple* if all the toric hyper planes are in general position. One can check that in case of simple arrangements every interval of the associated intersection poset is a Boolean algebra. For simple toric arrangements we have $f_{\mathcal{A}}(x) = f_0(x+1)^l$ hence

$$f_k = f_0 \binom{l}{l-k}.$$

3. Some Face Enumeration Formulas

In this section we focus our attention to arrangements in the 2-torus $\mathbb{T}^2$ with the aim to explore relationship between the face numbers f_0, f_1, f_2 and n the number of subtori in an arrangement.

The projection of the straight line $ax + by = c$ in $\mathbb{R}^2$ under the canonical map π onto the torus, where $a, b \in \mathbb{Z}$, is said to be a *toric line*. Here we identify c with $\pi(c)$ and assume that the a and b are coprime.

Definition 9 — A toric line arrangement is a finite collection $\mathcal{A} = \{l_1, \ldots, l_n\}$ of toric lines in $\mathbb{T}^2$.

We denote a line l_j by an augmented matrix $[a_j, b_j \mid c_j]$ and say that the line is of type (a_j, b_j) if the intercept is not relevant. As before we will denote $\mathcal{A}$ by an augmented $n \times 3$ matrix $[A \mid \mathbf{c}]$. Where rows of A are coefficients of equations of toric lines. We also assume that these arrangements are essential, i.e., we do not consider the arrangements in which all the lines are parallel.

The intersection two toric lines, of type say (a_i, b_i) and (a_j, b_j), is a finite set of points. The cardinality of the intersection is the absolute value of the determinant $\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$. This follows from the Smith normal form. Recall that the Smith normal formal form of a 2×2 matrix A is a diagonal matrix with 1 in the $(1, 1)$ position and $\det A$ in the $(2, 2)$ position. The subgroup of the torus corresponding to A is then a finite abelian group of order $|\det A|$ as the normal form is obtained by unimodular transformations (see [7, Lemma 1] for a geometric proof).

We now turn to the faces of an arrangement. For simplicity we call 0-dimensional faces vertices, 1-dimensional faces edges and 2-dimensional faces chambers; their numbers are denoted by f_0, f_1, f_2 respectively. These face numbers clearly satisfy the Euler relation $f_0 - f_1 + f_2 = 0$. It tells us that f_1 is redundant; hence we characterize pairs of natural numbers which appear as (f_0, f_2) for some toric arrangement.

Let L_0 denote the set of all vertices. The number of lines in a toric line arrangement, that pass through a vertex v is called the *degree* of that vertex and is denoted by $\deg(v)$. The following is a straightforward application of Theorem 8.

Lemma 10 —

$$f_1 = \sum_{v \in L_0} \deg(v).$$

For $j \geq 2$ denote by t_j the number of vertices v with $\deg v = j$.

Definition 11 — Let $\pi\colon \mathbb{R}^2 \to T^2$ denote the canonical projection. A subset $C \subseteq T^2$ is a *toric k-gon* if there exists a k-gon C_0 in $\mathbb{R}^2$ such that $\pi(C_0) = C$ and the restriction of π to the interior of C_0 is a homeomorphism onto the image.

For a toric arrangement $\mathcal{A}$ let p_k denote the number of chambers that are toric k-gons for $k \geq 3$.

Lemma 12 — The following identities hold for any toric arrangement:

$$f_0 = \sum_j t_j, f_1 = \sum_j jt_j, f_2 = \sum_k p_k, f_2 = \sum_j (j-1)t_j.$$

Lemma 13 — For a toric arrangement we have

$$2f_1 = \sum_k kp_k.$$

PROOF : Given a chamber C of $\mathcal{A}$ its lift $\pi^{-1}(C)$ consists of polygons in $\mathbb{R}^2$. The right hand side of the above equation is obtained by counting the edges bounding a polygon in that lift. Since each such edge of this polygon projects downstairs to an edge in the arrangement, $\sum_{k\geq 3} kp_k$ counts every edge in the arrangement a certain number of times.

Let an edge e be counted j times in the above manner. Since e is arbitrary, if we show that $j = 2$ we are done. Observe that a small enough neighbourhood U of a point on e is the union of j semi-disks identified along their diameters. For a point $p \in e$ we have

$$H_2(\mathbb{T}^2, \mathbb{T}^2 - p) \cong H_2(U, U - p) \cong H_1(U - p) \cong H_1(\bigvee_{j-1} S^1) \cong Z^{j-1}.$$

Since $\mathbb{T}^2$ is a closed manifold, we already have $H_2(\mathbb{T}^2, \mathbb{T}^2 - p) \cong \mathbb{Z}$ so that $j = 2$. □

Lemma 14 — For any toric line arrangement, we have the following:

$$t_2 = \sum_{j\geq 3}(j-3)t_j + \sum_{k\geq 3}(k-3)p_k, \tag{1}$$

$$p_3 = \sum_{j\geq 2} 2(j-2)t_j + \sum_{k\geq 4}(k-4)p_k. \tag{2}$$

PROOF : The proof is a simple application of the Euler relation and definitions:

$$RHS - t_2 = \sum_{j\geq 2}(j-3)t_j + \sum_{k\geq 3}(k-3)p_k = f_1 - 3f_0 + 2f_1 - 3f_2 = 0$$

and for Equation 2 we have

$$RHS - p_3 = \sum_{j\geq 2} 2(j-2)t_j + \sum_{k\geq 3}(k-4)p_k = 2f_1 - 4f_0 + 2f_1 - 4f_2 = 0. \qquad \square$$

We now turn our attention to Shnurnikov's result that characterizes the numbers that can occur as the number of chambers of a toric line arrangement. We reproduce the proof for the benefit of the reader.

***Theorem* 15** — *[7, Theorem 1]. Denote by $F(\mathbb{T}^2, n)$ the set of all possible values of f_2 that correspond to an arrangement of n toric lines. Then*

$$F(\mathbb{T}^2, n) = \{n-1\} \cup \{l \in \mathbb{N} : l \geq 2n-4\}.$$

PROOF : We first prove that $F(\mathbb{T}^2, n) \supseteq \{n-1\} \cup \{l \in \mathbb{N} : l \geq 2n-4\}$ by constructing arrangements with specified f_2. In order get $f_2 = n-1$ consider the following arrangement:

$$\mathcal{A} = \{[1,0 \mid 0], [0,1 \mid c_j] : 1 \leq j \leq n-1, c_i \neq c_j \text{ for } i \neq j\}.$$

Figure 3 below illustrates the construction for $n = 6$. The boundary edges of the fundamental domain correspond to toric lines $[1,0 \mid 0]$ and $[0,1 \mid 0]$. Any of these lines are shown dotted if they are not part of the arrangement.

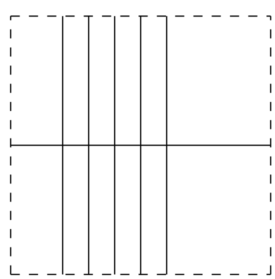

Figure 3: $(n, f_2) = (6, 5)$

We now construct an arrangement $\mathcal{A}$ with $f_2 = 2n - 4 + a$ for an arbitrary whole number a. Consider the arrangement consisting of the toric lines of the following types:

1. $[0,1 \mid 0]$;
2. $[a+1, -1 \mid 0]$;
3. $[1,0 \mid 0]$, $\left[1,0 \mid \frac{1}{a+2}\right], \left[1,0 \mid \frac{1}{a+3}\right], \ldots, \left[1,0 \mid \frac{1}{a+n-2}\right]$.

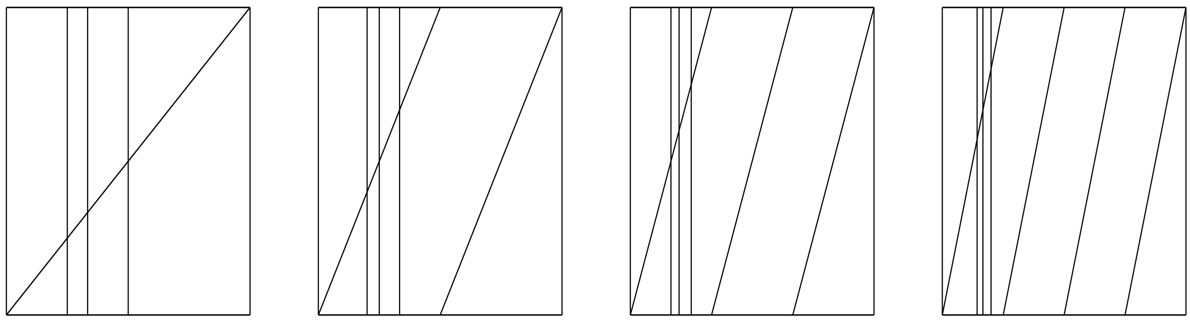

Figure 4: $(n, f_2) = (6, 8), (6, 9), (6, 10)$ and $(6, 11)$

The construction is illustrated in Figure 4 for $n = 6$.

In order to prove the reverse containment assume that $\mathcal{A}$ contains at most m toric lines of the same type (i.e., m is the maximal number of parallel lines). If $m = n-1$ then f_2 is a multiple of $n-1$. To see this observe that the nth line intersects all the previous lines in the same number of points.

For $2 \leq m \leq n-2$, each of the remaining $n-m$ lines intersect the m parallel lines in at least m points. Hence we have $f_2 \geq m(n-m)$. An easy exercise in calculus shows that the function $x(n-x)$ attains its bounds on the interval $[2, n-2]$ and the minima is $2(n-2)$ which implies that

$$f_2 \geq 2n - 4.$$

The last case is that of having no parallel lines in $\mathcal{A}$. Here one has to consider several sub-cases. First, assume that any two of the lines intersect in at least two points. So, if $n = 2$ then $f_2 \geq 2$. By induction on n assume that for $n-1$ lines $f_2 \geq 2(n-1)-2$. If the nth line creates a new vertex then t_2 goes up by 1 and if it passes through an existing vertex then t_j goes down by one and t_{j+1} goes up by one for some j. In either case, using Lemma 12, we conclude that f_2 increases by at least 2.

Assume that some two lines, say l_i, l_j, meet in exactly one point. If all the remaining $n-2$ lines meet $l_i \cup l_j$ in at least two points then $f_2 \geq 2n-3$. Otherwise we claim that there exists a third line l_k which passes through the same intersection point. In order to prove the claim assume that l_i is of type $(1, 0)$ and l_j is of type $(0, 1)$ then as the line l_k should intersect both these lines in point it has to be of the type $(1, 1)$ or $(1, -1)$. In either case the intersection $l_i \cap l_j \cap l_k$ is singleton. Furthermore it is easy to prove that if l' is any line which is not parallel to either l_i, l_j or l_k then it intersects these three lines in at least two points. In this case $f_2 \geq 2n-4$; the proof is on the same lines as that of the first sub-case. $\square$

It is well-known that the face numbers f_0, f_2 of projective line arrangements satisfy linear inequalities. These inequalities are such that their convex hull is a cone in the (f_0, f_2)-plane. However not all lattice point in that cone are realizable as face numbers of projective line arrangements (see [3, page 401] for details). The face numbers of 3-polytopes also satisfy similar inequalities and determine a cone. However, every pair (f_0, f_2) satisfying these inequalities indeed corresponds to some polytope (see [3, page 190] for details). The case of toric arrangements is not very different as we shall now prove. We say that a toric line arrangement is *simplicial* if all the chambers are triangles. Note that this is equivalent to saying that the induced stratification defines a Δ-complex structure.

***Theorem* 16** — *Given $f_0, f_2 \in \mathbb{N}$, there exists a toric arrangement $\mathcal{A}$ with f_0 vertices and f_2 faces if and only if*

$$f_0 \leq f_2 \leq 2f_0.$$

Equality on the left holds if and only if $\mathcal{A}$ is simple; equality on the right holds if and only if $\mathcal{A}$ is simplicial.

PROOF : We prove the 'only if' part first. Using Lemma 12 we see that:

$$f_0 = \sum_{j\geq 2} t_j \leq \sum_{j\geq 2}(j-1)t_j = f_2.$$

The second inequality can be written as $f_2 \leq 2f_1 - 2f_2$, or equivalently as $2f_1 \geq 3f_2$ which follows from Lemmas 12 and 13:

$$2f_1 = \sum_{k\geq 3} kp_k \geq \sum_{k\geq 3} 3p_k = 3f_2.$$

The 'if' part on the other hand can be proved constructively. Consider the arrangement $\mathcal{A}$ which contains the following $n = f_2 - f_0 + 2$ toric lines :

1. $[0, 1 \mid 0]$;
2. $[f_0, -1 \mid 0]$;
3. $\left[1, 0 \mid \frac{r}{f_0}\right]$ for all $0 \leq r < f_2 - f_0$.

See Figures 5, 6 for illustrations.

Now assume that $\mathcal{A}$ is an arrangement with $f_2 = f_0$. Then $\sum_{j\geq 2}(j-1)t_j = \sum_{j\geq 2} t_j$ implies that

$$\sum_{j\geq 2}(j-2)t_j = 0.$$

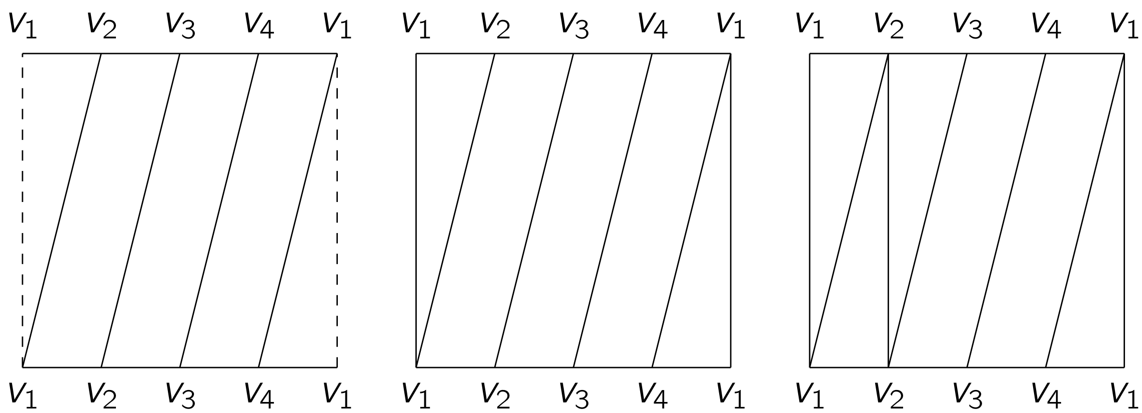

Figure 5: From left: toric arrangements with f-vectors (4,8,4), (4,9,5) and (4,10,6)

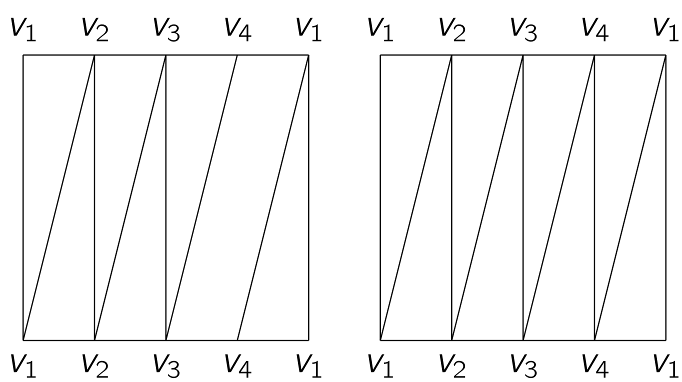

Figure 6: From left: toric arrangements with f-vectors (4,11,7) and (4,12,8)

Since $(j-2) > 0$ for every $j \geq 3$ we have $t_j = 0$ for those j's. Consequently there are only degree 2 vertices; equivalently the arrangement is simple. Converse of this statement is also clear.

Now assume that $f_2 = 2f_0$. Using the equation

$$\sum_{k\geq 3} p_k = 2\sum_{k\geq 3}(\frac{k}{2}-1)p_k$$

we get $p_k = 0$ for $k \geq 4$. Converse can be proved analogously. □

Combining above inequality with Theorem 15 we see that for an arrangement of n toric lines $f_0 \in \{\lfloor\frac{n-1}{2}\rfloor\} \cup \{l : l \geq n-2\}$. As f_0 is not bounded above there is no hope for a complete characterization of the pairs (n, f_0). Instead we focus on the triples (n, f_0, f_2). More precisely we would like to characterize all such triples of natural numbers for which there exists an arrangement $\mathcal{A}$

of n lines, with f_0 vertices and f_2 faces. For $n \geq 2$, let

$$\mathcal{C}'(n) := \{(f_0, f_2) \mid f_0 \leq f_2 \leq 2f_0, f_2 \geq n-1\}.$$

There is an obvious chain of inclusions $\mathcal{C}'(2) \supset \cdots \mathcal{C}'(n) \supset \mathcal{C}'(n+1) \supset \cdots$. Our aim is to characterize elements of $\mathcal{C}'(n)$ that are realizable as face numbers of toric arrangement. We denote this subset by $\mathcal{C}(n)$. We start with the easiest case.

Lemma 17 — For toric arrangements of 2 lines we have

$$\mathcal{C}(2) = \{(f, f) \mid f \in \mathbb{N}\}.$$

Equivalently, all toric arrangements of 2 lines are simple and their f-vectors are of the form $(f, 2f, f)$ for all natural numbers f.

PROOF : Since there are only 2 lines, all vertices have degree 2. This ensures, $f_0 = t_2 = f_2$. On the other hand, if $f_0 = f_2$ is given, then indeed consider $\mathcal{A}$ with two toric lines, one each of the types $(0, 1)$ and $(f_0, -1)$. □

Next we completely describe the set $\mathcal{C}(3)$. First we show that as in the 2 lines case, the points of the type (f_0, f_0) are completely realizable for simple arrangements of 3 toric lines. Then we characterize the simplicial arrangements of the type $(f_0, 2f_0)$ that are realizable. Finally we look at the remaining arrangements.

***Theorem* 18** — *There exists an arrangement of 3 lines with $f_0 = f_2$ if and only if $f_0 \geq 2$.*

PROOF : The 'only if' part follows at once from Theorem 15.

For the 'if' part set $\mathcal{A} = \{[1, 0 \mid 0], [0, 1 \mid 0], [f_0 - 2, -1 \mid r]\}$ where r is a fixed irrational number. The irrationality of r ensures that the line $[f_0 - 2, -1 \mid r]$ does not pass through the intersection of the other two lines. The reader can easily check that there are f_0 vertices, all of which have degree 2, so that $f_2 = f_1 - f_0 = 2t_2 - t_2 = t_2 = f_0$. □

***Theorem* 19** — *There exists an arrangement of 3 lines with f_0 vertices and $f_2 = 2f_0$ chambers if and only if f_0 is an odd number.*

PROOF : Start by assuming that f_0 is even. Since there are only 3 lines, it is clear that

$$f_2 = t_2 + 2t_3.$$

As $2f_0 = 2t_2 + 2t_3$ we have $t_2 = 0$ and consequently all vertices are of degree 3. Whence all the three lines l_1, l_2, l_3 in the arrangement pass through all the vertices. This shows that any two of the

l_i's intersect at f_0 many points. Without loss of generality assume that l_1 is of type $(0,1)$ and l_2 of type $(f_0, -a)$, where of course a, f_0 are coprime. Since l_3 intersects l_1 as well as l_2 at f_0 vertices, l_3 must be of the type $(f_0, -a \pm 1)$.

Now if f_0 is even then a must be an odd number coprime to f_0. This means that $-a \pm 1$ is even and hence the line of type $(f_0, -a \pm 1)$ has two components, which is a contradiction, for it gives an arrangement of 4 lines.

Conversely, if f_0 is odd, say $2k-1$, consider the arrangement of 3 lines one each of the types $(k, -(k-1))$, $(k-1, -k)$ and $(1,1)$. This gives an arrangement with $2k-1$ vertices and $4k-2$ faces. □

Theorem 20 — $\mathcal{A} = \{l_1, l_2, l_3\}$ *is an arrangement of* 3 *toric lines such that* $f_0 < f_2 < 2f_0$ *(i.e., it is neither simplicial nor simple) if and only if* $f_2 - f_0$ *divides* f_0 *and one of the following holds:*

1. $\dfrac{f_0}{f_2 - f_0}$ is even;
2. f_0 is odd.

PROOF : The necessary condition can be shown in two parts:

1. Show $f_2 - f_0$ divides f_0.
2. If $\dfrac{f_0}{f_2 - f_0}$ is odd, then f_0 must be odd.

First we express f_0 and $f_2 - f_0$ in terms of the types of lines in the arrangement. Without loss of generality, one of the lines l_1 can be assumed of type $(1,0)$. The other two l_2 and l_3 are, say, of types (a,b) and (c,d) respectively. We know that for arrangements of 3 lines,

$$f_0 = |l_1 \cap l_2| + |l_2 \cap l_3| + |l_3 \cap l_1| - 2|l_1 \cap l_2 \cap l_3|.$$

Now, formulate f_0 in terms of a, b, c, d by evaluating the above cardinalities in terms of a, b, c, d, i.e., $|l_1 \cap l_2| = b, |l_1 \cap l_3| = d, |l_2 \cap l_3| = |ad - bc|$. The intersection of all the lines (which is non-empty otherwise the arrangement is simple) is an intersection of cosets of two finite cyclic subgroups of a copy of S^1 in $\mathbb{T}^2$. Their intersection is of cardinality $\gcd(b,d)$. Hence, putting it all together:

$$f_0 = b + d + |ad - bc| - 2(b,d)$$

Since $f_2 - f_0 = t_3$ is by definition the cardinality of the grand intersection, we have the constraint $(b,d) = f_2 - f_0$. The part (1) above is an easy consequence of the fact that $\gcd(b,d)$ divides b and d.

Let $b = mt_3$ and $d = nt_3$ where $\gcd(m, n) = 1$. Hence

$$f_0 = (m + n + |an - cm| - 2)t_3.$$

Now the proof of part (2). By assumption we have $f_0 = 2^k r$ and $f_2 - f_0 = 2^k s$ where $r \mid s$; we need to show that $k = 0$. Since t_3 divides f_0 the above equation can be rewritten as

$$\frac{f_0}{f_2 - f_0} + 2 = m + n + |an - cm|$$

where the left hand side is odd. Since f_0 is odd $\iff k = 0 \iff f_2 - f_0$ is odd, we need to show t_3 is odd. If we show that either a or c is even (in which case ac is even) we are done, for $\gcd(a, b) = \gcd(a, mt_3) = 1 = \gcd(c, d) = \gcd(c, nt_3)$ thereby showing that $(ac, t_3) = 1$. But since ac is even, $t_3 = f_2 - f_0$ then has to be odd. Further, if m and n are both odd then the left hand side is odd and $m + n$ is even thereby forcing $an - cm$ to be odd. So we have either an or cm is even. But m, n being odd, a or c must be even.

Hence proving part (2) boils down to showing that m and n are both odd. We prove it by contradiction. Without loss of generality let m be even. Then $\gcd(m, n) = 1 = \gcd(a, mt_3)$ forces a and n to be odd, whereas cm is even. Hence the right hand side is now even which is a contradiction.

For the converse part of the theorem, we provide the following construction. It is understood that all the lines in the arrangements below pass through $\mathbf{0} \in \mathbb{T}^{\mathbf{2}}$. Let $f_2 - f_0 \mid f_0$:

(I) Assume f_0 and f_2 are such that condition (1) in the theorem holds. Indeed consider the arrangement $\mathcal{A}$ with 3 lines, one each of the types $(1, 0), \left(1, f_2 - \frac{f_0}{2}\right)$ and $(1, f_2 - f_0)$. One checks that $f_0(\mathcal{A}) = f_0$ and $t_3(\mathcal{A}) = f_2 - f_0$. Since $f_0(\mathcal{A}) + t_3(\mathcal{A}) = f_2(\mathcal{A})$, the arrangement $\mathcal{A}$ satisfies our requirement.

(II) Assume f_0 and f_2 satisfy condition (2) of the theorem: $\mathcal{A}$ contains three lines, one each of the types $(1, 0), (a, f_2 - f_0)$ and $(c, f_2 - f_0)$ where $\gcd(a, f_2 - f_0) = \gcd(c, f_2 - f_0) = 1$ and $a - c = \frac{f_0}{f_2 - f_0}$. To complete the proof we need to only show that given k, d odd, one can get a such that $\gcd(a, k) = 1 = \gcd(a + d, k)$. By the Chinese remainder Theorem, get an a such that $a \neq 0, -d \mod p \quad \forall p \mid k$. Check that $\gcd(a, k) = \gcd(a + d, k) = 1$. As f_0 is odd, clearly $f_2 - f_0$ and $\frac{f_0}{f_2 - f_0}$ are odd too, so one can use the above for $k = f_2 - f_0$ and $d = \frac{f_0}{f_2 - f_0}$. □

In general, complete characterization of $\mathcal{C}(n)$ seems to be a hard problem. We end this section by a result that characterizes arrangements for which the degree of the vertices is constant.

Proposition 21 — Let $\mathcal{A}$ be a toric arrangement of n lines such that $\deg(v) = k, \forall v \in L_0$ then $\mathcal{A}$ is either simple or simplicial.

PROOF : We know that $t_k \neq 0$ for some $k \geq 2$ and all other t_i's are zero. Therefore,

$$f_2 = (k-1)t_k = (k-1)f_0 \leq 2f_0.$$

Thus k is either 2 or 3. The $k = 2$ case implies that no three lines are concurrent which means that the arrangement is simple. Whereas for $k = 3$ using Lemma 14 we see that $p_k = 0$ for $k \geq 4$. □

4. THE CASE OF HIGHER GENUS SURFACES

We end the paper by a brief discussion about possible directions for future research. One direction is to look at arrangements in surfaces of higher genus (see [7, §6]) and the other direction is to study these problems in higher-dimensional tori. For $g > 1$ we consider the genus g surface M_g as the quotient of the hyperbolic plane $\mathbb{H}^2$ by a certain discrete subgroup G of the isometry group of $\mathbb{H}^2$. A *simple closed geodesic* in M_g is defined to be the image of a geodesic line in $\mathbb{H}^2$ under the covering projection $\mathbb{H}^2 \to \mathbb{H}^2/G$

Definition 22 — A finite collection of simple closed geodesics in M_g inducing a polytopal cell structure is known as a *geodesic arrangement* in a genus-g surface.

We analogously define the intersection poset $L(\mathcal{A})$ and the face numbers f_0, f_1, f_2 for geodesic arrangements. Some of the results proved in Section 3 easily generalize in this case. They remain unchanged except for an additional term $\chi(M_g)$, the Euler characteristic. We enumerate such results now.

***Theorem* 23** — *For a geodesic arrangement $\mathcal{A}$ in M_g we* have

$$f_1 = \sum_{v \in L_0} \deg v.$$

PROOF : By [1, Theorem 4.2]:

$$f_1 = \sum_{\dim Y = 1} \left(- \sum_{\substack{Z \in L(\mathcal{A}) \\ Y \leq Z}} \mu(Y, Z)\chi(Z) \right).$$

Since $\chi(Y) = 0$ we could replace $Y \leq Z$ with $Y < Z$ above. The summation then is over all Z

that cover Y, so that all μ values are -1. So the summation becomes:

$$f_1 = \sum_{\dim Y=1} \sum_{\substack{Z \in L(\mathcal{A}) \\ Y<Z}} \chi(Z).$$

Finally, $\chi(Z) = 1$ for the Z are just points.

$$f_1 = \sum_{\substack{\dim Z=0 \\ Z \in L(\mathcal{A})}} \left(\sum_{\substack{\dim Y=1 \\ Y<Z}} 1 \right).$$

But the inner summation is the degree of the vertex Z and we are done. □

Definition 24 — Let $\pi \colon \mathbb{H}^2 \to M_g$ denote the covering map. A subset $C \subseteq M_g$ is a k-gon if there exists a geodesic k-gon C_0 of $\mathbb{H}^2$ such that $\pi(C_0) = C$ and restriction of π to the interior of C_0 is a homeomorphism onto the image.

As before t_j stands for the number of degree j vertices and p_k stands for the number of k-gons.

Lemma 25 — The following results hold for a geodesic arrangement on a genus g surface:

$$f_0 = \sum_j t_j, f_1 = \sum_j jt_j, f_2 = \sum_k p_k,$$

$$2f_1 = \sum_k kp_k,\ f_2 - \chi(M_g) = \sum_j (j-1)t_j,\ 2(f_0 - \chi(M_g)) = \sum_k (k-2)p_k.$$

PROOF : The first three identities follow from definitions. The fourth identity is proved analogously as in Section 3. The fifth equation follows from the first two and the Euler relation. The last identity is a consequence of the third, fourth and the Euler relation. □

Denote by d_j the number of vertices that are incident with j edges. Note that since all the edges are formed by intersections of lines we have $t_j = 0$ if either $j = 2$ or j is odd.

Lemma 26 — For a geodesic arrangement in a genus-g surface, we have the following:

$$t_2 - 3\chi(M_g) = \sum_{j \geq 3} (j-3)t_j + \sum_{k \geq 3} (k-3)p_k,$$

$$p_3 - 4\chi(M_g) = \sum_{j \geq 2} 2(j-2)t_j + \sum_{k \geq 4} (k-4)p_k,$$

$$3p_3 + 2p_4 + p_5 - 6\chi(M_g) = \sum_{j \geq 4} (j-3)d_j + \sum_{k \geq 7} (k-6)p_k.$$

PROOF : The proof follows from the application of the Euler relation and Lemma 25 above. □

We now state a partial analogue of Theorem 16; the proof is on the similar lines.

***Theorem* 27** — *For an arrangement on a genus-g surface, we have the following:*

$$f_0 + \chi(M_g) \leq f_2 \leq 2(f_0 - \chi(M_g)).$$

The equality on the left holds if and only if the arrangement is simple (i.e., $t_j = 0$ for $j \geq 3$) whereas the equality on the right holds if and only if the arrangement is simplicial (i.e., $p_k = 0$ for $k \geq 4$).

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